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A REMARK ON A DISTORTION THEOREM IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. A distortion theorem on a homogeneous bounded domain in $\mathbb{C}^n$ is obtained which is the generalization of Schwarz lemma.

1. PRELIMINARIES

We denote a point z of $\mathbb{C}^n$ by the column vector $z = (z_1, \ldots, z_n)'$. We denote a mapping $f(z)$ from a domain D in $\mathbb{C}^n$ to $\mathbb{C}^n$ by the column vector $f(z) = (f_1(z), \ldots, f_n(z))'$. The mapping $f(z)$ is said to be holomorphic in D if each component function is holomorphic in D. We denote the Jacobian matrix of the mapping $f(z)$ by

$$\frac{\partial f}{\partial z}(z) \left(:= \frac{\partial}{\partial z} \times f(z) \right),$$

where

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right).$$

Let D be a bounded domain in $\mathbb{C}^n$. $K_D(z, z)$ denotes the Bergman kernel function of D.

Let

$$T_D(z, z) = \frac{\partial^2}{\partial z^* \partial z} \log K_D(z, z),$$

where

$$\frac{\partial}{\partial z^*} = \left(\frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}} \right)'.$$

We define as follows: ([5])

$$K_{D,(p,q)}(z,z) = K_D^p(z,z)(\det T_D(z,z))^q,$$

$$T_{D,(p,q)}(z,z) = \frac{\partial^2}{\partial z^* \partial z} \log K_{D,(p,q)}(z,z), (p,q \geq 0).$$

When $p = 1$ and $q = 0$, $K_{D,(p,q)}(z,z)$ and $T_{D,(p,q)}(z,z)$ denote the ordinary Bergman kernel function $K_D(z,z)$ and the Bergman metric tensor $T_D(z,z)$ respectively.

We have the following relative biholomorphic invariant formula:

Let F be a biholomorphic mapping from D onto $F(D)(:= \Delta)$. Then

(1) $$K_{D,(p,q)}(z,z) = \left(\overline{\det \frac{\partial F}{\partial z}(z)}\right)^{p+q} K_{\Delta,(p,q)}(F(z),F(z)) \left(\det \frac{\partial F}{\partial z}(z)\right)^{p+q},$$

(2) $$T_{D,(p,q)}(z,z) = \left(\frac{\partial F}{\partial z}(z)\right)^* T_{\Delta,(p,q)}(F(z),F(z)) \left(\frac{\partial F}{\partial z}(z)\right).$$

Throughout this paper, the symbols $\prime, *and\times$ stand for transposition,conjugated transposition and Kronecker product, respectively.

We say the bounded domain D is a (p,q)-minimal domain with center at $\tau \in D$ if $K_{D,(p,q)}(z,\tau) = K_{D,(p,q)}(\tau,\tau), \forall z \in D$ holds. For $p = 1$ and $q = 0$, this concept coincides with the minimal domain in the sense of Maschler.

After Hahn ([3]), we define as follows:

$$c(D) := \left\{ t \in D \middle| K_D(t,t) = \frac{1}{vol(D)} \right\},$$

$$m(D) := \left\{ t \in D \middle| K_{D,(p,q)}(t,t) \leq \min_{z \in D} K_{D,(p,q)}(z,z) \right\}.$$

The following facts are known: ([3],[8],[10]).

If $K_D(z,z)$ becomes infinite everywhere on ∂D, then $m(D) \neq \emptyset$ and $m(D) \supset c(D)$. For example,if D is a homogeneous bounded domain,then $K_D(z,z)$ becomes infinite everywhere on ∂D,and so $m(D) \neq \emptyset$ and $m(D) \supset c(D)$. The set $c(D)$ consists of at most one point of D, and is non-empty if and only if $c(D) = m(D)$ for $p = 1$ and $q = 0$. D is a minimal domain with center at t in the sense of Maschler if and only if $\{t\} = c(D) \neq \emptyset$.

2. Distortions on a homogeneous bounded domain

At first we give the following Proposition obtained by Carathéodory and Cartan.

Proposition ([7]). *Let D be a bounded domain in $\mathbb{C}^n$, and let $f : D \longrightarrow D$ be holomorphic. Let $p \in D$, and suppose that $f(p) = p$. Then*

$$\left|\det \frac{\partial f}{\partial z}(p)\right| \leq 1.$$

If $\left|\det \frac{\partial f}{\partial z}(p)\right| = 1$, then f is an automorphism of D

Using the above Proposition and the biholomorphic invariant formulas (1) and (2), we have the following:

Theorem 1. *Let D be a homogeneous bounded domain in $\mathbb{C}^n$. Let F be a biholomorphic map from D onto $F(D) := \Delta$. Let f be a holomorphic map from D into Δ. Then*

$$\left|\det \frac{\partial f}{\partial z}(z)\right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z,z)}{K_{\Delta,(p,q)}(f(z),f(z))},$$

$$\left|\det \frac{\partial f}{\partial z}(z)\right|^{2} \leq \frac{\det T_{D,(p,q)}(z,z)}{\det T_{\Delta,(p,q)}(f(z),f(z))}, z \in D, p,q \geq 0.$$

Proof. Put $f(t) = \alpha$, $F(t) = \beta$, $t \in D$. Let $\phi(w)$ be an automorphism of the homogeneous bounded domain Δ such that $\phi(\alpha) = \beta$.

Let $g := F^{-1} \circ \phi \circ f$. Then g is a holomorphic map from D into itself with $g(t) = t$. From the Proposition, we have

$$\left|\det \frac{\partial g}{\partial z}(t)\right| = \left|\det \left(\frac{\partial}{\partial z}(F^{-1} \circ \phi \circ f)(t)\right)\right| \leq 1.$$

Noting that

$$\frac{\partial F^{-1}}{\partial w} = \left(\frac{\partial F}{\partial z}(z)\right)^{-1},$$

where $w = F(z)$, by chain rule, we have

$$\left|\det \frac{\partial f}{\partial z}(t)\right| \leq \frac{\left|\det \frac{\partial F}{\partial z}(t)\right|}{\left|\det \frac{\partial \phi}{\partial w}(\alpha)\right|}.$$

The biholomorphic relative invariants of $K_{D,(p,q)}(z,z)$ and $T_{D,(p,q)}(z,z)$ give us the following:

$$K_{D,(p,q)}(t,t) = K_{\Delta,(p,q)}(\beta,\beta)\left|\det \frac{\partial F}{\partial z}(t)\right|^{2(p+q)},$$

$$K_{\Delta,(p,q)}(\alpha,\alpha) = K_{\Delta,(p,q)}(\beta,\beta)\left|\det \frac{\partial \phi}{\partial w}(\alpha)\right|^{2(p+q)},$$

$$\det T_{D,(p,q)}(t,t) = \det T_{\Delta,(p,q)}(\beta,\beta)\left|\det \frac{\partial F}{\partial z}(t)\right|^{2},$$

$$\det T_{\Delta,(p,q)}(\alpha,\alpha) = \det T_{\Delta,(p,q)}(\beta,\beta)\left|\det \frac{\partial \phi}{\partial w}(\alpha)\right|^{2}.$$

Therefore the proof is completed, since we may take t to be an arbitrary point in D.

Remark. Since $K_{D,(p,q)}(z,z)$ and $T_{D,(p,q)}(z,z)$ are the ordinary Bergman kernel function and the Bergman metric tensor for $p = 1$ and $q = 0$, we have

$$\left|\det \frac{\partial f}{\partial z}(z)\right|^2 \leq \frac{K_D(z,z)}{K_\Delta(f(z),f(z))} = \frac{\det T_D(z,z)}{\det T_\Delta(f(z),f(z))}.$$

In particular, since the Bergman kernel function of the unit ball

$$B_n = \left\{ z \in \mathbb{C}^n \,\middle|\, |z|^2 = \sum_{j=1}^{n} |z_j|^2 < 1 \right\}$$

is

$$K_{B_n}(z,z) = \frac{n!}{\pi^n} \frac{1}{(1-|z|^2)^{n+1}},$$

we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq \left(\frac{1-|f(z)|^2}{1-|z|^2} \right)^{n+1}.$$

In the case of n=1 (i.e. for the unit disc), we have

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2},$$

which is the well-known Schwarz Lemma.

Corollary ([2],[6]). *Let f be a holomorphic map of a homogeneous bounded domain D into itself. Then we have*

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z,z)}{K_{D,(p,q)}(f(z),(f(z))}.$$

In particular, $\tau_0 \in m(D)$,which is non-empty, we have

$$\left| \det \frac{\partial f}{\partial z}(\tau_0) \right| \leq 1.$$

Remark. In Theorem 1, since Δ is a homogeneous bounded domain, there exists $\tau_0 \in m(\Delta)$. Then we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z,z)}{K_{\Delta,(p,q)}(\tau_0,\tau_0)}, z \in D.$$

In particular for $p = 1$ and $q = 0$, if τ_0 belongs to $c(\Delta)$, we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq K_D(z,z)vol(\Delta), z \in D.$$

Theorem 2. *Let D be a bounded domain with $t_0 \in m(D)$. Let F be a biholomorphic map from D onto $F(D) =: \Delta$ with $\tau_0 = F(t) \in m(\Delta)$ for $t \neq t_0$. Then we have*

$$\begin{aligned}\left|\det \frac{\partial F}{\partial z}(t)\right|^{2(p+q)} &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \left|\det \frac{\partial F}{\partial z}(t_0)\right|^{2(p+q)}.\end{aligned}$$

In particular, if D is a homogeneous bounded domain and if f is a holomorphic map from D into $F(D) =: \Delta$, then we have

$$\left|\det \frac{\partial F}{\partial z}(t)\right| \geq max\left\{\left|\det \frac{\partial f}{\partial z}(t)\right|, \left|\det \frac{\partial f}{\partial z}(t_0)\right|\right\}. \tag{3}$$

Proof. Noting that $t_0 \in m(D)$ and $\tau_o \in m(\Delta)$, we have, for $\tau = F(t_0)$,

$$\begin{aligned}\left|\det \frac{\partial F}{\partial z}(t)\right|^{2(p+q)} &= \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau, \tau)} \\ &= \left|\det \frac{\partial F}{\partial z}(t_0)\right|^{2(p+q)}.\end{aligned}$$

If D is a homogeneous domain with $m(D) \neq \phi$, then $F(D) =: \Delta$ is also homogeneous with $m(\Delta) \neq \phi$. Therefore we have, for $\tau_0 = F(t)$,

$$\begin{aligned}\left|\det \frac{\partial F}{\partial z}(t)\right|^{2(p+q)} &= \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(f(z), f(z))} \\ &= \frac{K_{D,(p,q)}(t, t)}{K_{D,(p,q)}(z, z)} \cdot \frac{K_{D,(p,q)}(z.z)}{K_{\Delta,(p,q)}(f(z), f(z))} \\ &\geq \frac{K_{D,(p,q)}(t, t)}{K_{D,(p,q)}(z, z)} \left|\det \frac{\partial f}{\partial z}(z)\right|^{2(p+q)}.\end{aligned}$$

Since $K_{D,(p,q)}(z,z) \geq K_{D,(p,q)}(t_0,t_0)$, we have (3).

From Theorem 2 the following Corollary easily follows.

Corollary. *Let D be a bounded minimal domain with center at $t_0 \in c(D)$ in the sense of Maschler. Let F be a biholomorphic map from D onto $F(D) =: \Delta$ with $\tau_0 = F(t)$. Let $F(D) =: \Delta$ be a bounded minimal domain with center at $\tau_0 \in c(\Delta)$. Then we have*

$$\left|\det \frac{\partial F}{\partial z}(t)\right|^2 \geq \frac{vol(F(D))}{vol(D)} \geq \left|\det \frac{\partial F}{\partial z}(t_0)\right|^2,$$

where the equality signs hold if and only if $t = t_0$. In particular, if F is a volume preserving biholomorphic map, then we have

$$\left|\det \frac{\partial F}{\partial z}(t)\right| \geq 1 \geq \left|\det \frac{\partial F}{\partial z}(t_0)\right|.$$

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