数理解析研究所講究録
918巻 1995年 178-195

On embedding of classical substructural logics

Ken-etsu Fujita
Kyushu Institute of Technology
email: fujiken@ai.kyutech.ac.jp

27 March, 1995

Abstract

There is an intimate connection between proofs of the natural deduction systems and typed lambda calculus. It is well-known that in simply typed lambda calculus, the notion of formulae-as-types makes it possible to find fine structure of the implicational fragment of intuitionistic logic, i.e., relevance logic, BCK-logic and linear logic. In this paper, we investigate classical substructural logics consisting of implication and negation. However our method would be general to be applied to rich systems beyond the fragment. We show that proofs in Parigot's $\lambda\mu$-calculus with proper constraints exactly correspond to proofs of substructural logics of Gentzen's LK. Moreover we discuss three embedding of classical substructural logics into the corresponding intuitionistic substructural logics.

1 Introduction and motivation

In the implicational fragment of Hilbert style intuitionistic logic, one can find three substructural logics, i.e., BCI, BCK and Relevance logic which are characterized by some structural rules in terms of sequent calculus. The proofs of these intuitionistic substructural logics of Hilbert style correspond to BCI-λ-terms, BCK-λ-terms and λI-terms respectively via the Curry-Howard isomorphism. Usually the classical system of Hilbert style is obtained by adding Peirce's law to the intuitionistic system. However it might be known that adding Peirce's law makes each intuitionistic substructural logic classical. In other words, we can derive both the axiom K $(A \supset B \supset A)$ and W $((A \supset A \supset B) \supset A \supset B)$ in the implicational fragment of LJ with the left exchange rules, the cut rules and Peirce's law as axioms where the cut rules in the proofs could not be removed.
On the other hand, H.Ono [Ono90] proved that $GL_x \vdash \Gamma \rightarrow A$ iff $FL_{ex} \vdash \Gamma \rightarrow A$ with proviso that Γ and A contain neither multiplicative disjunction nor multiplicative constant 0. Here x denotes empty, c (adding contraction rules) or w (adding weakening rules). FL is called full lambek calculus, and GL is a classical logic with neither contraction nor weakening rules, which is same as Girard's linear logic. Since we usually define negations using the multiplicative 0, this theorem might mean that we cannot expect classical substructural logics without negations, and that even if we take classical substructural logics without negations, they are essentially intuitionistic.
This paper investigates classical substructural logics consisting of implication and negation. However our discussion is not available only for the fragment. As a natural extension of restricted λ-terms (BCI-λ-terms, BCK-λ-terms and λI-terms), we define the corresponding classical proof terms (GL_X-$\lambda\mu$-terms where X is nil, C or W) in terms of Parigot's $\lambda\mu$-term with proper restrictions. It is proved that these classical terms exactly represent proofs of Gentzen's LK without weakening rules or contraction rules, i.e., the well-known notion of Curry-Howard isomorphism with respect to classical substructural logics, and that there exists a principal type scheme. We show that three double negation translations (Gödel, Kolmogorov and Kuroda) give embeddings of the classical substructural logics into the corresponding intuitionistic substructural logics. As corollaries of the embedding theorem, it is obtained that every GL_W-$\lambda\mu$-term is stratified, typability and inhabitation are decidable, well-typed GL_X-$\lambda\mu$-terms are strongly normalizable.

2 The $\lambda\mu$-Calculus

Originally $\lambda\mu$-calculus was invented by M.Parigot [Pari92-2] as a multiple-consequence natural deduction system in order to give a naturally computational meaning to classical proofs via the Curry-Howard

isomorphism [How80]. Here we introduce the modified version in which inference rules have one consequence like in NJ and naming rules are treated as a form of applications. The syntax of the $\lambda\mu$-term M is defined by λ-variables x and μ-variables α:
$M ::= x|MM|\lambda x.M|\alpha M|\mu\alpha.M$.
The set of λ-free variables and λ-bound variables in M are usually defined, which are respectively denoted by $\lambda FV(M)$ and $\lambda BV(M)$. The set of μ-free variables and μ-bound variables in M are also naturally defined, which are denoted by $\mu FV(M)$ and $\mu BV(M)$ respectively. If $\lambda FV(M) = \phi$, then we call M as λ-closed. If $\mu FV(M) = \phi$, then we call M as μ-closed. When M is λ-closed and μ-closed, we call M as closed.

We have two kinds of types, types indexed with λ-variables and negated types indexed with μ-variables. In the following $\neg\Delta$ is a set of μ-indexed negated types and distinct types never have the same index, and Γ denotes the usual set of types with λ-variables. The set of type assignment rules $TA_{\lambda\mu}$ is defined as follows together with the rule that infer $\Gamma, \neg\Delta \vdash x : A$ from $x : A \in \Gamma$.

$$\frac{\Gamma, x : A_1, \neg\Delta \vdash M : A_2}{\Gamma, \neg\Delta \vdash \lambda x.M : A_1 \to A_2} (\to I) \qquad \frac{\Gamma_1, \neg\Delta_1 \vdash M_1 : A_2 \to A_1 \quad \Gamma_2, \neg\Delta_2 \vdash M_2 : A_2}{\Gamma_1, \Gamma_2, \neg\Delta_1, \neg\Delta_2 \vdash M_1 M_2 : A_1} (\to E)$$

$$\frac{\Gamma, \neg\Delta, \alpha : \neg A \vdash M : \bot}{\Gamma, \neg\Delta \vdash \mu\alpha.M : A} (\bot E) \qquad \frac{\alpha : \neg A \in \neg\Delta \quad \Gamma, \neg\Delta \vdash M : A}{\Gamma, \neg\Delta \vdash \alpha M : \bot} (\bot I)$$

The first two rules are called logical rules and the latter two are called naming rules. When there is a $TA_{\lambda\mu}$ deduction of a statement $\Gamma, \neg\Delta \vdash M : A$ where $\neg\Delta$ is a set of negation types indexed with μ-variables and Γ is a set of undischarged assumptions indexed by λ-variables, we say M is stratified. Let Γ be $x_1 : A_1, \cdots, x_m : A_m$ and $\neg\Delta$ be $\alpha_1 : \neg A_1, \cdots, \alpha_n : \neg A_n$, then a set of λ-variables $\lambda Subjects(\Gamma)$ is defined by $\{x_1, \cdots, x_m\}$ and a set of μ-variables $\mu Subjects(\neg\Delta)$ is $\{\alpha_1, \cdots, \alpha_n\}$. Sometimes we need a minor modification of the Parigot's $\lambda\mu$-calculus with respect to the treatment of negations. For instance, closed $\lambda\mu$-terms in our usual sense might contain free μ-variables which are indexes of $\bot$. P.de Groote gives one modification adding negation rules to overcome this problem. The above system of one version of the $\lambda\mu$-calculus also avoids this kind of problem.

The one step reduction rules with respect to μ-abstraction are usually defined as follows.
Structural reduction: contract $(\mu\alpha.M)M_1$ to $(\mu\alpha.M)[\alpha \Leftarrow M_1]$ where
(1) $x[\alpha \Leftarrow M_1] = x$;
(2) $(\lambda x.M)[\alpha \Leftarrow M_1] = \lambda x.M[\alpha \Leftarrow M_1]$;
(3) $(MM')[\alpha \Leftarrow M_1] = M[\alpha \Leftarrow M_1]M'[\alpha \Leftarrow M_1]$;
(4) $(\mu\beta.M)[\alpha \Leftarrow M_1] = \mu\beta.M[\alpha \Leftarrow M_1]$;
(5) $(\alpha M)[\alpha \Leftarrow M_1] = \alpha(M[\alpha \Leftarrow M_1]M_1)$;
(6) $(\beta M)[\alpha \Leftarrow M_1] = \beta M[\alpha \Leftarrow M_1]$ if $\beta \not\equiv \alpha$.
The second reduction is $(S1)$ called a renaming reduction.
$(S1)$: contract $\alpha\mu\beta.M$ to $M[\beta := \alpha]$.
Other reductions are called $(S2)$ and $(S3)$ respectively.
$(S2)$: contract $\mu\alpha.\alpha M$ to M if $\alpha \notin \mu FV(M)$.
$(S3)$: contract $\mu\alpha.M$ to $\lambda x.\mu\alpha.M[\alpha \Leftarrow x]$ if M contains a subterm of the form $\alpha\lambda y.M'$ for some M'.
The binary relations $\triangleright$, $\triangleright^*$ and $=_\mu$ on the set of $\lambda\mu$-terms are defined with the usual β-reductions, structural reductions and $(S2)$.
1-1. $(\lambda x.M_1)M_2 \triangleright M_1[x := M_2]$;
1-2. $(\mu\alpha.M_1)M_2 \triangleright \mu\alpha.M_1[\alpha \Leftarrow M_2]$;
1-3. $\mu\alpha.\alpha M \triangleright M$ if $\alpha \notin \mu FV(M)$;
1-4. If $M_1 \triangleright M_2$, then $MM_1 \triangleright MM_2$, $M_1M \triangleright M_2M$, $\lambda x.M_1 \triangleright \lambda x.M_2$, $\mu\alpha.M_1 \triangleright \mu\alpha.M_2$ and $\alpha M_1 \triangleright \alpha M_2$.

2-1. $M \triangleright^* M$;
2-2. If $M_1 \triangleright M_2$, then $M_1 \triangleright^* M_2$;
2-3. If $M_1 \triangleright^* M_2$ and $M_2 \triangleright^* M_3$, then $M_1 \triangleright^* M_3$.

3-1. If $M_1 \triangleright^* M_2$, then $M_1 =_\mu M_2$;
3-2. If $M_1 =_\mu M_2$, then $M_2 =_\mu M_1$;
3-3. If $M_1 =_\mu M_2$ and $M_2 =_\mu M_3$, then $M_1 =_\mu M_3$.

Similarly we define $\triangleright_-$, $\triangleright^*_-$ and $=_{\mu-}$ without $(S2)$. $\triangleright_+$, $\triangleright^*_+$ and $=_{\mu+}$ are defined with all the above rules. $\triangleright_\beta$, $\triangleright^*_\beta$ and $=_\beta$ are used for the usual binary relation on λ-terms respectively obtained by the one step β-reduction, the reflexive and transitive closure, and equality relation. We implicitly use α-conversion.

Lemma 1 (Basis Lemma) .
(1) If $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ and $\Gamma, \neg\Delta \vdash M : A$, then $\Gamma', \neg\Delta' \vdash M : A$.
(2) If $\Gamma, \neg\Delta \vdash M : A$, then $\lambda FV(M) \subseteq \lambda Subjects(\Gamma)$ and $\mu FV(M) \subseteq \mu Subjects(\neg\Delta)$.
(3) If $\Gamma, \neg\Delta \vdash M : A$, then $\Gamma \uparrow \lambda FV(M), \neg\Delta \uparrow \mu FV(M) \vdash M : A$.

Proof. By induction on the derivation of $M : A$.

By the Basis Lemma, it is clearly remarked that $\Gamma, \neg\Delta \vdash M : A$ in the above system iff $\Gamma \vdash M : A, \Delta$ in Parigot's original $\lambda\mu$-calculus where $\Gamma, \neg\Delta \vdash M : \bot$ in the above system is identified with $\Gamma \vdash M : \Delta$ in Parigot's $\lambda\mu$-calculus, and for the name δ of $\bot$, $\mu\delta.M$ is identified with M and $[\delta]M$ is M.

Lemma 2 (Generation Lemma) .
(1) If $\Gamma, \neg\Delta \vdash x : A$, then $x : A \in \Gamma$.
(2) If $\Gamma, \neg\Delta \vdash M_1 M_2 : A$ where M_1 is not a μ-variable, then $\Gamma_1, \neg\Delta_1 \vdash M_1 : B \to A$ and $\Gamma_2, \neg\Delta_2 \vdash M_2 : B$ for some B where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = \Delta_1 \cup \Delta_2$.
(3) If $\Gamma, \neg\Delta \vdash \alpha M : A$, then A is $\bot$ and $\Gamma, \neg\Delta \vdash M : B$ for some B where $\alpha : \neg B \in \neg\Delta$.
(4) If $\Gamma, \neg\Delta \vdash \lambda x.M : A$, then $\Gamma, x : B, \neg\Delta \vdash M : C$ for some B, C where $A \equiv B \to C$.
(5) If $\Gamma, \neg\Delta \vdash \mu\alpha.M : A$, then $\Gamma, \neg\Delta, \alpha : \neg A \vdash M : \bot$.

Proof. By induction on the length of the derivation.

Lemma 3 (Substitution Lemma) .
(1) If $\Gamma, \neg\Delta \vdash M : A$, then $\Gamma\sigma, \neg\Delta\sigma \vdash M : A\sigma$ where σ is a substitution replacing a type variable with a type.
(2) If $\Gamma_1, x : A, \neg\Delta_1 \vdash M_1 : B$ and $\Gamma_2, \neg\Delta_2 \vdash M_2 : A$, then $\Gamma_1, \Gamma_2, \neg\Delta_1, \neg\Delta_2 \vdash M_1[x := M_2] : B$.
(3) If $\Gamma, \neg\Delta, \beta : \neg A \vdash M : B$ and $\alpha : \neg A \in \neg\Delta$, then $\Gamma, \neg\Delta \vdash M[\beta := \alpha] : B$.
(4) If $\Gamma_1, \neg\Delta_1 \vdash M_1 : A$ and $\Gamma_2, \neg\Delta_2 \vdash M_2 : B$ where $\alpha : \neg(B \to C) \in \neg\Delta_1$, then $\Gamma_1, \Gamma_2, \Delta_1 - \{\alpha : \neg(B \to C)\}, \alpha : \neg C, \neg\Delta_2 \vdash M_1[\alpha \Leftarrow M_2] : A$.

Proof. (1) By induction on the derivation of $\Gamma, \neg\Delta \vdash M : A$.
(2) By induction on the derivation of $\Gamma_1, x : A, \neg\Delta_1 \vdash M_1 : B$.
(3) By induction on the derivation.
(4) When $\alpha \notin \lambda FV(M_1)$, we have $M_1[\alpha \Leftarrow M_2] = M_1$, and apply Basis Lemma. Otherwise by the Generation Lemma, there is $\Gamma_3, \neg\Delta_3 \vdash \alpha M_3 : \bot$ for some Γ_3, M_3 and Δ_3, and hence $\Gamma_3, \neg\Delta_3 \vdash M_3 : B \to C$ where $\alpha : \neg(B \to C) \in \neg\Delta_3$. For such every subdeduction, we apply $\Gamma_2, \neg\Delta_2 \vdash M_2 : B$ to obtain $\Gamma_2, \Gamma_3, \neg\Delta_2, \neg\Delta_3 \vdash M_3 M_2 : C$ and hence $\Gamma_2, \Gamma_3, \neg\Delta_2, \neg\Delta_3 - \{\alpha : \neg(B \to C)\}, \alpha : \neg C \vdash \alpha M_3 M_2 : \bot$. Thus a derivation of $\Gamma_1, \Gamma_2, \neg\Delta_1 - \{\alpha : \neg(B \to C)\}, \alpha : C, \neg\Delta_2 \vdash M_1[\alpha \Leftarrow M_2]$ is obtained where $\alpha : B \to C \in \Delta_1$ was replaced with $\alpha : C$.

It is proved in [Pari92-2][Pari93] that for untyped or typed $\lambda\mu$-terms, the reduction rules have Confluent Property, Type preservation property and Strong normalization property.

Putting proper restrictions on $\lambda\mu$-terms makes it possible to define the notions of GL_X-$\lambda\mu$-terms (X is nil, C or W) which would correspond to proofs of classical substructural logics respectively. We give the definitions bellow, which all are natural extension of the well-known intuitionistic cases.

Definition 1 (*GL*-$\lambda\mu$-terms) .
1. Every λ-variable is GL-$\lambda\mu$-term.
2. If M_1 and M_2 are GL-$\lambda\mu$-terms, $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$, then so is $M_1 M_2$.
3. If M_1 and M_2 are GL-$\lambda\mu$-terms, $x \in \lambda FV(M_1)$ and $\alpha \in \mu FV(M_2)$, then so are $\lambda x.M_1$ and $\mu\alpha.M_2$.
4. If M is a GL-$\lambda\mu$-term and $\alpha \notin \mu FV(M)$, then so is αM.

The clause 2 forbids the left and right contraction rules on applications, and the right contractions are not allowed by the clause 4 in the other cases. The clause 3 excludes the left and right weakening rules on abstractions.

Definition 2 (GL_C-$\lambda\mu$-terms) .
1. Every λ-variable is GL_C-$\lambda\mu$-term.
2. If M_1 and M_2 are GL_C-$\lambda\mu$-terms, then so is $M_1 M_2$.
3. If M_1 and M_2 are GL_C-$\lambda\mu$-terms, $x \in \mu FV(M_1)$ and $\alpha \in \mu FV(M_2)$, then so are $\lambda x.M_1$ and $\mu\alpha.M_2$.
4. If M is a GL_C-$\lambda\mu$-term, then so is αM.

Definition 3 **(GL_W-$\lambda\mu$-terms)** .
1. Every λ-variable is GL_W-$\lambda\mu$-term.
2. If M_1 and M_2 are GL_W-$\lambda\mu$-terms, $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$, then so is $M_1 M_2$.
3. If M is a GL_W-$\lambda\mu$-term, then so are $\lambda x.M$ and $\mu\alpha.M$.
4. If M is a GL_W-$\lambda\mu$-term and $\alpha \notin \mu FV(M)$, then so is αM.

When no conditions are applied on $\lambda\mu$-terms, we call the terms as GL_{CW}-$\lambda\mu$-terms, which are exactly $\lambda\mu$-terms.

3 GL_X-$\lambda\mu$-terms are proofs of GL_X

Following [Ono90], we call the implicational and negational fragment of Gentzen's LK without the contraction rules and the weakening rules as GL. We show that GL_X-$\lambda\mu$-terms correspond to proofs of GL_X. In other words, according to the notion of formulae-as-types [How80], the types of GL_X-$\lambda\mu$-terms are provable in GL_X for each X. We define GL as the following sequent calculus system with the right and left exchange rules.

$$A \to A$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}\ (\neg \to) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}\ (\to \neg)$$

$$\frac{\Gamma_1 \to \Delta_1, A_1 \quad A_2, \Gamma_2 \to \Delta_2}{A_1 \supset A_2, \Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\ (\supset\to) \qquad \frac{A_1, \Gamma \to \Delta, A_2}{\Gamma \to \Delta, A_1 \supset A_2}\ (\to\supset).$$

$$\frac{\Gamma_1 \to \Delta_1, A \quad A, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\ (cut)$$

We define GL_C as the above GL plus the right and left contraction rules, and GL_W as GL together with the right and left weakening rules. GL_{CW} is defined as GL with all structural rules. $\tilde{\Gamma}$ is defined as a set of types with distinct λ-variables for the sequent Γ. $\tilde{\Delta}$ is a set of types for the sequent Δ, which consists of types with distinct μ-variables. We refer to the following two theorems from [Ono90].

Theorem 1 **(Grišin, Wroński-Krzystek)** .
The cut elimination theorem holds for GL, GL_C, GL_W and GL_{CW}.

Theorem 2 .Let FL_{EX} be an intuitionistic fragment of GL_X, i.e., based on LJ. Then $GL_X = FL_{EX} + \neg\neg A \supset A$ where X is either empty or any of C, W and CW.

Now we prove that GL_X proofs are represented as GL_X-$\lambda\mu$-terms.

Theorem 3 **(GL_X proofs as GL_X-$\lambda\mu$-terms)** .
(1) If $\Gamma \to \Delta, A$ in GL or GL_C, then there exists a GL-$\lambda\mu$-term or GL_C-$\lambda\mu$-term M respectively such that $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : A$ in $TA_{\lambda\mu}$ where $\lambda FV(M) = \lambda Subjects(\tilde{\Gamma})$ and $\mu FV(M) = \mu Subjects(\tilde{\Delta})$.
(2) If $\Gamma \to \Delta, A$ in GL_W or GL_{CW}, then there exists a GL_W-$\lambda\mu$-term or GL_{CW}-$\lambda\mu$-term M respectively such that $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : A$ in $TA_{\lambda\mu}$.

Proof. By induction on the number of sequents contained in the cut-free derivation of GL_X and case analysis on the last rule. We show only the following cases.
(1) Case1. $(\supset\to)$:
By the induction hypotheses, we have $\tilde{\Gamma}_1, \neg\tilde{\Delta}_1 \vdash M_1 : A_1$ and $x : A_2, \tilde{\Gamma}_2, \neg\tilde{\Delta}_2 \vdash M_2 : \perp$ for some GL_X-$\lambda\mu$-terms M_1 and M_2 such that $\lambda FV(M_1) = \lambda Subjects(\tilde{\Gamma}_1)$, $\mu FV(M_1) = \mu Subjects(\tilde{\Delta}_1)$, $\lambda FV(M_2) = \lambda Subjects(\tilde{\Gamma}_2) \cup \{x\}$ and $\mu FV(M_2) = \mu Subjects(\tilde{\Delta}_2)$. We assume that it is possible to take λ-variables and μ-variables such that $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$. This harmless assumption is also used in the following cases. Now we can take a new variable z with the type $A_1 \to A_2$. Since $\lambda x.M_2$ is a GL_X-$\lambda\mu$-term, so is $(zM_1)\lambda x.M_2$. Thus $z : A_1 \to A_2, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \neg\tilde{\Delta}_1, \neg\tilde{\Delta}_2 \vdash (zM_1)\lambda x.M_2 : \perp$ is obtained by $(\to E)$ where $\lambda FV((zM_1)\lambda x.M_2) = \lambda Subjects(\tilde{\Gamma}_1) \cup \lambda Subjects(\tilde{\Gamma}_2) \cup \{z\}$ and $\mu FV((zM_1)\lambda x.M_2) = \mu Subjects(\tilde{\Delta}_1) \cup \mu Subjects(\tilde{\Delta}_2)$ are satisfied.
Case2. $(\to C)$:
The induction hypothesis gives $\tilde{\Gamma}, \neg\tilde{\Delta}, \alpha : \neg A \vdash M : A$ for some GL_C-$\lambda\mu$-term M where $\lambda FV(M) =$

$\lambda Subjects(\tilde{\Gamma})$ and $\mu FV(M) = \mu Subjects(\tilde{\Delta}) \cup \{\alpha\}$. By ($\perp I$) and ($\perp E$), we have $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash \mu\alpha.\alpha M : A$ where the proof term is a GL_C-$\lambda\mu$-term.

(2) Case1. ($\supset\rightarrow$):
By the induction hypotheses, we have $\tilde{\Gamma}_1, \neg\tilde{\Delta}_1 \vdash M_1 : A_1$ and $x : A_2, \tilde{\Gamma}_2, \neg\tilde{\Delta}_2 \vdash M_2 : \perp$ for some GL_X-$\lambda\mu$-terms M_1 and M_2. Since we choose variables such that $(zM_1)\lambda x.M_2$ is a GL_X-$\lambda\mu$-term, $z : A_1 \rightarrow A_2, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \neg\tilde{\Delta}_1, \neg\tilde{\Delta}_2 \vdash (zM_1)\lambda x.M_2 : \perp$ is obtained by ($\rightarrow E$) where z is a fresh variable.
Case2. ($\rightarrow W$):
The induction hypothesis gives that $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : \perp$ where M is a GL_W or GL_{CW}-$\lambda\mu$-term. Now we take a new variable α such that $\alpha \notin \mu FV(M)$, then $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash \mu\alpha.M : A$ by Basis Lemma and ($\perp E$) where the proof term is a GL_W or GL_{CW}-$\lambda\mu$-term.

For $\Gamma, \neg\Delta \vdash M : A$, sequents of formulae Γ^* and Δ^* are defined as follows:
$\{\ \}^* = <\ >$;
$(\{x : A\} \cup \Gamma)^* = A, \Gamma^*$ if $x \in \lambda FV(M)$; $(\{x : A\} \cup \Gamma)^* = \Gamma^*$ if $x \notin \lambda FV(M)$;
$(\{\alpha : A\} \cup \Delta)^* = A, \Delta^*$ if $\alpha \in \mu FV(M)$; $(\{\alpha : A\} \cup \Delta)^* = \Delta^*$ if $\alpha \notin \mu FV(M)$.
By Basis Lemma, we have $\Gamma \uparrow \lambda FV(M), \neg\Delta \uparrow \mu FV(M) \vdash M : A$ from $\Gamma, \neg\Delta \vdash M : A$. Here Γ^* and Δ^* are the sequents obtained by omitting λ-variables and μ-variables from $\Gamma \uparrow \lambda FV(M)$ and $\Delta \uparrow \mu FV(M)$ respectively. On the inverse direction of the above theorem, we prove that GL_X-$\lambda\mu$-terms are represented as GL_X proofs.

Theorem 4 **(GL_X-$\lambda\mu$-terms as GL_X proofs)** .
Let M be a GL_X-$\lambda\mu$-term. If $\Gamma, \neg\Delta \vdash M : A$ in $TA_{\lambda\mu}$, then $\Gamma^* \rightarrow \Delta^*, A$ in GL_X.

Proof. By induction on the number of types contained in the $TA_{\lambda\mu}$ deductions and case analysis on the last rule. Only the following cases are mentioned.
Case of ($\rightarrow E$), i.e., M is $M_1 M_2$.
Case1-1. $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$:
By the induction hypotheses, there are $\Gamma_1^* \rightarrow \Delta_1^*, A_1 \supset A_2$ and $\Gamma_2^* \rightarrow \Delta_2^*, A_1$ in GL_X. Hence, using ($\supset\rightarrow$), we have $A_1 \supset A_2, \Gamma_2^* \rightarrow \Delta_2^*, A_2$, and then $\Gamma_1^*, \Gamma_2^* \rightarrow \Delta_1^*, \Delta_2^*, A_2$ by (cut).
Case1-2. $\lambda FV(M_1) \cap \lambda FV(M_2) \neq \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$, i.e., M is GL_C or GL_{CW}-$\lambda\mu$-term:
Following the above Case1-1 and use the left contraction rules.
Case1-3. $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) \neq \phi$, i.e., M is GL_C or GL_{CW}-$\lambda\mu$-term:
Following the Case1-1 and use the right contraction rules.
Case1-4. $\lambda FV(M_1) \cap \lambda FV(M_2) \neq \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) \neq \phi$, i.e., M is GL_C or GL_{CW}-$\lambda\mu$-term:
Following the Case1-1 and use both the left and right contraction rules.
Case of ($\perp E$), i.e., M is $\mu\alpha.M_1$.
Case2-1. $\alpha \in \mu FV(M_1)$:
The induction hypothesis gives that $\Gamma^* \rightarrow \Delta^*$ in GL_X where $\{\alpha : A\}^*$ is to be in Δ^*, which is what is to be proved.
Case2-2. $\alpha \notin \mu FV(M_1)$, i.e., M is GL_W or GL_{CW}-$\lambda\mu$-term:
By the induction hypothesis, we have $\Gamma^* \rightarrow \Delta^*$ in GL_X where $\{\alpha : A\}^*$ is not in Δ^*. The use of the right weakening rules leads to $\Gamma^* \rightarrow \Delta^*, A$ in GL_W or GL_{CW}.
Case of ($\perp I$), i.e., M is αM_1.
Case3-1. $\alpha \in \mu FV(M_1)$, i.e., M is a GL_C or GL_{CW}-$\lambda\mu$-term:
There is a deduction of $\Gamma^* \rightarrow \Delta^*, A$ by the induction hypothesis where $\{\alpha : A\}^*$ is a member of Δ^*. Hence the application of the right contraction rules yields to $\Gamma^* \rightarrow \Delta^* - \{A\}, A$ in GL_C or GL_{CW}.
Case3-2. $\alpha \notin \mu FV(M_1)$:
The induction hypothesis is what we need.

According to Theorem3 and 4, we can identify GL_X-$\lambda\mu$-terms as GL_X proofs. Hence with help of Theorem2, the set of types inhabited by closed GL_X-$\lambda\mu$-terms corresponds to the set of theorems in $FL_{EX}+ \rightarrow \neg\neg A \supset A$. Let BCI be the Hilbert-type system (axioms-based logic) consisting of modus ponens and substitution rules together with axioms (I): $A \supset A$, (C): $(A \supset B \supset C) \supset B \supset A \supset C$, and (B): $(A \supset B) \supset (C \supset A) \supset C \supset A$. Let $BCIW$ be BCI with axioms (W): $(A \supset A \supset B) \supset A \supset B$. Let $BCIK$ be BCI with (K): $A \supset B \supset A$. Let $BCIKW$ be BCI with axioms (K) and (W). Then from the correspondence between Hilbert systems and sequent systems, i.e., BCI and FL_E, $BCIW$ and FL_{EC}, $BCIK$ and FL_{EW}, $BCIKW$ and FL_{ECW}, the statement in the Corollary is followed.

Corollary 1 .For each the corresponding pair of X and Y,
$\{A \mid \vdash_{\lambda\mu} M : A$ for some GL_X-$\lambda\mu$ term $M\} = \{A \mid A$ is a theorem in $BCIY + \neg\neg D \supset D\}$.

4 Principal type scheme in GL, GL_C, GL_W and GL_{CW}

We prove the existence of a principal type scheme if GL_X-$\lambda\mu$-term is stratified. With respect to substitution, the most general type assignment for a $\lambda\mu$-term is defined as a principal type scheme.

Definition 4 (Principal type scheme and principal pair) .
1. For a closed $\lambda\mu$-term M, a type A is a principal type scheme (p.t.s.) of M iff $\vdash M : A'$ for a type A' such that $A\theta$ equals A' for some substitution θ.
2. A pair $< \Gamma, \neg\Delta; A >$ is a principal pair (p.p.) of M iff $\Gamma', \neg\Delta' \vdash M : A'$ such that $(\Gamma, \neg\Delta)\theta$ equals $\Gamma', \neg\Delta$ and $A\theta$ equals A' for some substitution θ. The deduction of $\Gamma, \neg\Delta \vdash M : A$ is called a principal deduction.

In order to prove Principal Deduction Theorem, we use the Composition-Extension Lemma especially in the case of applications.

Lemma 4 (Composition-Extension Lemma) .
Let θ be $\theta_1 \cup \theta_2$, $\mathbf{A} = Domain(\theta_1)$ and $\mathbf{B} = Domain(\theta_2)$. Let
(1) $\mathbf{A} \cap \mathbf{B} = \phi$; (2) $\theta_1 = \rho \circ \tau$ where $Domain(\tau) = \mathbf{A}$ and $Domain(\rho) \subset Range(\tau)$;
(3) $Range(\tau) \cap \mathbf{B} = \phi$. Then there exists a ρ' such that $\theta = \rho' \circ \tau$.

Proof. Take ρ' as $\theta_2 \cup \rho$. See [Hind88].

Theorem 5 (Principal deduction theorem of GL_X-$\lambda\mu$-terms) .Let X be nil, C, W or CW.
(1) If a GL_X-$\lambda\mu$-term M is stratified, then M has a principal pair $< \Gamma, \neg\Delta; A >$ where $\lambda Subjects(\Gamma) = \lambda FV(M)$ and $\mu Subjects(\neg\Delta) = \mu FV(M)$.
(2) There is a recursive algorithm which decides a GL_X-$\lambda\mu$-term M is stratified and which outputs the principal pair $< \Gamma, \neg\Delta; A >$ where $\lambda Subjects(\Gamma) = \lambda FV(M)$ and $\mu Subjects(\neg\Delta) = \mu FV(M)$, if M is stratified.

Proof. Along the line of [Hind88], by case analysis on the term M.

5 Reductions of GL_X-$\lambda\mu$-terms

In this section, we prove that each GL_X-$\lambda\mu$-term is closed under the reductions. As a corollary, we obtain subject reduction property of GL_X-$\lambda\mu$-terms.

Lemma 5 .Let M be a GL or GL_W-$\lambda\mu$-term.
If $x \in \lambda FV(M)$ and $\alpha \in \mu FV(M)$, then x and α occur exactly onece in M.

Proof. By induction on the structure of M.

Lemma 6 .(1) Let M be a GL or L_C-$\lambda\mu$-term.
If $M \rhd_+ N$, then $\lambda FV(M) = \lambda FV(N)$ and $\mu FV(M) = \mu FV(N)$.
(2) Let M be a GL_W or L_{CW}-$\lambda\mu$-term.
If $M \rhd_+ N$, then $\lambda FV(N) \subseteq \lambda FV(M)$ and $\mu FV(N) \subseteq \mu FV(M)$.

Proof. By induction on the derivation of $M \rhd_+ N$.

By the above Lemma, we straightforwardly derive the following.

Corollary 2 .
(1) Let M_1 and M_2 be GL_X-$\lambda\mu$-terms s.t. $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$.
If $M_1 \rhd_+ N_1$ and $M_2 \rhd_+ N_2$, then $\lambda FV(N_1) \cap \lambda FV(N_2) = \phi$ and $\mu FV(N_1) \cap \mu FV(N_2) = \phi$.
(2) Let M be GL or GL_C-$\lambda\mu$-term such that $x \in \lambda FV(M)$ and $\alpha \in \mu FV(M)$.
If $M \rhd_+ N$, then $x \in \lambda FV(N)$ and $\alpha \in \mu FV(N)$.
(3) Let M be a GL_X-$\lambda\mu$-term such that $\alpha \notin \mu FV(M)$.
If $M \rhd_+ N$, then $\alpha \notin \mu FV(N)$.

Lemma 7 (Reduction of GL_X-$\lambda\mu$-terms) .Let M be a GL_X-$\lambda\mu$-term.
If $M \rhd_+^* N$, then N is also a GL_X-$\lambda\mu$-term.

Proof. By induction on the derivation of $M \rhd_+^* N$.

Lemma 8 (Subject reduction of GL_X-$\lambda\mu$-terms) .Let M be a GL_X-$\lambda\mu$-term.
If $\Gamma, \neg\Delta \vdash M : A$ and $M \triangleright_{+}^{*} N$, then $\Gamma, \neg\Delta \vdash N : A$ and N is also a GL_X-$\lambda\mu$-term.

Proof. By induction on the derivation of $M \triangleright_{+}^{*} M'$ and use Lemma7. We only show the base case.
1. $\Gamma, \neg\Delta \vdash (\lambda x.M_1)M_2 : A$
By Generation Lemma, $\Gamma, x : B, \neg\Delta \vdash M_1 : A$ and $\Gamma, \neg\Delta \vdash M_2 : B$ for some B. By Substitution Lemma, $\Gamma, \neg\Delta \vdash M_1[x := M_2] : A$.
2. $\Gamma, \neg\Delta \vdash (\mu\alpha.M_1)M_2 : A$
By Generation Lemma, $\Gamma_1, \neg\Delta_1 \vdash M_1 : \bot$ where $\alpha : \neg(B \to A) \in \neg\Delta_1$ and $\Gamma_2, \neg\Delta_2 \vdash M_2 : B$ where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = (\Delta_1 - \{\alpha : B \to A\}) \cup \Delta_2$. By Substitution Lemma, $\Gamma_1, \Gamma_2, \neg\Delta_1 - \{\alpha : \neg(B \to A)\}, \alpha : \neg A, \neg\Delta_2 \vdash M_1[\alpha \Leftarrow M_2] : \bot$, and hence $\Gamma, \neg\Delta \vdash \mu\alpha.M_1[\alpha \Leftarrow M_2] : A$.
3. We have $\Gamma, \neg\Delta, \alpha_2 : \neg A \vdash M : \bot$ where $\alpha_1 : \neg A \in \Delta$ from the assumption $\Gamma, \neg\Delta \vdash \alpha_1(\mu\alpha_2.M) : \bot$. By Substitution Lemma, $\Gamma, \neg\Delta \vdash M[\alpha_2 := \alpha_1] : \bot$ is derived.
4. By assumption $\Gamma, \neg\Delta \vdash \mu\alpha.\alpha M : A$ where $\alpha \notin \mu FV(M)$, we have $\Gamma, \neg\Delta, \alpha : \neg A \vdash M : \bot$. Hence $\Gamma \uparrow \lambda FV(M), \neg\Delta \uparrow \mu FV(M) \vdash M : \bot$, and also $\Gamma, \neg\Delta \vdash M : \bot$ by Basis Lemma.
5. From assumption $\Gamma, \neg\Delta \vdash \mu\alpha.M : A$ where M has a subterm of the form $\alpha\lambda y.M'$ for some M', we have $\Gamma, \neg\Delta, \alpha : \neg A \vdash M : \bot$, and A must be of the form $A_1 \to A_2$ for some A_1 and A_2. Hence by Substitution Lemma and $x : A_1 \vdash x : A_1$, we obtain $\Gamma, x : A_1, \neg\Delta, \alpha : \neg A_2 \vdash M[\alpha \Leftarrow x] : \bot$ and then $\Gamma, \neg\Delta \vdash \lambda x.\mu\alpha.M[\alpha \Leftarrow x] : A$.

6 Embedding GL_X into $\lambda\mu_I$ via μ-head form proofs

We have already observed that there exists a special form of classical propositional proofs [Fuji94-1],[Fuji94-2], which we call μ-head form proofs. In terms of $\lambda\mu$-calculus, the μ-head form proofs are represented as μ-closed $\mu\alpha.M$ where $\mu FV(M) \subseteq \{\alpha\}$ and $\alpha \notin \mu BV(M)$. This notion makes the four classes of GL_X-$\lambda\mu$-terms collapsed into the one class of GL_{CW}-$\lambda\mu$-terms.

Theorem 6 (μ-head form proofs in GL_{CW}) .Let M be a μ-closed GL_X-$\lambda\mu$-term.
If $\Gamma \vdash_{\lambda\mu} M : A$, then there exists a μ-head form proof M' as a GL_{CW}-$\lambda\mu$-term such that $\Gamma \vdash_{\lambda\mu} M' : A$.

Proof. [Fuji94-1][Fuji94-2] proved that there exists a μ-head form proof for arbitrary classical propositional proofs.

The GL_X-$\lambda\mu$-terms can represent proofs of classical substructural logics of LK. On the other hand, BCI-λ-terms, BCK-λ-terms and λI-terms are proofs of substructural logics of minimal intuitionistic logic, strictly speaking primitive logic. Now we consider the intuitionistic fragment of $\lambda\mu$-calculus with one consequence, which is simply typed lambda calculus together with the intuitionistic absurdity rule. Here μ-operators appear only in the form of $\mu\beta.M$ for some β and $\mu FV(M) = \phi$. The proof term $\mu\beta.M$ represents the intuitionistic absurdity rules, i.e., infer arbitrary B from $M : \bot$. In terms of Felleisen's λ_c, this term corresponds to $\mathbf{A}(M)$ using abort. We denote the intuitionistic fragment of $\lambda\mu$ by $\lambda\mu_I$.

According to the existence of μ-head form proofs, we can easily derive the following corollary that is embedding classical substructural logics into intuitionistic logic, which is well-known as Glivenko's theorem.

Corollary 3 (Embedding GL_X-$\lambda\mu$-terms into $\lambda\mu_I$ via μ-head form proofs) .
If a type A is inhabited by a μ-closed GL_X-$\lambda\mu$-term, then $\neg\neg A$ is inhabited in $\lambda\mu_I$.

Proof. Consider a μ-head form proof $\mu\alpha.M$ of type A corresponding to the GL_X-$\lambda\mu$-term by the above Theorem. We define the following translation F which gives a $\lambda\mu_I$-term of type $\neg\neg A$ from the μ-head form proof term.
$F(\mu\alpha.M) = \lambda\alpha.(F(M))$; $F(x) = x$; $F(M_1M_2) = F(M_1)F(M_2)$;
$F(\lambda x.M_1) = \lambda x.F(M_1)$; $F(\alpha M_1) = \alpha(F(M_1))$; $F(\mu\beta.M_1) = \mu\beta.(F(M_1))$.
The definition seems to give no essential translation, however $(\bot E)$ is repalced with $(\to I)$, and $(\bot I)$ is with $(\to E)$ in the proof.

From a proof of the double negation $\neg\neg A$ in $\lambda\mu_I$, conversely we can obtain a classical proof of A in $\lambda\mu$, which is to be a μ-head form proof.

Lemma 9 (μ-head form proofs) .Let H be $\lambda x.\mu\alpha.x(\lambda k.\alpha k)$.
If $\Gamma \vdash M : \neg\neg A$ in $\lambda\mu_I$, then $\Gamma \vdash H(M) : A$ in $\lambda\mu$ gives a μ-head form proof of A.

Proof. Using the intuitionistic proof M of $\neg\neg A$ in $\lambda\mu_I$, we eliminate the double negation to obtain the μ-head form proof of A in $\lambda\mu$.

$$\dfrac{\dfrac{M:\neg\neg A \qquad \dfrac{\dfrac{[\alpha:\neg A]^2 \quad [k:A]^1}{\alpha k:\bot}\,(\bot I)}{\lambda k.\alpha k:\neg A}\,(\to I)^1}{M(\lambda k.\alpha k):\bot}\,(\supset E)}{\mu\alpha.M(\lambda k.\alpha k):A}\,(\bot E)^2$$

Lemma 10 .Let $\mu\alpha.M$ be a μ-head form proof.
Then $M =_\mu F(M)[\alpha := \lambda k.\alpha k]$.

Proof. By induction on the structure of M. We show only the case M of αM_1.
$F(\alpha M_1)[\alpha := \lambda k.\alpha k] = (\alpha F(M_1))[\alpha := \lambda k.\alpha k] = (\lambda k.\alpha k)F(M_1)[\alpha := \lambda k.\alpha k] =_\beta \alpha(F(M_1)[\alpha := \lambda k.\alpha k])$ $=_\mu \alpha M_1$ by the induction hypothesis.

With the help of η-conversion: $\lambda x.Mx =_\eta M$ where $x \notin \lambda FV(M)$, it is obtained that both $F \circ H$ and $H \circ F$ are identity.

Lemma 11 (H is inverse of F and vice versa) .
$F \circ H =_\eta id$ and $H \circ F =_\mu id$.

Proof. (1) $F \circ H =_\eta id$:
For any $\lambda\mu_I$-term M, $\mu FV(M) = \phi$ and $F(M) = M$. Then $(F \circ H)M = F(H(M)) = F(\mu\alpha.M(\lambda k.\alpha k))$ $= \lambda\alpha.F(M)\lambda k.\alpha k =_\eta \lambda\alpha.F(M)\alpha =_\eta F(M) = M$.
(2) $H \circ F =_\mu id$:
For any μ-head form proof $\mu\alpha.M$,
$(H \circ F)\mu\alpha.M = H(\lambda\alpha.F(M)) = \mu\alpha'.(\lambda\alpha.F(M))\lambda k.\alpha' k =_\beta \mu\alpha'.F(M)[\alpha := \lambda k.\alpha' k] =_\mu \mu\alpha.M$ by the above Lemma.

On the other hand, we directly prove Glivenko's theorem which would be used for a translation to μ-head form proofs.

Definition 5 (Glivenko's embedding G) .
For a proof term M in $\lambda\mu$, the translation G is defined as follows:
(1) $G(x) = \lambda k.kx$; (2) $G(\lambda x.M) = \lambda k.k(\lambda x.\mu\beta.G(M)(\lambda u.k(\lambda v.u)))$;
(3) $G(M_1 M_2) = \lambda k.G(M_1)(\lambda x.G(M_2)\lambda y.k(xy))$;
(4) $G(\mu\alpha.M) = \lambda\alpha.G(M)\lambda x.x$; (5) $G(\alpha M) = \lambda v.\alpha G(M)$ where v is fresh.

Lemma 12 (Glivenko's theorem) .
If $\Gamma, \neg\Delta \vdash M : A$ in $\lambda\mu$, then $\Gamma, \neg\Delta \vdash G(M) : \neg\neg A$ in $\lambda\mu_I$.

Proof. By induction on the number of types contained in the derivation of $\lambda\mu$ and case analysis on the last rule. We show only the case of $(\to E)$.

$$\dfrac{\dfrac{G(M_1):\neg\neg(A\to B) \qquad \dfrac{\dfrac{G(M_2):\neg\neg A \qquad \dfrac{\dfrac{[k:\neg B]^3 \qquad \dfrac{[x:A\to B]^2 \quad [y:A]^1}{xy:B}\,(\to E)}{k(xy):\bot}\,(\to E)}{\lambda y.k(xy):\neg A}\,(\to I)^1}{G(M_2)\lambda y.k(xy):\bot}\,(\to E)}{\lambda x.G(M_2)\lambda y.k(xy):\neg(A\to B)}\,(\to I)^2}{G(M_1)(\lambda x.G(M_2)\lambda y.k(xy)):\bot}\,(\to E)}{\lambda k.G(M_1)(\lambda x.G(M_2)\lambda y.k(xy)):\neg\neg B}\,(\to I)^3$$

Now we obtain a procedure to provide a μ-head form proof from an arbitrary classical propositional proof in $\lambda\mu$.

Corollary 4 (Translation from classical proofs to μ-head form proofs) .
$H \circ G$ provides μ-head form proofs for arbitrary μ-closed classical propositional proofs in $\lambda\mu$.

Proof. By Lemma9 and Lemma12.

The above Corollary is another proof of the existence of μ-head form proofs thanks to Glivenko's Theorem. According to the Corollary, we can obtain the following composed translation μHF which gives a μ-head form proof from any μ-closed $\lambda\mu$-term.
(1) $\mu HF(x) = \mu\alpha.\alpha x$; (2) $\mu HF(\lambda x.M) = \mu\alpha.\alpha\lambda x.\mu\beta.G(M)(\lambda u.\alpha\lambda v.u)$;
(3) $\mu HF(M_1 M_2) = \mu\alpha.G(M_1)(\lambda x.G(M_2)\lambda y.\alpha(xy))$;
(4) $\mu HF(\mu\alpha.\alpha M) = \mu\alpha.\alpha G(M)$; (5) $\mu HF(\mu\alpha.M) = \mu\alpha.G(M)[\alpha := \lambda k.\alpha k]$.

Remarks 1 (The embedding G does not preserve substructural logics) .
According to the proof of Lemma12, GL_X-$\lambda\mu$-terms are embedded into GL_{CW}-$\lambda\mu_I$-terms where vacuous discharge is applied and applications are used for two terms which have common variables.

7 Gödel's translation

We investigate an embedding of classical substructural logics into the corresponding intuitionistic substructural logics, which is known as Gödel's translation modified by Gentzen [Szabo69], [Dumm77]. As a corollary, we obtain that every GL_W-$\lambda\mu$-term is stratified, and that typability and inhabitation are respectively decidable for $\lambda\mu$. The negative translation A^g is defined as follows:
$\perp^g = \perp$; $A^g = \neg\neg A$ for an atomic formula distinct from $\perp$;
$(A_1 \to A_2)^g = A_1^g \to A_2^g$.
The negative translation is naturally extended to a context Γ such that
$(\{\ \})^g = \{\ \}$;
$(\{x : A\} \cup \Gamma)^g = \{x : A^g\} \cup \Gamma^g$.
For μ-indexed set Δ, similarly $\neg\Delta^g$ is defined. We give the definition of the translation of $\lambda\mu$-terms together with the auxiliary function G. The translation produces a λ-term with two kinds of variables denoted by x and α.

Definition 6 (Gödel's translation for $\lambda\mu$-terms) .
(1) $\overline{x} = x$; (2) $\overline{\lambda x.M} = \lambda x.\overline{M}$; (3) $\overline{M_1 M_2} = \overline{M_1}\ \overline{M_2}$;
(4) $\overline{\mu\alpha.M} = G(\lambda\alpha.\overline{M}, n)$; (5) $\overline{\alpha M} = \alpha\overline{M}$
where G is defined for a λ-term M and a natural number
$G(M, 0) = M\lambda x.x$; $G(M, 1) = \lambda z.M(\lambda y.yz)$;
$G(M, n+2) = \lambda y.G(\lambda k.M(\lambda z.k(zy)), n)$.
A function f is defined for a type A such that
$f(\perp) = 0$; $f(A) = 1$ for an atomic formula distinct from $\perp$;
$f(A_1 \to A_2) = 2 + f(A_2)$.

Remarks 2 .By the definition of f, it is clearly remarked that
(1) $f(A) \geq 2$ iff A is a function type.
(2) $f(A)$ is odd iff the target type of A is atomic.
(3) $f(A)$ is even iff the target type of A is of the form of negation.

Remarks 3 .Latter it will be clear that the natural number n in $G(M, n)$ might correspond to the complexity of the type of M measured by f.

Remarks 4 .
The slight modified definition of the above definition would also be available for the following results.
g' is defined for a type:
$A^{g'} = \neg\neg A$ for an atomic formula; $(A_1 \to A_2)^{g'} = A_1^{g'} \to A_2^{g'}$.
$\overline{M}$ is defined similarly with a function G':
$\overline{\mu\alpha.M} = G'(\lambda\alpha.\overline{M}\lambda x.x, n)$; $\overline{\alpha M} = \lambda k.k(\alpha\overline{M})$.
$G'(M, 0) = \lambda z.M(\lambda y.yz)$; $G'(M, n+1) = \lambda y.G'(\lambda k.M(\lambda z.k(zy)), n)$.
f' is defined for a type:
$f'(A) = 0$ for an atomic formula; $f'(A_1 \to A_2) = 1 + f'(A_2)$.

Since A^g is negative, that is, atomic formulae occur only negated, $\neg\neg A^g \leftrightarrow A^g$ holds in minimal logic as follows.

Lemma 13 .Let M be a λ-term.
$\vdash_{\lambda\to} M : \neg\neg A^g$ if and only if $\vdash_{\lambda\to} G(M, f(A)) : A^g$.

Proof. By induction on the structure of the formula A. We give only the case A of $A_1 \to A_2$, i.e., $f(A) \geq 2$.
Only-if-part:
Let $M : \neg\neg(A_1^g \to A_2^g)$ and $y : A_1^g$. Then $\lambda k.M(\lambda z.k(zy)) : \neg\neg A_2^g$, and hence A_2^g is inhabited by $G(\lambda k.M(\lambda z.k(zy)), f(A_2))$ using the induction hypothesis. Now we have $\lambda y.G(\lambda k.M(\lambda z.k(zy)), f(A_2)) : A_1^g \to A_2^g$, whose proof term is equal to $G(M, f(A_1 \to A_2))$.
If-part:
Let $G(M, n+2) = \lambda y.G(\lambda k.M(\lambda z.k(zy)), n) : (A_1 \to A_2)^g = A_1^g \to A_2^g$. We have $y : A_1^g$ and $G(\lambda k.M(\lambda z.k(zy)), n) : A_2^g$, then the induction hypothesis gives $\lambda k.M(\lambda z.k(zy)) : \neg\neg A_2^g$. Hence $k : \neg A_2^g$ and $M(\lambda z.k(zy)) : \bot$. Now $M : \neg B$ and $\lambda z.k(zy) : B$ for some B. B must be $B_1 \to B_2$ such that $z : B_1$ and $k(zy) : B_2$. Here B_2 is $\bot$ from $k : \neg A_2^g$, and $zy : A_2^g$ which gives $z : B_1 = A_1^g \to A_2^g$. Hence $M : \neg B = \neg((A_1^g \to A_2^g) \to \bot) = \neg\neg(A_1 \to A_2)^g$.

Let Y-term be a BCI-λ-term, BCK-λ-term or λI-term. The corresponding pair of X and Y means the pair of GL-$\lambda\mu$-terms and BCI-λ-terms, etc.

Lemma 14 .Let M be a Y-term and n be a natural number.
If M is a Y-term, then so is $G(M, n)$.

Proof. By induction on n.

Lemma 15 .For each corresponding pairs of X and Y,
if M is a GL_X-$\lambda\mu$-term, then $\overline{M}$ is a Y-term.

Proof. By induction on the structure of M.

Theorem 7 (Gödel's embedding of the $\lambda\mu$-calculus) .Let M be a GL_X-$\lambda\mu$-term.
If $\Gamma, \neg\Delta \vdash_{\lambda\mu} M : A$, then $\Gamma^g, \neg\Delta^g \vdash_{\lambda\to} \overline{M} : A^g$.

Proof. By induction on the number of types contained the deduction of $\lambda\mu$ and case ananlysis on the last rule. The case of $(\bot E)$ is only given in the following.
By the induction hypothesis, we have $\Gamma^g, \neg\Delta^g, \alpha : \neg A^g \vdash \overline{M_1} : \bot$. Then Lemma13 gives $\Gamma^g, \neg\Delta^g \vdash G(\lambda\alpha.\overline{M_1}, f(A)) : A^g$ whose proof term is $\overline{\mu\alpha.M_1}$.

Corollary 5 (preserving substructural logics) .
The Gödel's embedding transforms proofs of classical substructural logics into those of the corresponding intuitionistic substructural logics.

Proof. By the above Theorem and Lemma15.

If $\Gamma \vdash_{\lambda\to} M : A$, then by straightforward induction, we can construct a deduction $\Gamma^g \vdash_{\lambda\to} M : A^g$ such that every type in the deduction has the form of Gödel's double negation.

Lemma 16 .Let M be a GL_X-$\lambda\mu$-term.
If $\Gamma^g, \neg\Delta^g \vdash_{\lambda\to} \overline{M} : A^g$ where every type in the deduction has the form of Gödel's double negation, then $\Gamma, \neg\Delta \vdash_{\lambda\mu} M : A$.

Proof. By induction on the structure of the $\lambda\mu$-term M. The condition in the if-part is necessary in the case of applications. If $\overline{M}$ is to be in β-normal, then this condition is redundant by the Subformula property [Praw65]. Case M of $\mu\alpha.M_1$ is in the below.
By the assumption, $\Gamma^g, \neg\Delta^g \vdash_{\lambda\to} G(\lambda\alpha.\overline{M_1}, n) : A^g$ where we take n as $f(A)$, and then $\Gamma^g, \neg\Delta^g \vdash_{\lambda\to} \lambda\alpha.\overline{M_1} : \neg\neg A^g$ by Lemma15. Generation Lemma gives that $\Gamma^g, \neg\Delta^g, \alpha : \neg A^g \vdash_{\lambda\to} \overline{M_1} : \bot$. Thus $\Gamma, \neg\Delta \vdash_{\lambda\mu} \mu\alpha.M_1 : A$ is obtained by the induction hypothesis $\Gamma, \neg\Delta, \alpha : \neg A \vdash_{\lambda\mu} M_1 : \bot$.

Corollary 6 (stratification of GL_W-$\lambda\mu$-terms) .
Every GL_W-$\lambda\mu$-term is stratified.

Proof. Any GL_W-$\lambda\mu$-term M is translated to BCK-λ-term $\overline{M}$, which is stratified. Because it is known that every linear λ-term (BCK-λ-term) is stratified by Theorem4.1 in [Hind87]. Hence $\Gamma, \neg\Delta \vdash_{\lambda\to} \overline{M} : A$ for some Γ, Δ and A, and we also have $\Gamma^g, \neg\Delta^g \vdash_{\lambda\to} \overline{M} : A^g$ where every type in the derivation has a form of Gödel's double negation. Hence $\Gamma, \neg\Delta \vdash_{\lambda\mu} M : A$ by the above Lemma, i.e., M is stratified.

Moreover according to the decidability of typability and inhabitation for $\lambda \to$ [Bare91], it is easy to obtain those for $\lambda\mu$ as follows.

Corollary 7 (typability is decidable for $\lambda\mu$) .
Given GL_X-$\lambda\mu$-term M, it is decidable to check whether there exists a type A such that $\vdash_{\lambda\mu} M : A$ or not.

Proof. Typability of $\overline{M}$ in $\lambda \rightarrow$ is decidable. Hence if $\vdash_{\lambda\rightarrow} \overline{M} : A$ for some A, then $\vdash_{\lambda\mu} M : A$ by Corollary5. Otherwise there is no A such that $\vdash_{\lambda\mu} M : A$ by Theorem7. Also see Principal deduction theorem (Theorem5).

Corollary 8 (inhabitation is decidable for $\lambda\mu$) .
Given A, it is decidable to check whether there exists a term GL_X-$\lambda\mu$-term M such that $\vdash_{\lambda\mu} M : A$ or not.

Proof. Since inhabitation of A^g in $\lambda \rightarrow$ is decidable, if $\vdash_{\lambda\rightarrow} M : A^g$ for some M, then clearly $\vdash_{\lambda\mu} M' : A$ for some M'. Otherwise there is no M such that $\vdash_{\lambda\mu} M : A$ by Theorem7.

Next we discuss the correctness of Gödel's embedding with respect to $=_{\mu-}$.

Lemma 17 .Let M and N be λ-terms, and n be a natural number.
$G(M,n)[x := N] = G(M[x := N], n)$.

Proof. By induction on n.

Lemma 18 .Let M and N be λ-terms, and n be a natural number.
$G(M,n)[\alpha := N] = G(M[\alpha := N], n)$.

Proof. By induction on n.

Lemma 19 .Let M, M_1 and M_2 be λ-terms. Let n be a natural number.
If $M_1 =_\beta M_2$ and $M =_\beta G(M_1, n)$, then $M =_\beta G(M_2, n)$.
If $M_1 \rhd_\beta^* M_2$ and $M \rhd_\beta^* G(M_1, n)$, then $M \rhd_\beta^* G(M_2, n)$.

Proof. By induction on n.

Lemma 20 .Let M and N be GL_X-$\lambda\mu$-terms.
Then $\overline{M}[x := \overline{N}] = \overline{M[x := N]}$.

Proof. By induction on the structure of M. The case M of $\mu\alpha.M_1$ is as follows.
$\overline{\mu\alpha.M_1}[x := \overline{N}] = G(\lambda\alpha.\overline{M_1}, n)[x := \overline{N}] = G(\lambda\alpha.\overline{M_1}[x := \overline{N}], n)$ by Lemma17. The induction hypothesis makes it equal to $G(\lambda\alpha.\overline{M_1[x := N]}, n) = \overline{\mu\alpha.M_1[x := N]}$.

Lemma 21 .Let M and N be GL_X-$\lambda\mu$-terms.
Then $\overline{M}[\alpha := \lambda z.\alpha(z\overline{N})] \rhd_\beta^* \overline{M[\alpha \Leftarrow N]}$.

Proof. By induction on the structure of M. We only show the case M of αM_1.
$\overline{\alpha M_1}[\alpha := \lambda z.\alpha(z\overline{N})] = (\alpha\overline{M_1})[\alpha := \lambda z.\alpha(z\overline{N})] = (\lambda z.\alpha(z\overline{N}))\overline{M_1}[\alpha := \lambda z.\alpha(z\overline{N})] \rhd_\beta^* (\lambda z.\alpha(z\overline{N}))\overline{M_1[\alpha \Leftarrow N]}$ $\rhd_\beta \alpha(\overline{M_1[\alpha \Leftarrow N]}\ \overline{N})$ by the induction hypothesis. Then it is definitionally equal to $\overline{\alpha M_1[\alpha \Leftarrow N]N} = \overline{(\alpha M_1)[\alpha \Leftarrow N]}$.

Lemma 22 .Let M and N be GL_X-$\lambda\mu$-terms, and $n \geq 2$.
$G(\lambda\alpha.\overline{M}, n)\overline{N} \rhd_\beta^* G(\lambda\alpha.\overline{M[\alpha \Leftarrow N]}, n-2)$.

Proof. For $n \geq 2$, by the definition we have $(G(\lambda\alpha.\overline{M}, n))\overline{N} = (\lambda y.G(\lambda k.(\lambda\alpha.\overline{M})(\lambda z.k(zy)), n-2))\overline{N} \rhd_\beta$ $G(\lambda k.(\lambda\alpha.\overline{M})(\lambda z.k(z\overline{N})), n-2) \rhd_\beta G(\lambda k.\overline{M}[\alpha := \lambda z.k(z\overline{N})], n-2) =_\alpha G(\lambda\alpha.\overline{M}[\alpha := \lambda z.\alpha(z\overline{N})], n-2)$. By the Lemma19 and 21, it is β-reduced to $G(\lambda\alpha.\overline{M[\alpha \Leftarrow N]}, n-2)$.

Theorem 8 (Correctness of Gödel's embedding of GL_X-$\lambda\mu$-terms) . Let M_1 and M_2 be well-typed GL_X-$\lambda\mu$-terms. If $M_1 =_{\mu-} M_2$, then $\overline{M_1} =_\beta \overline{M_2}$.

Proof. By induction on the derivation of $M_1 =_{\mu-} M_2$. The atomic cases are in the following.
Case of β-reductions:
$\overline{(\lambda x.M)N} = (\lambda x.\overline{M})\overline{N} \rhd_\beta \overline{M}[x := \overline{N}] = \overline{M[x := N]}$ by Lemma20.
Case of structural reductions:
$\overline{(\mu\alpha.M)N} = G(\lambda\alpha.\overline{M}, n)\overline{N} \rhd^*_\beta G(\lambda\alpha.\overline{M[\alpha \Leftarrow N]}, n-2) = \overline{\mu\alpha.M[\alpha \Leftarrow N]}$ by Lemma22 and Remark2.

Remarks 5 .For renaming reductions ($S1$), we cannot expect this property. In order to work with ($S2$), η-reduction is necessary to obtain the corresponding lemma to the above. The inverse direction of the above Theorem does not hold.

Theorem 9 (Simulation of one step μ-reduction by embedded β-reductions) .
Let M_1 and M_2 be well-typed GL_X-$\lambda\mu$-terms. If $M_1 \rhd_- M_2$, then $\overline{M_1} \rhd^*_\beta \overline{M_2}$.

Proof. By induction on the derivation of $M_1 \rhd_- M_2$. The atomic cases are same as those in the proof of Theorem8.

Corollary 9 (Strong normalization of well-typed GL_X-$\lambda\mu$-terms) .
Well-typed GL_X-$\lambda\mu$-terms are strongly normalizable with respect to structural reductions and β-reductions.

Proof. By the above theorem and the fact that $\lambda \rightarrow$ has strong normalization property [Bare91].

Remarks 6 .The result of the above Corollary and Lemma8 (Subject reduction) is not inconsistent with the fact that GL_X is cut-free (Theorem1).

8 Simplification on Gödel's embedding

In this section we discuss a simplification on the Gödel's embedding. As a consequence if we had renaming reduction ($S1$) and ($S3$) besides, then the translation gave a β-normal form $\overline{\overline{M}}$ if M is to be in normal in this sense. With information on types whose subjects are μ-variables, we define a simplified embedding based on Gödel's double negation translation, which also appears in [Pari92-1].

Definition 7 (simplified embedding) .
(1) Let n be a quotient of $(f(A)+1)$ divided by 2 for $\alpha : \neg A$. Let M be a GL_X-$\lambda\mu$-term.
$\overline{\overline{x}} = x$;
$\overline{\overline{M_1 M_2}} = \overline{\overline{M_1}}\,\overline{\overline{M_2}}$; $\qquad \overline{\overline{\lambda x.M}} = \lambda x.\overline{\overline{M}}$;
$\overline{\overline{\mu\alpha.M}} = \lambda x_1 \cdots x_n.\overline{\overline{M}}$; $\qquad \overline{\overline{\alpha M}} = \overline{\overline{M}} x_1 \cdots x_n$.
(2) Let A be a type.
$\bot^* = \{\ \}$; $\qquad A^* = \{x : \neg A\}$ if A is atomic distinct from $\bot$;
$(A_1 \rightarrow A_2)^* = \{x_1 : A_1^g\} \cup A_2^*$.
(3) Let Δ be a set of types indexed with μ-variables.
$\{\ \}^* = \{\ \}$; $\qquad (\{\alpha : A\} \cup \Delta)^* = A^* \cup \Delta^*$.

Theorem 10 (simplified Gödel's embedding) .Let M be a GL_X-$\lambda\mu$-term.
If $\Gamma, \neg\Delta \vdash_{\lambda\mu} M : A$, then $\Gamma^g, \Delta^* \vdash_{\lambda\rightarrow} \overline{\overline{M}} : A^g$.

Proof. By induction on the derivation. We show the following two cases.
Case1 of ($\bot E$), i.e., M is $\mu\alpha.M_1$:
Let the induction hypothesis be IH-1, i.e., $\Gamma^g, \Delta^*, (\alpha : A)^* \vdash_{\lambda\rightarrow} \overline{\overline{M}} : \bot$. By induction on $f(A)$.
Case1-1. $f(A) = 0$:
$(\alpha : \bot)^* = \{\ \}$ and $\Gamma^g, \Delta^* \vdash_{\lambda\rightarrow} \overline{\overline{M_1}} : \bot$ is directly derived by IH-1, and (f(A)+1) div 2 is 0.
Case1-2. $f(A) = 1$:
$(\alpha : A)^* = \{x : \neg A\}$ and $\Gamma^g, \Delta^*, x : \neg A \vdash_{\lambda\rightarrow} \overline{\overline{M_1}} : \bot$ by IH-1. Hence $\Gamma^g, \Delta^* \vdash_{\lambda\rightarrow} \lambda x.\overline{\overline{M_1}} : A^g$ is obtained, and (f(A)+1) div 2 is 1.
Case1-3. $f(A_1 \rightarrow A_2) > 1$:
Let the second induction hypothesis be IH-2. $(\alpha : A_1 \rightarrow A_2)^* = \{x_1 : A_1^g\} \cup A_2^*$ and $\Gamma^g, \Delta^*, x_1 : A_1, A_2^* \vdash_{\lambda\rightarrow} \overline{\overline{M_1}} : \bot$ by IH-1. Then by IH-2, $\Gamma^g, \Delta^*, x_1 : A_1^g \vdash_{\lambda\rightarrow} \lambda x_1 \cdots x_m.\overline{\overline{M_1}} : A_2^g$ where m is a quotient of $f(A_2)+1$ divided by 2. Hence we have $\Gamma^g, \Delta^* \vdash_{\lambda\rightarrow} \lambda x_0 \lambda x_1 \cdots x_m.\overline{\overline{M_1}} : (A_1 \rightarrow A_2)^g$, and $m+1 = ((f(A_2)+1)\ div\ 2)+1 = (f(A_1 \rightarrow A_2)+1)\ div\ 2$.
Case2 of ($\bot I$), i.e., M is αM_1:

Let IH-1 be $\Gamma^g, \Delta^* \vdash \overline{\overline{M}} : A^g$ where $\alpha : A \in \Delta$. By induction on $f(A)$.
Case2-1. $f(A) = 0$:
(f(A)+1) div 2 is 0, and IH-1 directly gives $\Gamma^g, \Delta^* \vdash_{\lambda\to} \overline{\overline{M}} : \bot$.
Case2-2. $f(A) = 1$:
(f(A)+1) div 2 = 1, and $\Gamma^g, \Delta^* \vdash_{\lambda\to} \overline{\overline{M}}x : \bot$ where $x : \neg A \in \Delta^*$.
Case2-3. $f(A_1 \to A_2) > 1$:
Since $x_0 : A_1^g \in \Delta^*$, we have $\Gamma^g, x_0 : A_1^g, A_2^* \vdash_{\lambda\to} \overline{\overline{M}}x_0 : A_2^g$ by IH-1. Thus by the second induction hypothesis, $\Gamma^g, \Delta^* \vdash_{\lambda\to} (\overline{\overline{M}}x_0)x_1 \cdots x_m : \bot$. Here $m = (f(A_2) + 1)\ div\ 2$, and hence $m + 1 = (f(A_1 \to A_2) + 1)\ div\ 2$.

Lemma 23 .Let M and N be GL_X-$\lambda\mu$-terms.
$\overline{\overline{M}}[x := \overline{\overline{N}}] = \overline{\overline{M[x := N]}}$

Proof. By induction on the structure of M.

Lemma 24 .Let M and N be well-typed GL_X-$\lambda\mu$-terms.
If $n = ((f(A) + 1)\ div\ 2)$ where $\alpha \in \mu FV(M)$ and $\alpha : \neg A$ in the derivation, then $n - 1 = ((f(A') + 1)\ div\ 2)$ where $\alpha \in \mu FV(M[\alpha \Leftarrow N])$ and $\alpha : \neg A'$.

Proof. n must be greater than 1, i.e., A is a function type $A_1 \to A_2$. Then A' is A_2 in the case of $N : A_1$, hence $((f(A_2) + 1)\ div\ 2) = ((f(A) + 1)\ div\ 2) - 1$.

Lemma 25 .Let M and N be GL_X-$\lambda\mu$-terms. Let $((f(A) + 1)\ div\ 2)$ be n which is greater than 1 for $\alpha : \neg A$.
$\overline{\overline{M}}[x_1^\alpha := \overline{\overline{N}}] = \overline{\overline{M[\alpha \Leftarrow N]}}$.

Proof. By induction on the structure of M. Only the case M of αM_1 is showed:
$\overline{\overline{\alpha M_1}}[x_1^\alpha := \overline{\overline{N}}] = (\overline{\overline{M_1}}x_1 \cdots x_n)[x_1 := \overline{\overline{N}}] = (\overline{\overline{M_1}}[x_1 := \overline{\overline{N}}])\overline{\overline{N}}x_2 \cdots x_n = \overline{\overline{M_1[\alpha \Leftarrow N]}}\ \overline{\overline{N}}x_2 \cdots x_n = \overline{\overline{M_1[\alpha \Leftarrow N]N}}x_2 \cdots x_n = \overline{\overline{\alpha(M_1[\alpha \Leftarrow N]N)}} = \overline{\overline{(\alpha M_1)[\alpha \Leftarrow N]}}$.

Lemma 26 .Let M_1 and M_2 be well-typed GL_X-$\lambda\mu$-terms.
If $M_1 \rhd_+ M_2$, then $\overline{\overline{M_1}} \rhd_\beta^* \overline{\overline{M_2}}$.

Proof. By induction on the derivation of $M_1 \rhd_+ M_2$. We show the following cases.
Case1 of β-reduction:
$\overline{\overline{(\lambda x.M)N}} = (\overline{\overline{\lambda x.M}})\overline{\overline{N}} \rhd_\beta \overline{\overline{M}}[x := \overline{\overline{N}}] = \overline{\overline{M[x := N]}}$.
Case2 of structural reduction:
$\overline{\overline{(\mu\alpha.M)N}} = (\lambda x_1 \cdots x_n.\overline{\overline{M}})\overline{\overline{N}} \rhd_\beta \lambda x_2 \cdots x_n.\overline{\overline{M}}[x_1 := \overline{\overline{N}}] = \lambda x_2 \cdots x_n.\overline{\overline{M[\alpha \Leftarrow N]}} = \overline{\overline{\mu\alpha.M[\alpha \Leftarrow N]}}$.
Case3 of renaming reduction $(S1)$:
$\overline{\overline{\alpha(\mu\beta.M)}} = \overline{\overline{\mu\beta.M}}x_1^\alpha \cdots x_n^\alpha = (\lambda x_1^\beta \cdots x_n^\beta.\overline{\overline{M}})x_1^\alpha \cdots x_n^\alpha \rhd_\beta^* \overline{\overline{M}}[x_1^\beta := x_1^\alpha] \cdots [x_n^\beta := x_n^\alpha]$.
Case4 of $(S3)$:
$\overline{\overline{\mu\alpha.M}} = \lambda x_1^\alpha \cdots x_n^\alpha.\overline{\overline{M}} =_\alpha \lambda x.\lambda x_2 \cdots x_n.\overline{\overline{M}}[x_1 := x] = \lambda x.\lambda x_2 \cdots x_n.\overline{\overline{M[\alpha \Leftarrow x]}} = \overline{\overline{\lambda x.\mu\alpha.M[\alpha \Leftarrow x]}}$.

Remarks 7 .Moreover in order to work with $(S2)$, we need η-reduction. Then it is obtained that the corresponding lemma to the above.

Corollary 10 .Well-typed GL_X-$\lambda\mu$-terms are strongly normalizable with respect to $\rhd_+$.

Lemma 27 .Let M be a μ-closed GL_X-$\lambda\mu$-term.
If M is to be in normal with respect to $\rhd_+$, then $\overline{\overline{M}}$ is in β-normal.

Proof. By induction on the structure of the normal form. Only the following two cases are mentioned.
Case1. M is M_1M_2:
M_1 and M_2 are also in normal, and hence $\overline{\overline{M_1}}$ and $\overline{\overline{M_2}}$ are in β-normal. Assume that $\overline{\overline{M_1M_2}}$ is not in β-normal. Then $\overline{\overline{M_1}}$ must be of the form λ-abstraction or μ-abstraction. In both cases, M_1M_2 contains a redex by β-reduction or structural reduction respectively, which is a contradiction.
Case2. M contains a substerm αM_1:
M_1 is also in normal and $\overline{\overline{M_1}}$ is in β-normal. Assume that $\overline{\overline{\alpha M_1}}$ is not in β-normal. Then $\overline{\overline{M_1}}$ is the form λ-abstraction or μ-abstraction. In the first case, since M is μ-closed, $\mu\alpha$ appears outside of αM_1 in M, which is a redex by $(S3)$. On the latter case, αM_1 contains a redex by renaming reduction. Both cases lead to a contradiction.

Remarks 8 .According to the above proof, it is clear that $(S3)$ and the condition of μ-closed are necessary to obtain a β-normal form. The following two examples are given.
(a). Consider two $\lambda\mu$-terms with the type $\neg\neg A \to A$ such that $M_1 = \lambda y.\mu\alpha.y(\lambda x.\alpha x)$ and $M_2 = \mu\alpha.\alpha(\lambda z.\mu\beta.z(\lambda x.\alpha\lambda y.x))$, in which the latter is called as a μ-head form proof in [Fuji94-1], [Fuji94-2]. Then $M_2 \triangleright M_1$ with the help of $(S3)$. Let $f(A) = 1$. Both $\overline{M_1} = \lambda y.\lambda z.((\mu\alpha.y(\lambda x.\alpha x))\lambda w.wz)$ and $\overline{\overline{M_2}}$ contain a β-redex. On the other hand, $\overline{\overline{M_1}} = \lambda y.\lambda z.y(\lambda x.xz)$ which is in β-normal.
(b). Let M be $\mu\beta.\alpha\lambda x.x$ which is in normal and not μ-closed. There is a deduction of $\alpha : \neg(A \to A) \vdash_{\lambda\mu} M : B$. For $f(A) = f(B) = 1$, we have $\overline{M} = \lambda z.\alpha\lambda x.x$ where $\alpha : \neg(A \to A)^g$, and $\overline{\overline{M}} = \lambda x_0.(\lambda x.x)x_1x_2 \triangleright_\beta \lambda x_0.x_1x_2$ where $x_1 : A^g$ and $x_2 : \neg A$. Hence $\overline{M} \neq_\beta \overline{\overline{M}}$, and $\overline{\overline{M}}$ is not in β-normal.

9 Kolmogorov's translation

We show Kolmogorov-style embedding that does not collapse substructural logics, i.e., embeds proofs of a classical substructural logic to those of the corresponding intuitionistic substructural logic. P. de Groote investigated the CPS-translation of $\lambda\mu$-terms in [Groo94-1]. However our translation is different from it in the following two points:
(a). The treatment of μ-abstraction and named terms are distinct, because of the different version of Kolmogorov's negative translation.
(b). β-reductions, structural reductions and $(S2)$ are considered here.

Definition 8 (Kolmogorov-style translation of $\lambda\mu$-terms) .
(1) $\underline{x} = \lambda k.xk$; (2) $\underline{\lambda x.M} = \lambda k.k(\lambda x.\underline{M})$; (3) $\underline{M_1M_2} = \lambda k.\underline{M_1}(\lambda m.m\underline{M_2}k)$;
(4) $\underline{\mu\alpha.M} = \lambda\alpha.\underline{M}(\lambda x.x)$; (5) $\underline{\alpha M} = \lambda k.k(\underline{M}\alpha)$.

Kolmogorov's negative translation k is defined for A such that
$A^k = \neg\neg A$ for an atomic formula; $(A_1 \to A_2)^k = \neg\neg(A_1^k \to A_2^k)$.
We define A^* as the formula such that $A^k \equiv \neg\neg A^*$. The negative translation is naturally extended to contexts indexed by λ-variables and μ-variables.

We obtain that the CPS-translation transforms proofs of a classical substructural logic to those of the corresponding intuitionistic substructural logic (minimal logic).

Theorem 11 (Preservation of substructural logics) .Let M be GL_X-$\lambda\mu$-term. For each corresponding pair of X and Y, if $\Gamma, \neg\Delta \vdash M : A$, then $\Gamma^k, \neg\Delta^* \vdash \underline{M} : A^k$ where $\underline{M}$ is a Y-term.

Proof. The outline of the proof is given by the observation on the definition of the CPS-translation of de Groote such that the translation itself is to be a BCI-λ-term. In the following proof, the intuitionistic absurdity rule plays no role and only the existence of the constant $\perp$ is essential. Hence the proof is done in the primitive logic with the constant $\perp$, i.e., in minimal logic.

We prove it by induction on the number of types contained in the deduction of $\Gamma, \neg\Delta \vdash M : A$ and case analysis on the last rule. We show only the case $(\to E)$, i.e., M is M_1M_2.
Case1. $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$:
By the induction hypotheses, there are Y-terms $\underline{M_1}$ and $\underline{M_2}$ such that $\Gamma_1^k, \neg\Delta_1^* \vdash \underline{M_1} : (A \to B)^k$ and that $\Gamma_2^k, \neg\Delta_2^* \vdash \underline{M_2} : A^k$. Hence the following deduction provides a Y-term $\lambda k.\underline{M_1}(\lambda m.m\underline{M_2}k)$.

$$\dfrac{\underline{M_1} : (A \to B)^k \qquad \dfrac{\dfrac{\dfrac{[m : A^k \to B^k]^2 \quad \underline{M_2} : A^k}{m\underline{M_2} : B^k}(\to E) \qquad [k : \neg B^*]^1}{m\underline{M_2}k : \perp}(\to E)}{\lambda m.m\underline{M_2}k : \neg(A^k \to B^k)}(\to I)^2}{\dfrac{\underline{M_1}(\lambda m.m\underline{M_2}k) : \perp}{\lambda k.\underline{M_1}(\lambda m.m\underline{M_2}k) : B^k}(\to I)^1}(\to E)$$

Case2. $\lambda FV(M_1) \cap \lambda FV(M_2) \neq \phi$ or $\mu FV(M_1) \cap \mu FV(M_2) \neq \phi$, i.e., M is either a GL_C-$\lambda\mu$-term or GL_{CW}-$\lambda\mu$-term:
Same as the previous case, we have a BCK-λ-term or λ-term $\lambda k.\underline{M_1}(\lambda m.m\underline{M_2}k)$ such that $\Gamma_1^k \cup \Gamma_2^k, \neg\Delta_1^* \cup \neg\Delta_2^* \vdash \lambda k.\underline{M_1}(\lambda m.m\underline{M_2}k) : B^k$.

We show the correctness of the translation along the line of de Groote.

Lemma 28 .Let M be a $\lambda\mu$-term where $k \notin \lambda FV(\underline{M})$.
$\lambda k.\underline{M}k \rhd_\beta \underline{M}$.

Proof. By induction on the structure of M.

Lemma 29 .Let M_1 and M_2 be $\lambda\mu$-terms.
$\underline{M_1}[x := \underline{M_2}] = \underline{M_1[x := M_2]}$.

Proof. By induction on the structure of M_1.

Lemma 30 .Let M_1 and M_2 be $\lambda\mu$-terms.
$\underline{M_1}[\alpha := \lambda m.m\underline{M_2}\alpha] =_\beta \underline{M_1[\alpha \Leftarrow M_2]}$.

Proof. By induction on the structure of M_1. The case M_1 of αM is in the bellow:
$\underline{\alpha M}[\alpha := \lambda m.m\underline{M_2}\alpha] = \lambda k.k(\underline{M}[\alpha := \lambda m.m\underline{M_2}\alpha]\lambda m.m\underline{M_2}\alpha) =_\beta \lambda k.k(\underline{M[\alpha \Leftarrow M_2]}\lambda m.m\underline{M_2}\alpha) =_\beta$
$\lambda k.k((\lambda k'.\underline{M[\alpha \Leftarrow M_2]}\lambda m.m\underline{M_2}k')\alpha) = \lambda k.k(\underline{M[\alpha \Leftarrow M_2]M_2}\alpha) = \underline{\alpha(M[\alpha \Leftarrow M_2]M_2)} = \underline{(\alpha M)[\alpha \Leftarrow M_2]}$.

Lemma 31 .Let M_1 and M_2 be $\lambda\mu$-terms.
If $M_1 =_\mu M_2$, then $\underline{M_1} =_\beta \underline{M_2}$.

Proof. By induction on the derivation of $M_1 =_\mu M_2$. The atomic cases are as follows:
Case of β-reductions:
$\underline{(\lambda x.M)M_1} \rhd_\beta^* \lambda k.(\lambda x.\underline{M})\ \underline{M_1}k \rhd_\beta \lambda k.\underline{M}[x := \underline{M_1}]k = \lambda k.\underline{M[x := M_1]}k \rhd_\beta \underline{M[x := M_1]}$.
Case of structural reductions:
$\underline{(\mu\alpha.M)M_1} \rhd_\beta \lambda\alpha.\underline{M}[\alpha := \lambda m.m\underline{M_1}\alpha]\lambda x.x =_\beta \lambda\alpha.\underline{M[\alpha \Leftarrow M_1]}\lambda x.x = \underline{\mu\alpha.M[\alpha \Leftarrow M_1]}$.
Case of $(S2)$ where $\alpha \notin \mu FV(M)$:
$\underline{\lambda\alpha.\alpha M} \rhd_\beta \lambda\alpha.\underline{M}\alpha \rhd_\beta \underline{M}$.

10 Kuroda's translation

We briefly show yet another double negation translation known as Kuroda's embedding. In order to work with Kuroda's negative translation, instead of β-reductions we adopt β_V-reductions in λ_V [Plot75]:
$(\lambda x.M)V \rhd_{\beta_V} M[x := V]$.
Here a value V is defined as follows:
$V ::= x|\lambda x.M|\alpha V$.
The negative translation and the translation with Ψ from values to values are defined.
$A^q = A$ for an atomic formula; $\quad (A_1 \to A_2)^q = A_1^q \to \neg\neg A_2^q$.

Definition 9 (Kuroda-style embedding for $\lambda\mu$-terms) .
$\underline{x} = \lambda k.kx$; $\quad \underline{\lambda x.M} = \lambda k.k(\lambda x.\underline{M})$;
$\underline{M_1M_2} = \lambda k.\underline{M_1}(\lambda n.\underline{M_2}(\lambda m.nmk))$;
$\underline{\alpha M} = \lambda k.k(\underline{M}\alpha)$; $\quad \underline{\mu\alpha.M} = \lambda\alpha.\underline{M}(\lambda x.x)$, and
$\Psi(x) = x$; $\quad \Psi(\lambda x.M) = \lambda x.\underline{M}$; $\quad \Psi(\alpha V) = \alpha\Psi(V)$.

Theorem 12 .Let M be a GL_X-$\lambda\mu$-term. For each corresponding pair of X and Y, if $\Gamma, \neg\Delta \vdash M : A$, then $\Gamma^q, \neg\Delta^q \vdash \underline{M} : \neg\neg A^q$ where $\underline{M}$ is a Y-term.

Proof. By induction on the number of types contained in the $\lambda\mu$ derivation. We show only three cases.
Case1 of $(\to I)$:

$$\dfrac{\dfrac{[k : \neg(A \to B)^q]^2 \quad \dfrac{\begin{array}{c}[x : A^q]^1 \\ \vdots \\ \underline{M} : \neg\neg B^q\end{array}}{\lambda x.\underline{M} : A^q \to \neg\neg B^q}\,(\to I)^1}{k(\lambda x.\underline{M}) : \bot}\,(\to E)}{\lambda k.k(\lambda x.\underline{M}) : \neg\neg(A \to B)^q}\,(\to I)^2$$

Case2 of $(\to E)$:

$$\dfrac{\underline{M_1} : \neg\neg(A \to B)^q \qquad \dfrac{\dfrac{\underline{M_2} : \neg\neg A^q \qquad \dfrac{\dfrac{\dfrac{[n : A^q \to \neg\neg B^q]^1 \quad [m : A^q]^2}{nm : \neg\neg B^q}\,(\to E) \qquad [k : \neg B^q]^3}{nmk : \bot}\,(\to E)}{\lambda m.nmk : \neg A^q}\,(\to I)^2}{\underline{M_2}(\lambda m.nmk) : \bot}\,(\to E)}{\lambda n.\underline{M_2}(\lambda m.nmk) : \neg(A^q \to \neg\neg B^q)}\,(\to I)^1}{\dfrac{\underline{M_1}\lambda n.\underline{M_2}(\lambda m.nmk) : \bot}{\lambda k.\underline{M_1}\lambda n.\underline{M_2}(\lambda m.nmk) : \neg\neg B^q}\,(\to I)^3}\,(\to E)$$

Case3 of $(\bot I)$:

$$\dfrac{\dfrac{[k : \bot \to \bot]^1 \qquad \dfrac{\underline{M} : \neg\neg A^q \quad \alpha : \neg A^q}{\underline{M}\alpha : \bot}\,(\to E)}{k(\underline{M}\alpha) : \bot}\,(\to E)}{\lambda k.k(\underline{M}\alpha) : \neg\neg\bot^q}\,(\to I)$$

Lemma 32 .Let M be a $\lambda\mu$-term where $k \notin \lambda FV(\underline{M})$.
$\lambda k.\underline{M}k \rhd_\beta \underline{M}$.

Proof. By induction on the structure of M.

Lemma 33 .Let V be a value.
$\underline{V} \rhd_\beta^* \lambda k.k\Psi(V)$.

Proof. By induction on the structure of V. The case of αV is as follows using the induction hypothesis:
$\underline{\alpha V} = \lambda k.k(\underline{V}\alpha) \rhd_\beta^* \lambda k.k((\lambda k'.k'\Psi(V))\alpha) \rhd_\beta \lambda k.k(\alpha\Psi(V)) = \lambda k.k(\Psi(\alpha V))$.

Lemma 34 .Let M be a $\lambda\mu$-term and V be a value.
$\underline{M[x := V]} \rhd_\beta^* \underline{M}[x := \Psi(V)]$.

Proof. By induction on the structure of M. The case M of x is given using the above Lemma.
$\underline{x[x := V]} = \underline{V} \rhd_\beta^* \lambda k.k\Psi(V) = \lambda k.kx[x := \Psi(V)] = \underline{x}[x := \Psi(V)]$.

Lemma 35 . Let M_1 and M_2 be $\lambda\mu$-terms.
$\underline{M_1}[\alpha := \lambda n.\underline{M_2}(\lambda m.nm\alpha)] =_\beta \underline{M_1[\alpha \Leftarrow M_2]}$.

Proof. By induction on the structure of M_1. The case of αM is as follows.
$\underline{\alpha M}[\alpha := \lambda n.\underline{M_2}(\lambda m.nm\alpha)] = \lambda k.k(\underline{M}[\alpha := \lambda n.\underline{M_2}(\lambda m.nm\alpha)]\lambda n.\underline{M_2}(\lambda m.nm\alpha)) =_\beta$
$\lambda k.k(\underline{M[\alpha \Leftarrow M_2]}\lambda n.\underline{M_2}(\lambda m.nm\alpha)) =_\beta \lambda k.k((\lambda k'.\underline{M[\alpha \Leftarrow M_2]}\lambda n.\underline{M_2}(\lambda m.nmk'))\alpha) =$
$\lambda k.k(\underline{M[\alpha \Leftarrow M_2]M_2\alpha}) = \underline{\alpha(M[\alpha \Leftarrow M_2]M_2)} = \underline{(\alpha M)[\alpha \Leftarrow M_2]}$.

Lemma 36 .Let $=_{\mu_V}$ be a congruence relation obtained by β_V, structural reductions and $(S2)$. Let M_1 and M_2 be $\lambda\mu$-terms.
If $M_1 =_{\mu_V} M_2$, then $\underline{M_1} =_\beta \underline{M_2}$.

Proof. By induction on the derivation of $=_{\mu_V}$. The atomic cases are in the bellow.
Case of β-reductions:
$\underline{(\lambda x.M)V} = \lambda k.k(\lambda k'.k'(\lambda x.\underline{M}))(\lambda n.\underline{V}(\lambda m.nmk)) \rhd_\beta \lambda k.\underline{V}(\lambda m.\underline{M}[x := m]k) \rhd_\beta^*$
$\lambda k.(\lambda k'.k'\Psi(V))(\lambda m.\underline{M}[x := m]k) \rhd_\beta^* \lambda k.\underline{M}[x := \Psi(V)]k =_\beta \lambda k.\underline{M[x := V]}k \rhd_\beta \underline{M[x := V]}$.
Case of structural reductions:
$\underline{(\mu\alpha.M)N} = \lambda k.(\lambda\alpha.\underline{M}(\lambda x.x))(\lambda n.\underline{N}(\lambda m.nmk)) \rhd_\beta \lambda k.\underline{M}[\alpha := \lambda n.\underline{N}(\lambda m.nmk)](\lambda x.x) =_\alpha$
$\lambda\alpha.\underline{M}[\alpha := \lambda n.\underline{N}(\lambda m.nm\alpha)](\lambda x.x) =_\beta \mu\alpha.\underline{M[\alpha \Leftarrow N]}(\lambda x.x) = \underline{\mu\alpha.M[\alpha \Leftarrow N]}$.
Case of $(S2)$ where $\alpha \notin \mu FV(M)$:
$\underline{\mu\alpha.\alpha M} = \lambda\alpha.((\lambda k.k(\underline{M}\alpha))\lambda x.x) \rhd_\beta^* \lambda\alpha.\underline{M}\alpha \rhd_\beta \underline{M}$.

11 Extension to the second order classical natural deduction

We extend our discussion to the full $\lambda\mu$-calculus, i.e., second order classical natural deduction. We only show the extension for our Gödel's embedding. In the case of Kuroda's embedding, the extension is quite straightforward, and also see [Groo94-1] for Kolmogorov's embedding. In the following, only the additions are given to the inference rules and embedding.

$$\frac{\Gamma, \neg\Delta \vdash M : A[y/x]}{\Gamma, \neg\Delta \vdash \lambda x.M : \forall x.A}\ (\forall I)^* \qquad \frac{\Gamma, \neg\Delta \vdash M : \forall x.A}{\Gamma, \neg\Delta \vdash Mt : A[t/x]}\ (\forall E)$$

$$\frac{\Gamma, \neg\Delta \vdash M : A[Y/X]}{\Gamma, \neg\Delta \vdash \lambda X.M : \forall X.A}\ (\forall^2 I)^* \qquad \frac{\Gamma, \neg\Delta \vdash M : \forall X.A}{\Gamma, \neg\Delta \vdash MB : A[B/X]}\ (\forall^2 E)$$

where * denotes the eigenvariable condition.
$(\forall x.A)^g = \forall x.A^g$; $\qquad (\forall X.A)^g = \forall X.A^g$.
$f(\forall x.A) = 2 + f(A)$; $\qquad f(\forall X.A) = 2 + f(A)$.

Theorem 13 (Gödel's embedding of the full $\lambda\mu$-calculus) .Let M be $\lambda\mu$-term.
If $\Gamma, \neg\Delta \vdash M : A$ in $\lambda\mu$, then $\Gamma^g, \neg\Delta^g \vdash \overline{M} : A^g$ in $\lambda P2$ à la Curry.

12 Concluding remarks

We have defined proof term assignment to the classical resource logics consisting of implication and negation in terms of the $\lambda\mu$-calculus. According to these notions, we can classify the $\lambda\mu$-terms into four categories, i.e., GL_X-$\lambda\mu$-terms here X is either nil, C, W or CW. It is shown that GL_X-$\lambda\mu$-terms exactly correspond to GL_X proofs which are proofs of Gentzen's LK without some structural rules, that a closed GL_X-$\lambda\mu$-term has a principal type scheme if stratified, and that GL_X-$\lambda\mu$-terms have subject reduction property. On embeddings, we first discussed a translation of GL_X-$\lambda\mu$-terms into full intuitionistic logic via μ-head form proofs. As a corollary, well-known Glivenko's theorem is obtained. With the help of the Glivenko's theorem, it is derived that an algorithm which gives μ-head form proofs from arbitrary classical propositional proofs. Moreover we have investigated the three embeddings (Gödel, Kolmogorov and Kuroda) of classical substructural logics into the corresponding intuitionistic substructural logics. As corollaries of the embedding, we obtained that every GL_W-$\lambda\mu$-term is stratified, typability and inhabitation for $\lambda\mu$ are decidable, and that well-typed GL_X-$\lambda\mu$-terms are strongly normalizable.

Recently the computational aspects of classical proofs are actively investigated in [Grif90], [Murt91-2], [Pari92-2] and [Pari93] along the natural line of [How80]. The above classical systems except Parigot's $\lambda\mu$ are based on Felleisen's λ_c-calculus [FFKD86]. Murthy stated in [Murt91-1] that different double negation translations fix the order of evaluation in a functional programming. That is, a call-by-name evaluation is adopted by Kolmogorov's translation and a call-by-value evaluation is by Kuroda's translation. P.de Groote has shown the CPS translation from $\lambda\mu$ to $\lambda_\rightarrow$, which adopts Kolmogorov's negative translation in logical interpretation. On the other hand, we investigated another CPS translation cooperating with Kuroda's translation. Relating to Murthy's theorem, it is interesting to study translation and simulation property of the CPS translation with respect to not only λ_c but also $\lambda\mu$.

We finally remark technical distinction among three translations used with Gödel, Kolmogorov and Kuroda's double negation embedding. Let M' be a translated λ-term by one of them for $\lambda\mu$-term M.
1. By Gödel's embedding, we obtained Theorem9 and Lemma26, that is, if $M \triangleright_{\pm} N$, then $M' \triangleright_\beta^* N'$, by which strong normalization is obtained. However we cannot expect this property with respect to neither the CPS translations used with Kolmogorov's embedding nor Kuroda's embedding. Because Gödel's embedding establishes Lemma21. On the other hand, since the others need β-expansion rules to prove the corresponding lemma, i.e., Lemma30 and 35, they satisfy the corresponding lemma not with β-reductions but with β-conversions.
2. P. de Groote proved [Groo94-1] that if $M' =_\beta N'$, then $M =_\mu N$. On the other hand, with respect to Gödel's embedding, we cannot obtain the inverse direction of the Theorem8. For instance, let M be $\lambda y.\mu\alpha.y(\lambda x.\alpha x)$ of the type $\neg\neg A \rightarrow A$. Let A be an atomic formula distinct from $\perp$, i.e., $f(A) = 1$. Then M' is $\lambda y.G(\lambda\alpha.y(\lambda x.\alpha x), 1)$, and take N' as $\lambda yz.y(\lambda x.xz)$ which is the β-normal form of M'. We do not have $M =_\mu N$ where $N \equiv N'$.
3. All of the three simple translations cannot give that if M is to be in μ-normal, then M' is in β-normal. However the modified CPS translation by de Groote establishes this property. We also give a simplified Gödel's translation with this property for μ-closed $\lambda\mu$-terms. This kind of translation is also investigated

in [Pari92-1] here one would not prove it without the condition of μ-closed $\lambda\mu$-terms.
4. The CPS translations which are defined with Kolmogorov's or Kuroda's embedding work for untyped $\lambda\mu$-terms. On the other hand, our translation based on Gödel involves type information.
5. With the Kolmogorov-style translation, de Groote establishes the correctness of the translation with respect to a congruence relation by β-reductions, structural reductions and $(S1)$. On the other hands, our Kolmogorov-style and Kuroda-style translations also support the correctness with respect to structural reductions, $(S2)$ and either β-reductions or β_V-reductions.

References

[Bare91] H.P.Barendregt. *Lambda Calculi with Types.* Handbook of Logic in Computer Science. Oxford University Press (1991).

[Dumm77] M.Dummett. *Elements of Intuitionism.* Clarendon Press (1977).

[Fuji94-1] K.Fujita. *μ-head form proofs.* The 11-th meeting on Symbolic Logic and Computer Science (SLACS'94), Kagawa, November 6-8 (1994).

[Fuji94-2] K.Fujita. *Extending NJ with two consequences.* The Workshop on Non-Standard Logic and Logical Aspects of Computer Science (NSL'94), Kanazawa, Japan, December 5-8 (1994).

[Fuji95] K.Fujita. *Proof term assignment to classical substructural logics.* The Second Workshop on Non-Standard Logic and Logical Aspects of Computer Science (NSL'95), Irkutsk, Russia, June 15-18 (1995).

[FFKD86] M.Felleisen, D.P. Friedman, E.Kohlbecker and B.Duba. *Reasoning with continuations.* Proc. of LICS, pp.131-141, 1986.

[Grif90] T.G.Griffin. *A formulae-as-types notion of control.* Proc. of POPL, pp.47-58, 1990.

[Groo94-1] P.de Groote. *A CPS-Translation of the $\lambda\mu$-Calculus.* Proc. of the Colloquim on Trees and Algebra and Programming, 1994.

[Groo94-2] P.de Groote. *On the relation between the $\lambda\mu$-calculus and the syntactic theory of sequential control.* Proc. of the Logic Programming and Automated Reasoning, 1994.

[Hind87] J.R.Hindley. *BCK-combinators and linear λ-terms have types.* Mathematics Division, University College, Swansea. August, 1987.

[Hind88] J.R.Hindley. *The principal-type-scheme algorithm revisited: principal deductions in λ-calculus.* Mathematics Division, University College, Swansea. 20 July, 1988.

[How80] W.Howard. *The formulae-as-types notion of construction.* To H.B.Curry: Essays on combinatory logic, lambda-calculus, and formalizm. Academic Press, pp.479-490, 1980.

[Murt91-1] C.R.Murthy. *Finding computational content in classical proofs.* , Logical Frameworks. Cambridge University Press, pp.341-362, 1991.

[Murt91-2] C.R.Murthy. *An Evaluation semantics for classical proofs.* Proc. of LICS, pp.96-107, 1991.

[Ono90] H.Ono. *Structural Rules And A Logical Hierarchy.* Proc. of Heyting'88 Conference, Plenum Press, pp.95-104, 1990.

[Pari92-1] M.Parigot. *Classical Proofs as Programs.* TYPES Workshop at Båstad in June 1992.

[Pari92-2] M.Parigot. *$\lambda\mu$-Calculus: An Algorithmic Interpretation of Classical Natural Deduction.* Proc. International Conference on Logic Programming and Automated Reasoning, Springer LNCS 624, pp.190-201, (1992).

[Pari93] M.Parigot. *Strong Normalisation for Second Order Classical Natural Deduction.* Proc. LICS (1993).

[Plot75] G.D.Plotkin. *Call-by-name, call-by-value and the λ-calculus.* Theoretical Computer Science 1, pp.125-159, (1975).

[Praw65] D.Prawitz. *Natural Deduction: A Proof-Theoretical Study.* Almqvist&Wiksell, Stockholm (1965).

[Szabo69] G.Gentzen. *The collected papers of Gerhard Gentzen.* Edited by M.E.Szabo, North-Holland, 1969.