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A new proof of Chew’s theorem

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Abstract
We present a new proof of Chew’s theorem, which states that normal forms are unique up to conversion in compatible term rewriting systems.

1 Introduction
A term rewriting system (TRS) $R$ is compatible if for each pair of rules in $R$, there exist appropriate linearizations and they are almost non-overlapping. Chew’s theorem [Che81] states that the unique normal form property (UN) holds in a compatible TRS, i.e., normal forms are unique up to conversion. The theorem is important since compatibility is a syntactic condition and the class partly contains non-left-linear non-terminating TRSs. However, there is a general feeling of doubt about the original proof in [Che81]. In fact, there is a gap in the proof of a key lemma.

There have been several attempts at a new proof, and partial answers have been obtained [dV90, Oga92, TO94]. De Vrijer showed that UN of a TRS $R$ can be reduced to the Church-Rosser property (CR) of its conditional linearization, $R^L$ [dV90]. In $R^L$, reductions are associated with subproofs that solve equivalence constraint. If $R^L$ is non-overlapping, $R^L$ is CR [BK86]. De Vrijer [dV90] showed that a particular compatible TRS CL-pc (combinatory logic with parallel-conditional) is UN by the following: (1) construct a model, (2) show CL-pc to be semantically non-overlapping, (3) CL-pc is thus CR. However, it is generally not easy to find such an appropriate model for a compatible TRS. Ogawa proved that UN holds for so-called weakly compatible TRSs [Oga92]. This class contains CL-pc, but is incomparable with the class of Chew’s compatible TRSs. Toyama and Oyamaguchi [TO94] introduced a variant of conditional linearization and gave a sufficient condition of UN for non-duplicating TRSs. In this paper, we will give a new proof of the entire statement of Chew’s theorem itself in a complete form.

Let us briefly outline the methodology of our proof. Given a compatible TRS $R$, we transform it into conditional linearization $\hat{R}$ with extra variables [TO94]. Similar to what de Vrijer observed, it is sufficient to prove that CR holds for $\hat{R}$ in order to conclude that $R$ is UN. We will prove CR of $\hat{R}$ by a peak elimination process. Given a proof $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ in $\hat{R}$, the peak elimination replaces a peak $t_{i-1} \rightarrow t_i \rightarrow t_{i+1}$ in this proof with a conversion $t_{i-1} \leftrightarrow^{*} t_{i+1}$ in $\hat{R}$ according to the peak elimination rules. If all peaks are eliminated by applying the rules to the given proof repeatedly, i.e., if the peak elimination process eventually terminates, then we find a term $s$ such that $t_1 \rightarrow^{*} s \rightarrow^{*} t_n$ as shown in figure 1. (Section 3)

We say a reduction is in a proof $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ not only for the reductions $t_i \rightarrow t_{i+1}$ (or $t_i \leftarrow t_{i+1}$) but also for the ones in the subproofs. When a proof $A'$ is obtained by applying a peak elimination to a proof $A$, any reduction $\alpha'$ in $A'$ can be regarded as a descendant of a reduction in $A$. Unfortunately, if a peak is made with overlapping reductions, the peak elimination may cause multiple descendants of a reduction. That is, if the reductions $\gamma_1 : t_{i-1} \leftarrow t_i$ and $\gamma_2 : t_i \rightarrow t_{i+1}$ are overlapping and a reduction $\alpha$ is in a subproof of $\gamma_1$ (or $\gamma_2$), then multiple descendants of $\alpha$ can be caused by eliminating the peak made with $\gamma_1$ and $\gamma_2$. In this case, $\alpha$ is said to be duplicated by $\gamma_1$ ($\gamma_2$). This makes it difficult to prove termination of peak elimination processes.

However, we can estimate how many times eliminations of overlapping peak occur by examining which reduction

\footnote{See section 2.2 for details.}
or descendants of it) can not go into a subproof of reductions making a overlapping peak during the process. Therefore, we introduce a binary relation independence on the reductions that satisfies the following properties:

1. Independence is preserved during a peak elimination process. (Theorem 4.1)

2. A reduction is not independent of its subproofs. (Lemma 4.4)

3. If two reductions are independent, their subproofs are also independent. (Lemma 4.5)

4. Two descendants of a reduction are independent of each other. (Lemma 4.6)

One of the candidates of independence is "parallelness" formally defined as $\perp_{1}$ in this paper. In fact, if the conditional linearization $\hat{R}$ is non-overlapping, "parallelness" satisfies the above properties. However, for a compatible TRS this is not enough. Consider the following compatible TRS$^{2}$:

$$R = \{ f(x, a) \rightarrow x, \ f(a, y) \rightarrow y \}$$

Note that $R$ is overlapping. Its conditional linearization $\hat{R}$ is:

$$\hat{R} = \{ f(x', a) \rightarrow x \text{ if } x' = x \cdots (1), \ f(a, y') \rightarrow y \text{ if } y' = y \cdots (2) \}$$

Suppose that $t_{1} \rightarrow^{*} a$ and $a \leftrightarrow^{*} t_{2}$ in $\hat{R}$, then there is a peak of the form:

$$t_{1} \rightarrow^{*} a \quad a \leftrightarrow^{*} t_{2} \quad : \text{subproofs}$$

$$t_{1} \quad \overset{(1)}{\leftarrow} \quad f(a, a) \quad \overset{(2)}{\rightarrow} \quad t_{2} \quad : \text{peak}$$

where $t_{1} \rightarrow^{*} a$ contracts with the first argument of $f(a, a)$, and $a \leftrightarrow^{*} t_{2}$ with the second argument, so they are "parallel". A peak elimination replaces the peak with:

$$t_{1} \rightarrow^{*} a \leftrightarrow^{*} t_{2}$$

and "parallelness" is not preserved. However, in this case the term $a$ is a normal form (since $\hat{R}$ contains only root-to-root overlap and $a$ is a proper subterm of the LHS of a rule) and it "splits" the proof into $t_{1} \rightarrow^{*} a$ and $a \leftrightarrow^{*} t_{2}$. By exploiting this observation, independence must be defined for the conditional linearizations of compatible TRSs. (Section 4)

We then introduce initial labeling and a descendant forest for a peak elimination process. Each reduction in the process is labeled an initial label, which indicates the ancestor of the reduction in the starting proof. For each reduction in the starting proof of the process, a descendant tree in the forest is associated; the reduction in the starting proof is the root vertex of the tree. Each path of the tree traces the descendants of the starting reduction, and non-leaf vertices represent applications of peak eliminations that repetitions of reductions of the descendant occurred in. From the property of independence, if a reduction $a$ is duplicated by another reduction labeled $\gamma$, then any descendants of $a$ will never be duplicated by the reductions labeled $\gamma$ any more. This proves that the descendant forest is finite, which leads to the termination of the peak elimination process. Therefore, CR of $\hat{R}$ is obtained, and so we complete the proof of Chew’s theorem. (Section 5)

---

$^{2}$This is an example which is compatible but not weakly compatible[Oga92].
2 Preliminaries

2.1 Abstract reduction systems and term rewriting systems

The definitions and terminologies of abstract reduction systems, terms, and term rewriting systems are taken from [Klo92].

Let $\rightarrow$ be an abstract reduction system that is a binary relation on some underlying domain. The symmetric closure, the reflexive transitive closure, and the reflexive transitive symmetric closure of $\rightarrow$ are written as $\leftrightarrow$, $\rightarrow^*$ and $\leftrightarrow^*$, respectively. If there is no $a'$ such that $a \rightarrow a'$, then $a$ is a normal form of the reduction system. A sequence $a_1 \rightarrow \cdots \rightarrow a_n$ is called a proof. A subsequence of the form $a' \rightarrow a \rightarrow a''$ is called a peak.

A reduction system $\rightarrow$ has the unique normal form property (UN) if $a \rightarrow^* a' \Rightarrow a \equiv a'$ for each pair of normal forms $a, a'$. We say $\rightarrow$ has the Church-Rosser property (CR) if, for any $a \rightarrow^* a'$, there exists $b$ such that $a \rightarrow^* b$ and $a' \rightarrow^* b$.

Let $F$ be a set of function symbols, and let $V$ be a countably infinite set of variables. The set of all terms built from $F$ and $V$ is defined as usual. The set of all terms occurring in a term $t$ is denoted by $V(t)$.

Let $\square$ be a fresh special constant symbol. A context $C[]$ is a term in $F \cup \square$ and $V$. When $C[]$ is a context with $n$ $\square$'s and $t_1, \ldots, t_n$ are terms, $C[t_1, \ldots, t_n]$ denotes the term obtained by replacing all $\square$ in $C[]$ with $t_i$ in a left-to-right manner.

Let $t$ be terms s.t. $t \equiv C[s]$ with a context $C[]$ and a non-variable term $s$. If $s$ and $t'$ are unifiable with a most general unifier $\theta$, then $C[s\theta]$ is called a superposition of $t$ and $t'$.

Positions of a term are encoded in the sequences of natural numbers. The set of positions of a term $t$ is denoted by $P(t)$. For a position $p \in P(t)$, $t[p \leftarrow s]$ is the term obtained by replacing the subterm at $p$ in $t$ with $s$.

For positions $p_1, p_2$, $p_1 \leq p_2$ if $p_1$ is a prefix of $p_2$. We write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. When neither $p_1 \leq p_2$ nor $p_2 \leq p_1$, $p_1$ and $p_2$ are said to be parallel (notation $p_1 \perp p_2$). The longest common prefix of $p_1$ and $p_2$ is denoted by $\lambda(p_1, p_2)$.

A term rewriting system (TRS) is a finite set $R$ of rewrite rules. A rewrite rule is a pair of terms denoted by $l \rightarrow r$ satisfying (1) $l$ is not a variable and (2) $V(l) \supseteq V(r)$.

The reduction system $\rightarrow_R$ on the set of terms is defined from a TRS $R$ as follows:

$$\rightarrow_R = \{ C[l\theta] \rightarrow_R C[r\theta] \mid C[] \text{ is a context, } \theta \text{ is a substitution, and } l \rightarrow r \in R \}.$$ 

A term $l\theta$ is called a redex of $R$ if $l \rightarrow r \in R$. For a reduction $\alpha : C[l\theta] \rightarrow_R C[r\theta]$, the position of the redex $l\theta$ in $C[l\theta]$ is denoted by $p(\alpha)$.

When we think of a pair of rules $S$ and $S'$, we assume that $S$ and $S'$ are standardized apart, i.e., the variables in $S$ and $S'$ are renamed appropriately so that $S$ and $S'$ do not share variables.

Let $C[]$ be a context with $n$ $\square$'s, and let $t_1 \rightarrow_R t'_1$ be proofs in $R$ for $1 \leq i \leq n$. The embedding of the proofs into $C[]$ is the following:

$$C[t_1, t_2, \ldots, t_n] \leftrightarrow_R C[t'_1, t_2, \ldots, t_n] \leftrightarrow_R C[t'_1, t'_2, \ldots, t_n] \leftrightarrow_R \cdots \leftrightarrow_R C[t'_1, t'_2, \ldots, t'_n],$$

which is denoted by $C[t_1, t_2, \ldots, t_n] \Rightarrow_R C[t'_1, t'_2, \ldots, t'_n]$.

Rewrite rules $S$ and $S'$ are overla y if a superposition of $l$ and $l'$ exists only in a root-to-root case, i.e., the context $C[]$ in the definition of superposition is $\square$. If $S$ and $S'$ are overlay and $r\sigma \equiv r'\sigma$ for all unifiers $\sigma$ of $l$ and $l'$, then $S$ and $S'$ are almost non-overlapping.

Definition 2.1 A term $\bar{t}$ is a linearization of a term $t$ if (1) $\bar{t}$ is linear, and (2) there is a substitution $\sigma$ s.t. $\bar{t}\sigma = t$ and $x\sigma \in V$ for all $x \in V$. For a rewrite rule $l \rightarrow r$, $\bar{l} \rightarrow \bar{r}$ is called a linearization of $l \rightarrow r$, if the following properties hold:

- $\bar{l}$ is a linearization of $l$ s.t. $\bar{l}\sigma = l$, and
- $\bar{r}\sigma = r$.

Definition 2.2 ([Che81, dV90]) Rewrite rules $S$ and $S'$ are said to be compatible\footnote{De Vrijer's terminology [dV90] is used here. The corresponding notion in Chew's original paper is "strongly non-overlapping and compatible".} if there exist linearizations $\bar{S}$, $\bar{S}'$ of $S$, $S'$ such that $\bar{S}$ and $\bar{S}'$ are almost non-overlapping. A TRS $R$ is compatible if each pair of rules is compatible.
Example 2.1 Combinatory logic CL can be regarded as a TRS if the function application is expressed explicitly by a binary symbol, e.g., $\alpha$. A compatible TRS CL-\pc is the union of CL and the following parallel-conditional rules.

\[
\begin{align*}
\text{SKI} & \\
Sxy & \rightarrow x(yz) \\
Kxy & \rightarrow x \\
Ixy & \rightarrow x
\end{align*}
\begin{align*}
\text{parallel-conditional} & \\
CTxy & \rightarrow x \\
CFxy & \rightarrow y \\
Czxy & \rightarrow x
\end{align*}
\]

The aim of this paper is the proof of Chew's theorem [Che81].

Theorem 2.1 A compatible TRS is UN.

2.2 Scenario of Chew's original proof

Let $R$ be a TRS, and let $R'$ be the set of all linearizations of all rules in $R$. For example, if $g(h(x,x)) \rightarrow h(x,x) \in R$, then $g(h(x_1, x_2)) \rightarrow h(x_1, x_1)$, $g(h(x_1, x_2)) \rightarrow h(x_1, x_2)$, $g(h(x_1, x_2)) \rightarrow h(x_2, x_1)$ and $g(h(x_1, x_2)) \rightarrow h(x_2, x_2)$ are in $R'$. For a reduction $\rightarrow$, $t \xrightarrow{R'} t'$ denotes reduction sequence preserving the root symbol of $t$. In order to avoid difficulty caused by non-left-linearity, Chew introduced close-up marker $4$.

The close-up $\xrightarrow{R}$ of $\xrightarrow{R}$ with respect to $R'$ is (inductively) defined as the following conditional TRS obtained from $R'$:

\[
\begin{align*}
& g(h(x_1, x_2)) \rightarrow h(x_1, x_1) \\
& g(h(x_1, x_2)) \rightarrow h(x_1, x_2) \\
& g(h(x_1, x_2)) \rightarrow h(x_2, x_1) \\
& g(h(x_1, x_2)) \rightarrow h(x_2, x_2)
\end{align*}
\]

Two fresh symbols $\alpha$ and $\beta$ called markers (corresponding to the right direction and the left direction respectively, as will become clear) are introduced to represent "all the possible choices of variables in the linearization" in one rewrite rule. For example, $g(h(x,x)) \rightarrow h(x,x)$ is transformed into the following rule using $\alpha$:

\[
g(h(x_1, x_2)) \rightarrow \alpha(h(x_1, x_1), h(x_1, x_2), h(x_2, x_1), h(x_2, x_2)).
\]

The TRS obtained by such a transformation from $R$ is denoted by $\alpha R$. $\beta R$ is defined similarly using the symbol $\beta$. To simulate $\xrightarrow{R}$, the following additional reductions are also introduced, i.e., copying reductions $\rightarrow_{\alpha+}$, $\rightarrow_{\beta+}$, selecting reductions $\rightarrow_{\alpha-}$, $\rightarrow_{\beta-}$, and distributing reductions $\leftrightarrow_{\alpha \sigma}$, $\leftrightarrow_{\beta \sigma}$. For instance,

\[
\begin{align*}
& h(t_1, t_2) \rightarrow_{\alpha+} \alpha(h(t_1, t_2), h(t_1, t_2)), \\
& \alpha(h(t_1, t_2), h(t_3, t_4)) \rightarrow_{\alpha-} h(t_1, t_2) \text{ or } h(t_3, t_4), \\
& g(\alpha(h(t_1, t_2), h(t_3, t_4))) \rightarrow_{\leftrightarrow_{\alpha \sigma}} g(\alpha(h(t_1, t_3), \alpha(t_2, t_4))).
\end{align*}
\]

A reduction $\rightarrow_{\alpha R^c}$ (resp.) is the closure of $R$ with respect to $\alpha R$ ($\beta R$) using $\rightarrow_{\alpha R^c}$ in the conditional part, where $\rightarrow R = \rightarrow_{\alpha R} \cup \rightarrow_{\beta R} \cup \rightarrow_{\alpha+} \cup \rightarrow_{\beta+} \cup \rightarrow_{\alpha-} \cup \rightarrow_{\beta-} \cup \rightarrow_{\alpha \sigma} \cup \rightarrow_{\beta \sigma} \cup \rightarrow_{\alpha \sigma}$. Let $\rightarrow S = \rightarrow_{\alpha R} \cup \rightarrow_{\alpha+} \cup \rightarrow_{\beta+} \cup \rightarrow_{\alpha-} \cup \rightarrow_{\beta-} \cup \rightarrow_{\alpha \sigma} \cup \rightarrow_{\beta \sigma}$. The outline of Chew's original proof is the following. At first, similar to what de Vrijer observed, UN of $\rightarrow R$ is reduced to CR of $\rightarrow R$. Next, $\rightarrow S$ and $\rightarrow T$ are shown to be commutative. Finally, CR of $\rightarrow R$ is proved by the following steps: given a proof $t \xrightarrow{R} t'$,

1. transform it into $t \xrightarrow{R} t'$ (since $\rightarrow R$ and $\rightarrow R$ are the same in convertibility),
2. replace each $\rightarrow R$ with $\rightarrow_{\alpha R^c} \cdot \rightarrow_{\alpha+} (\in \rightarrow S \cdot \rightarrow T)$ and replace each $\rightarrow R$ with $\rightarrow_{\beta R^c} \cdot \rightarrow_{\beta+} (\in \rightarrow S \cdot \rightarrow T)$,
3. $t \xrightarrow{R} t'$ from the commutativity of $\rightarrow S$ and $\rightarrow T$,
4. $t \xrightarrow{R} t'$ by "stripping" $\alpha$'s and $\beta$'s.

The key lemma 6.1 in [Che81] is necessary in the final step. It states that if $A$ is a redex of $\rightarrow_{\alpha R^c}$ (by definition, this means that there exists a redex $B$ of $R$ such that $B \rightarrow_{\alpha R} A$), then any $\rightarrow_{\alpha-} \cup \rightarrow_{\beta-}$-normal form

\footnote{Notations and definitions are slightly different from the original. We use $\rightarrow R$, $\rightarrow R^c$, $\rightarrow_{\alpha R^c}$, $\rightarrow_{\beta R^c}$ instead of the original notations $\rightarrow G$, $\rightarrow G^c$, $\rightarrow_{\alpha G^c}$, $\rightarrow_{\beta G^c}$.}
\[ A \text{ of } A \text{ is a redex of } \rightarrow_{R}. \text{ The "proof" of the lemma is due to the induction on the length of } B \rightarrow\!\!\!→^{*}_{R} A. \text{ However, here is a gap which seems to be difficult to remedy.} \]

The induction does not work for \( \rightarrow_{od} \) [vO94]. Let us consider the following example:

\[ B \rightarrow_{R} g(\alpha(h(t_{1}, t_{2}), h(t_{3}, t_{4}))) \rightarrow_{od} g(h(\alpha(t_{1}, t_{2}), \alpha(t_{2}, t_{4}))), \]

where \( t_{i} \) are arbitrary terms containing neither \( \alpha \) nor \( \beta \). Removing the markers by \( \rightarrow_{\alpha-} \) and \( \rightarrow_{\beta-} \), we obtain \( C_{B'} = \{g(h(t_{1}, t_{2}), g(h(t_{3}, t_{4}))) \} \) from \( B' \) and \( C_{A} = \{g(h(t_{1}, t_{2}), g(h(t_{3}, t_{4}))) \} \) from \( A \). In the induction step, it must be shown that for each \( s_{A} \in C_{A} \), there exists \( s_{B'} \in C_{B'} \) such that \( s_{B'} \rightarrow_{R} s_{A} \); this is impossible in the case \( s_{A} = g(h(t_{1}, t_{4})) \) or \( g(h(t_{3}, t_{2})) \).

### 2.3 A property of compatible rewrite rules

In this section, we establish some properties of compatible systems used in the later sections.

**Definition 2.3** The set of non-common positions \( NC_{t,t'} \) of terms \( t \) and \( t' \) is the set of all minimal elements in \( \{p \mid \text{Root}(t/p) \neq \text{Root}(t'/p)\} \) wrt \( \leq \), where \( \text{Root}(s) \) is the root symbol of the term \( s \). The common context \( C_{t,t'} \) of \( t \) and \( t' \) is \( t|p \leftarrow \square | p \in NC_{t,t'} \) (\( \equiv t'|p \leftarrow \square | p \in NC_{t,t'} \)).

**Definition 2.4** For terms \( t, t' \), a relation \( \sim_{t,t'} \) is defined as follows:

\[ s \sim_{t,t'} s' \text{ iff } s \equiv t/p \text{ and } s' \equiv t'/p \text{ for some } p \in NC_{t,t'}. \]

**Lemma 2.1** Let \( t, t' \) be terms without shared variables. Assume \( s \sim_{t,t'} C[u] \) and \( u \sim_{t,t'} u' \). Then \( u \) is a ground term.

**Lemma 2.2** Let \( t, t' \) be linear terms without shared variables. Suppose that \( t \) and \( t' \) are unifiable. Then the substitution defined as below is a unifier of \( t \) and \( t' \):

\[ \theta_{t,t'} = \{x := s' \mid x \sim_{t,t'} s'\} \cup \{x' := s \mid s \sim_{t,t'} x' \text{ and } s \notin V\}. \]

**Lemma 2.3** Let \( S : l \rightarrow r, S' : l' \rightarrow r' \) be compatible rewrite rules with unifiable linearizations of left-hand sides, i.e., there exist linearizations \( S : l \rightarrow \bar{r}, S' : l' \rightarrow \bar{r}' \) of \( S, S' \) respectively such that \( \bar{l} \) and \( \bar{l}' \) are unifiable and \( \bar{r} \equiv \bar{r}' \sigma \) for each unifier \( \sigma \) of \( \bar{l} \) and \( \bar{l}' \). Then for all \( q \in NC_{r,r'} \), either of the following holds:

1. \( \bar{r}/q \in V \), and there exist a context \( C_q'[\cdot] \) with \( m' \sqcup s \) (\( m' \geq 0 \)), ground terms \( g_{1}', \ldots, g_{m'} \), and variables \( x_{1}', \ldots, x_{m'} \) in \( S' \) s.t.
   - \( \bar{r}/q \sim_{l,l'} C_q'[g_{1}', \ldots, g_{m'}], \)
   - \( g_{k} \sim_{l,l'} x_{k}' \) for all \( k = 1, \ldots, m' \), and
   - \( \bar{r}/q \equiv C_q'[x_{1}', \ldots, x_{m'}]. \)
2. \( \bar{r}/q \in V \), and there exist a context \( C[\cdot] \) with \( n' \sqcup s \) (\( n' \geq 0 \)), ground terms \( g_{1}', \ldots, g_{n'} \), and variables \( x_{1}, \ldots, x_{n'} \) in \( S \) s.t.
   - \( C_q'[g_{1}', \ldots, g_{n'}] \sim_{l,l'} \bar{r}/q, \)
   - \( x_{k} \sim_{l,l'} g_{k}' \) for all \( k = 1, \ldots, n' \), and
   - \( \bar{r}/q \equiv C_q'[x_{1}, \ldots, x_{n'}]. \)

**Proof** Since \( \bar{r} \) and \( \bar{r}' \) are unifiable, \( \bar{r}/q \in V \) or \( \bar{r}'/q \in V \). We only check the former case. The other case is treated similarly. Let \( C_q'[\cdot] = \bar{r}/q[x := \square | \text{if } x \theta_{l,l'} \neq x] \), where \( \theta_{l,l'} \) is the unifier defined in lemma 2.2. Since \( \theta_{l,l'} \) is a unifier of \( \bar{r} \) and \( \bar{r}' \), there are terms \( g_{1}', \ldots, g_{m'} \) and variables \( x_{1}', \ldots, x_{m'} \) satisfying the three conditions. From lemma 2.1, \( g_{k} \) is a ground term for \( k = 1, \ldots, m' \).
3 Conditional linearization and peak elimination

3.1 Left-right separated CTRS and conditional linearization

Definition 3.1 A left-right separated conditional term rewriting system is a finite set of conditional rewrite rules with extra variables of the form \( l \rightarrow r \Leftarrow x_1 = y_1, \ldots, x_n = y_n \) satisfying the following conditions:

1. \( l \) is left-linear, \( V(l) = \{x_1, \ldots, x_n\} \),
2. \( V(r) \subseteq \{y_1, \ldots, y_n\} \),
3. \( \{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \emptyset \), and
4. \( x_i \neq x_j \) if \( i \neq j \).

\( l \rightarrow r \) is called the unconditional part and \( x_1 = y_1, \ldots, x_n = y_n \) is called the condition part of \( l \rightarrow r \Leftarrow x_1 = y_1, \ldots, x_n = y_n \). For convenience:

1. A condition part is often abbreviated by \( Q, Q' \), etc.

2. Variables \( x_1, \ldots, x_n \) are assumed to appear in the left-to-right order in \( l \), e.g. \( l = f(x_1, g(x_2, x_3)) \).

Definition 3.2 Let \( \hat{R} \) be a left-right separated CTRS. The reduction \( \nabla_{\hat{R}_i} \) is inductively defined as follows:

\[
\begin{align*}
\nabla_{\hat{R}_0} &= \emptyset, \\
\nabla_{\hat{R}_{i+1}} &= \{C[\hat{\theta}] \nabla_{\hat{R}_{i+1}} C[\hat{\theta}] \mid i \rightarrow \hat{\theta} \Leftarrow x_1 = y_1, \ldots, x_n = y_n \in \hat{R} \text{ and } x_j \nabla_{\hat{R}_i} y_j\theta \text{ for } i = 1, \ldots, n\}.
\end{align*}
\]

Then, \( \nabla_{\hat{R}} = \cup_i \nabla_{\hat{R}_i} \).

Proofs \( x_j \nabla_{\hat{R}}^* y_j\theta \) are called subproofs associated with \( C[\hat{\theta}] \nabla_{\hat{R}_{i+1}} C[\hat{\theta}] \). Subproofs of an \( R_1 \) reduction are called trivial subproofs, and we eventually denote \( \nabla_{\hat{R}_1} \) as \( \nabla_{\hat{R}_1} \).

When a reduction \( t \nabla_{\hat{R}} t' \) is done by a rewrite rule \( \hat{S} \in \hat{R} \), it is also denoted by \( t \nabla_{\hat{S}} t' \). For a reduction \( C[\hat{\theta}] \nabla_{\hat{R}} C[\hat{\theta}] \), \( \hat{\theta} \) is called a redex.

Reductions are often treated as more than a relation; we assume a reduction in \( \hat{R} \) is associated with the following “information” implicitly: the rule used, the position, and the subproofs.

Similarly, a rewrite proof \( A : t_1 \nabla_{\hat{R}} \cdots \nabla_{\hat{R}} t_n \) is regarded as a hierarchical object. Reductions \( t_i \nabla_{\hat{R}} t_{i+1} \) (or \( t_i \nabla_{\hat{R}} t_{i+1} \)) themselves are top-level components, reductions in subproofs of them are second-level components, etc. A reduction \( \alpha \) is in \( A \) when \( \alpha \) is a component of the hierarchical object. Moreover, the top-level component is called the top-level reduction.

Definition 3.3 For a rewrite rule \( S : l \rightarrow r \), a conditional linearization \( \hat{S} : \hat{l} \rightarrow \hat{r} \Leftarrow Q \) is a left-right separated conditional rewrite rule constructed as follows:

1. \( \hat{l} \) is a linearization of \( l \) s.t. \( \hat{l} \sigma = l \) and \( V(\hat{l}) \cap V(l) = \emptyset \),
2. \( \hat{r} \equiv r \), and
3. add \( x\sigma = x \) to the condition part \( Q \) for all \( x \in V(l) \).

Note that conditional linearizations of \( S \) are unique up to renaming of variables in \( \hat{l} \). In the rest of this paper, \( R \) denotes the TRS and \( \hat{R} \) denotes the collection of conditional linearizations of all rules in \( R \), called the conditional linearization of \( R \).

Example 3.1 \( \hat{R} \) is the conditional linearization of \( R \).

\[
R = \left\{ \begin{array}{ccc}
d(x, x) & \rightarrow & 0 \\
f(y) & \rightarrow & d(y, f(y)) \\
1 & \rightarrow & f(1) \\
\end{array} \right\}.
\]

\[
\hat{R} = \left\{ \begin{array}{ccc}
d(x_1, x_2) & \rightarrow & 0 \\
f(y_1) & \rightarrow & d(y_1, f(y)) \\
1 & \rightarrow & f(1) \\
\end{array} \right\}.
\]

\( y_i \equiv y_j \) may hold for \( i \neq j \).
The following theorem appeared in [TO94] with the condition of non-duplicating. The expansion to the general case is straightforward.

**Theorem 3.1 ([TO94])** If \( \hat{R} \) is CR, then \( R \) is UN.

For left-right separated conditional rewrite rules \( \hat{S}, \hat{S}', \hat{S} \) and \( \hat{S}' \) are said to be non-overlapping (almost non-overlapping, overlay) if their unconditional parts are overlapping (almost non-overlapping, overlay). A left-right separated CTRS \( \hat{R} \) is non-overlapping (almost non-overlapping, overlay) when every pair of rules in \( \hat{R} \) is non-overlapping (almost non-overlapping, overlay). A left-right separated CTRS \( \hat{R} \) is compatible if there exists a compatible TRS \( R \) such that \( \hat{R} \) is the conditional linearization of \( R \).

**Definition 3.4** A term \( t \) is a head normal form of \( \hat{R} \) if \( s \) is not a redex of \( \hat{R} \) for all \( s \) such that \( t \rightarrow^* \hat{R} s \). A term \( t \) is a quasi-ground normal form of \( \hat{R} \) wrt \( q \in P(t) \) if

1. for each \( q' \leq q \), \( t/q' \) is a head normal form of \( \hat{R} \), and
2. \( t/q \) is a ground normal form of \( \hat{R} \).

**Lemma 3.1** Let \( \hat{I} \rightarrow \hat{r} \Leftarrow Q \in \hat{R} \). Suppose that \( \hat{R} \) is compatible. Then for each non-variable proper subterm \( t \) of \( I \) and substitution \( \theta, t\theta \) is a head normal form of \( \hat{R} \).

### 3.2 Conditional peak elimination

In this section, the following notations will be established:

1. \( R \) is a compatible TRS.
2. \( S : I \rightarrow r, S' : l' \rightarrow r' \in R \).
3. \( \hat{S} : \hat{I} \rightarrow \hat{r}, \hat{S}' : \hat{P} \rightarrow r' \) are linearizations of \( S \) and \( S' \) s.t. \( \hat{r} = \hat{r}' \sigma \) for all unifiers \( \sigma \) of \( \hat{I} \) and \( \hat{P} \).
4. \( \hat{S} : \hat{I} \rightarrow \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n, \hat{S}' : \hat{P} \rightarrow \hat{r}' \Leftarrow x'_1 = y'_1, \cdots, x'_m = y'_m \) are conditional linearizations of \( S \) and \( S' \) s.t. \( \hat{I} \equiv \hat{I} \) and \( \hat{P} \equiv \hat{P} \).

**Definition 3.5** Suppose that there is a peak of the form \( C[\theta] \mathop{\Sigma}_{\hat{S}} C[\theta] \equiv C[\theta'] \mathop{\Sigma}_{\hat{S}} C[\theta'] \). For \( p \in NC_{i,p} \), the left connecting proof \( A_p \) of the peak is defined as follows:

\[
A_p = \begin{cases} 
    y_\theta \mathop{\Sigma}_{\hat{R}} x_\theta \equiv C_p[x'_\theta, \cdots, x'_{j+\theta}] \mathop{\Sigma}_{\hat{S}} C'_{p'}[y'_\theta, \cdots, y'_{j+\theta}] & \text{if } \hat{I}/p \equiv x_i \text{ and } \hat{P}/p = C'_p[x'_1, \cdots, x'_{j+\theta}], \\
    C_p[y_\theta, \cdots, y_{i+\theta}] \mathop{\Sigma}_{\hat{R}} C_p[x_\theta, \cdots, x_{i+\theta}] \equiv x'_\theta \mathop{\Sigma}_{\hat{S}} y'_\theta & \text{if } \hat{I}/p = C_p[x_i, \cdots, x_{i+\theta}] \notin V \text{ and } \hat{P}/p = x'_j,
\end{cases}
\]

where \( x_k \mathop{\Sigma}_{\hat{R}} y_k \) and \( x'_k \mathop{\Sigma}_{\hat{R}} y'_k \) are the subproofs of \( C[\theta] \mathop{\Sigma}_{\hat{S}} C[\theta] \equiv C[\theta'] \mathop{\Sigma}_{\hat{S}} C[\theta'] \). \( V(\hat{I}/p) = \{x_i, \cdots, x_{i+\theta}\} \) and \( V(\hat{P}/p) = \{x'_1, \cdots, x'_{j+\theta}\} \).

**Definition 3.6** For a rewrite rule \( \hat{S} : \hat{I} \rightarrow \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n \), \( T_{\hat{S}} \) is a substitution defined as follows:

\[
T_{\hat{S}} = \{x_1 := y_1, \cdots, x_n := y_n\}.
\]

**Lemma 3.2** Suppose that there is a peak of the form \( C[\theta] \mathop{\Sigma}_{\hat{S}} C[\theta] \equiv C[\theta'] \mathop{\Sigma}_{\hat{S}} C[\theta'] \). Assume \( t \sim_{l, p}, t' \).

Then there exits \( p \in NC_{i,p} \) such that \( tT_{\hat{S}} \mathop{\Sigma}_{\hat{R}} t'T_{\hat{S}}, \theta \) is the left connecting proof \( A_p \).

**Lemma 3.3** Suppose that there is a peak of the form \( C[\theta] \mathop{\Sigma}_{\hat{S}} C[\theta] \equiv C[\theta'] \mathop{\Sigma}_{\hat{S}} C[\theta'] \). For \( q \in NC_{p,p} \), either of the following holds:

1. \( q \in V \) and there exist a context \( C_q[\theta] \) with \( m' \cap q \) s.t. \( (m' \cap q) = 0 \), ground terms \( g_1, \cdots, g_{m'} \) and variables \( y_{j_1}, \cdots, y_{j_{m'}} \). If

   - \( q \sim_{\hat{R}} C_q[\theta] \mathop{\Sigma}_{\hat{S}} C_q[\theta]T_{\hat{S}}, \theta \) is a left connecting proof of the peak,
   - \( g_k \mathop{\Sigma}_{\hat{S}} y_{j_k} \theta \) are left connecting proofs of the peak for all \( k = 1, \cdots, m' \),
Proof. We only check the former case. The other case is treated similarly. The first three conditions are satisfied by lemmas 2.3 and 3.2, \( r \equiv r S \), and \( r' \equiv r'S \). The last condition follows from lemma 3.1 and the fact that \( C'[g_1, \ldots, g_m] \) is a proper subterm of \( \tilde{r} \).

Definition 3.7 Suppose there is a peak of the form \( C[r\theta] \subset S C[l\theta] \equiv C[l\theta] \subset S C[r\theta] \). For \( q \in NC_\tilde{r}, \tilde{s} \), the right connecting proof \( B_q \) of the peak is a proof connecting \( \tilde{r}/\theta \) and \( \tilde{r}'/\theta \) described in the previous lemma, i.e.,

\[
B_q = \begin{cases} 
\tilde{r}/\theta \equiv C'[g_1, \ldots, g_m]T_S\theta \equiv \tilde{S}' C'[g_1, \ldots, g_m]T_S\theta & \text{if } \tilde{r}/\theta \in V, \\
C_q[y_1, \ldots, y_k]T_S\theta \equiv \tilde{S}' C_q[y_1, \ldots, y_k]T_S\theta & \text{if } \tilde{r}/\theta \notin V \text{ and } \tilde{r}'/\theta \in V.
\end{cases}
\]

Definition 3.8 For a proof \( A : t_i \rightarrow_R \cdots \rightarrow_R t_n \) in \( \tilde{R} \), a peak elimination is a transformation of \( A \) where a peak in \( A \), e.g., \( t_{i-1} \rightarrow_R t_i \rightarrow_R t_{i+1} \), is replaced with the sequence defined below. If \( A' \) is obtained from \( A \) by a conditional peak elimination of \( A \), we write \( A \rightarrow A' \).

There are three peak elimination rules corresponding to the relative position of the reductions making the peak.

\((P_{\perp})\) If two reductions making the peak occur at parallel positions, then the replacement sequence \( t_{i-1} \rightarrow_R t_i \rightarrow_R t_{i+1} \) is obtained by exchanging the order of reductions making the peak.

\((P_<)\) Suppose that two reductions making the peak are nesting; e.g., \( s \rightarrow_R \tilde{S} \tilde{s}' \) is a subproof of \( t_i \rightarrow_R t_{i+1} \), \( t_{i-1} \rightarrow_R t_i \) occurs below the substitution part \( s \) of \( t_i \rightarrow_R t_{i+1} \), and \( t_{i-1} \equiv C[u] \rightarrow_R C[s] \equiv t_i \). The replacement sequence is \( t_{i-1} \rightarrow_R t_{i+1} \rightarrow_R t_i \) which has the same subproofs as \( t_i \rightarrow_R t_{i+1} \) except for the modified subproof \( u \rightarrow_R \tilde{S} \tilde{s}' \). It is similar when \( t_{i-1} \rightarrow_R t_i \) occurs above \( t_i \rightarrow_R t_{i+1} \).

\((P_c)\) Suppose that two reductions making the peak overlap. Since \( \tilde{R} \) is compatible, the reductions making the peak occur at the same position in \( t_i \). Assume the peak is of the form \( C[r\theta] \subset S C[l\theta] \equiv C[l\theta] \subset S C[r\theta] \). The replacement sequence is

\[
t_{i-1} \equiv C[C[p,s_1, \ldots, s_k]] \rightarrow_R C[C[p, s_1', \ldots, s_k']] \equiv t_{i+1},
\]

where \( s_j \rightarrow_R \tilde{s}_j \) are right connecting proofs of the peak.

Example 3.2 Let \( \hat{R} \) be that of example 3.1. Suppose that \( 1 \rightarrow_R \hat{s} \) and that \( t \rightarrow_R \hat{s} \). There is a peak of the form \( d(f(1), t) \rightarrow_R \hat{S} d(1, t) \rightarrow_R 0 \), where the left-oriented reduction is by the third rule and the right-oriented reduction by the first rule. By \( P_c \), it is replaced with \( d(f(1), t) \rightarrow_R 0 \) as shown in figure 2.

Example 3.3 Let \( \hat{R} \) be the following:

\[
\hat{R} = \begin{cases} 
\hat{S} : f(x_1, a) \rightarrow g_1 = x_1 = y_1, \\
\hat{S}' : f(g(x_1', a, a), x_2') \rightarrow g(y_1', y_2, y_2') = x_1' = y_1', x_2' = y_2'.
\end{cases}
\]
Suppose that $t \vdash_{\alpha'} t_1', a \vdash_{\alpha'} s_2'$ and that $g(t,a,a) \vdash_{\alpha'} s_1$. Then there is a peak of the form $s_1 \vdash_{\beta} f(g(t,a,a),a) \vdash_{\delta} g(s_1',s_2',s_2')$. By $P_C$, it is replaced with $s_1 \vdash_{\beta} g(t,a,a) \vdash_{\alpha'} g(s_1',a,a)$ and $a \vdash_{\alpha'} s_2'$ are left connecting proofs. Note that $g(s_1',a,a)$ is a quasi-ground normal form wrt $p_1$ and wrt $p_2$, where $p_i$ are the positions of $a^i$ in $g(t,a^1,a^2)$.

**Definition 3.9** Let $A : t_1 \vdash_{\alpha'} \cdots \vdash_{\alpha'} t_n$ be a proof in $\mathcal{R}$. Suppose that $A \rightarrow A'$, where a peak $t_{i-1} \vdash_{\alpha'} t_i \vdash_{\alpha'} t_{i+1}$ is eliminated. For a reduction $\alpha'$ in $A'$, the ancestor $\alpha$ in $A$ of $\alpha'$ is defined as follows. We also say $\alpha'$ is a descendant of $\alpha$ if $\alpha$ is the ancestor if $\alpha'$.

- If $\alpha'$ is not in the replacement sequence of the eliminated peak, then the ancestor is the same reduction in $A$ as $\alpha'$.
- If $\alpha'$ is in the replacement sequence of the eliminated peak, then we distinguish the following cases according to which peak elimination rule is applied.

$(P_L)$ Notations are the same as those used in the definition of $P_L$. If $\alpha'$ is $t_{i-1} \vdash_{\alpha'} t_i'$, the ancestor is $t_i \vdash_{\alpha'} t_{i+1}$. Since $t_{i-1} \vdash_{\alpha'} t_i'$ and $t_i \vdash_{\alpha'} t_{i+1}$ have the same subproofs, if $\alpha'$ is in a subproof of $t_{i-1} \vdash_{\alpha'} t_i'$, the ancestor is defined from the natural correspondence. It is similar when $\alpha'$ is $t_i' \vdash_{\alpha'} t_{i+1}$ or is in a subproof of it.

$(P_C)$ Notations are the same as those used in the definition of $P_C$. Here we consider the case $t_{i-1} \vdash_{\alpha'} t_i$ occurs below a substitution part $s$ of $t_i \vdash_{\alpha'} t_{i+1}$. The other cases are treated similarly. If $\alpha'$ is $t_{i-1} \vdash_{\alpha'} t_{i+1}$, then the ancestor is $t_i \vdash_{\alpha'} t_{i+1}$. Since $t_{i-1} \vdash_{\alpha'} t_{i+1}$ and $t_i \vdash_{\alpha'} t_{i+1}$ have the same subproofs except for $u \vdash_{\alpha'} s$ -part, if $\alpha'$ is in a subproof of $t_{i-1} \vdash_{\alpha'} t_i'$ and not in $u \vdash_{\alpha'} s$, the ancestor is defined from the natural correspondence. If $\alpha$ is $u \vdash_{\alpha'} s$, the ancestor is $t_{i-1} \vdash_{\alpha'} t_i$. Since $u \vdash_{\alpha'} s$ and $t_{i-1} \vdash_{\alpha'} t_i$ have the same subproofs, the ancestor is defined naturally if $\alpha'$ is in a subproof of $u \vdash_{\alpha'} s$. The dash lines in figure 2 illustrate the ancestor-descendant relation.

$(P_C)$ Notations are the same as those used in the definition of $P_C$. In this case, the replacement sequence is an embedding of left connecting proofs. Moreover, each left connecting proof is a collection of right
connecting proofs, and each right connecting proof is a collection of subproofs of the reductions making the peak as described in definition 3.7 and 3.5. Therefore, for all \( \alpha' \) in the replacement sequence, there is a segment of the sequence such that \( \alpha' \) is in the segment and the segment corresponds to a subproof of the reductions making the peak. The ancestor is defined from the natural correspondence. The dash lines in figure 3 illustrate the ancestor-descendant relation.

Moreover, for a peak elimination process \( A_1 \mapsto \cdots \mapsto A_n \) and reductions \( \alpha_i \) in \( A_i \), we say \( \alpha_1 \) is the ancestor of \( \alpha_n \) and \( \alpha_n \) is a descendant of \( \alpha_1 \) if \( \alpha_{i+1} \) is a descendant of \( \alpha_i \) for each \( 1 \leq i < n \).

Note that, when \( P_C \) is applied, the top-level reductions making the eliminated peak have no descendant.

**Definition 3.10** Let \( A : t_1 \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} t_n \) be a proof in \( \hat{R} \). Suppose that \( A \mapsto A' \), where a peak made with the reductions \( \gamma : t_{i-1} \overset{R}{\mapsto} t_i \) and \( t_i \overset{R}{\mapsto} t_{i+1} \) is eliminated with \( P_C \). Assume that a reduction \( \alpha \) is in a subproof of \( \gamma_1 (\gamma_2, \text{resp}) \). Then, \( \alpha \) said to be duplicated by \( \gamma_1 (\gamma_2) \).

**Lemma 3.4** Let \( \hat{R} \) be compatible. If a peak elimination process \( A_1 \mapsto A_2 \mapsto \cdots \) terminates for every proof \( A_1 \) in \( \hat{R} \), then \( \hat{R} \) is CR.

### 4 Independence of reductions

#### 4.1 Flattening and independence

In this section, the notion of independence is introduced. Independence is first defined for reductions in a proof in \( \hat{R}_1 \). It is then lifted up to any proof in \( \hat{R} \) by flattening.

**Lemma 4.1** For each non-\( \hat{R}_1 \) reduction \( t \overset{R}{\mapsto} t' \), there is a proof \( t \equiv C[s_1, \ldots, s_m] \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} C[s', \ldots, s_m] \overset{R}{\mapsto} t' \)

satisfying

1. \( s_i \overset{R}{\mapsto} s'_i \) are the subproofs of \( t \overset{R}{\mapsto} t' \), and
2. in the reduction \( C[s'_1, \ldots, s'_m] \overset{R}{\mapsto} t'_i \), the same rule is used at the same position as in \( t \overset{R}{\mapsto} t' \).

**Proof** Let \( l \rightarrow r \leftarrow x_1 = y_1, \ldots, x_m = y_m \) be the rewrite rule for the reduction \( t \overset{R}{\mapsto} t' \), \( t \equiv C'[l] \) and \( t' \equiv C'[r] \). Let \( C''[\cdot] \) be a context such that \( C''[x_1, \ldots, x_m] \equiv l \). The result follows by setting \( C[\cdot] = C''[C''[\cdot]] \).

**Definition 4.1** For a non-\( \hat{R}_1 \) reduction \( t \overset{R}{\mapsto} t' \), the proof \( t \equiv C[s_1, \ldots, s_m] \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} C[s'_1, \ldots, s'_m] \overset{R}{\mapsto} t' \)
described in lemma 4.1 is called the flattening of \( t \overset{R}{\mapsto} t' \). The flattening of a proof \( A : t_1 \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} t_n \) at the \( i \)-th non-\( \hat{R}_1 \) reduction is obtained by replacing \( t_i \overset{R}{\mapsto} t_{i+1} \) with its flattening.

**Lemma 4.2** When a flattening operation is regarded as a reduction on the set of proofs, there exists a unique normal form for each proof \( P \). The normal form is called the flat proof of \( A \) and is denoted by \( A^b \).

**Proof** Since a flattening operation is WCR and SN, it is CR.

Note that \( A^b \) contains only \( \hat{R}_1 \) reductions.

**Definition 4.2** Let \( A : t_1 \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} t_n \) be a proof in \( \hat{R} \) and \( A' \) be a flattening of \( A \). Suppose \( A' \) is obtained by replacing \( t_i \overset{R}{\mapsto} t_{i+1} \) with \( t_i \equiv C[s_1, \ldots, s_m] \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} C[s'_1, \ldots, s'_m] \overset{R}{\mapsto} t_{i+1} \). The mapping flat is a bijection from reductions in \( A \) to ones in its flattening as follows.

1. If \( \alpha \) is the top-level reduction \( t_i \overset{R}{\mapsto} t_{i+1} \), then \( \text{flat}(\alpha) \) is \( C[s'_1, \ldots, s'_m] \overset{R}{\mapsto} t_{i+1} \).
2. If \( \alpha \) is in the \( i \)-th subproof \( s_i \overset{R}{\mapsto} s'_i \), \( \text{flat}(\alpha) \) is the corresponding reduction in \( C[s'_1, \ldots, s'_m] \overset{R}{\mapsto} \cdots \overset{R}{\mapsto} C[s'_i, \ldots, \cdots] \).
3. Otherwise, \( \text{flat}(\alpha) \) is the same reduction in \( A' \) as \( \alpha \).

When \( A' \) is obtained by replacing \( t_i \overset{R}{\mapsto} t_{i+1} \), flat is defined similarly. For a reduction \( \alpha \) in \( A \), \( \alpha^b \) in \( A^b \) is obtained by repeated applications of \( \text{flat} \).
Example 4.1 Let $\hat{R}$ be that in example 3.1. Let $A$ be a one-step proof of the form $A : d(d(1, f(1)), 1) \xrightarrow{\hat{R}} 0$ with subproofs $d(1, f(1)) \xrightarrow{\hat{R}_1} f(1)$ and $1 \xrightarrow{\hat{R}_1} f(1)$. Applying a flattening operation to $A$, we obtain $A^* : d(d(1, f(1)), 1) \xrightarrow{\hat{R}_1} d(f(1), 1) \xrightarrow{\hat{R}_1} d(f(1), f(1)) \xrightarrow{\hat{R}_1} 0$ as in figure 4. The dash-arrows illustrate the bijection flat.

Lemma 4.3 Let $A : t \xrightarrow{\hat{R}} t'$ be a one-step proof where the position of the reduction is $p$, and let $A^* : t \equiv t_1 \xrightarrow{\hat{R}_1} \cdots \xrightarrow{\hat{R}_1} t_n \xrightarrow{\hat{R}_1} t'$. For all $i \leq n$, there is a reduction $\alpha : t_i \xrightarrow{\hat{R}^*} t'$ satisfying $p(\alpha) = p$.

Definition 4.3 Let $A_1 : t_1 \xrightarrow{\hat{R}_1} \cdots \xrightarrow{\hat{R}_1} t_n$ be a proof in $\hat{R}_1$, and let $\alpha_i : t_i \xrightarrow{\hat{R}_1} t_{i+1}$. Suppose that $i \leq j$. We say $t_k$ is between

\[
\begin{align*}
t_i \text{ and } t_j \text{ (or } t_j \text{ and } t_i) & \text{ when } i \leq k \leq j, \\
\alpha_i \text{ and } t_j \text{ (or } t_j \text{ and } \alpha_i) & \text{ when } i + 1 \leq k \leq j, \\
\alpha_i \text{ and } t_j \text{ (or } t_j \text{ and } \alpha_i) & \text{ when } i \leq k \leq j, \\
\alpha_i \text{ and } \alpha_j \text{ (or } \alpha_j \text{ and } \alpha_i) & \text{ when } i + 1 \leq k \leq j.
\end{align*}
\]

A reduction $\alpha_k : t_k \xrightarrow{\hat{R}_1} t_{k+1}$ (or $\alpha_k : t_k \xrightarrow{\hat{R}_1} t_{k+1}$) is between terms (or a reduction and a term, or reductions) if both $t_k$ and $t_{k+1}$ are between terms (or a reduction and a term, or reductions).

Definition 4.4 Let $A_1$ be a proof in $\hat{R}_1$. Relations $\perp_1, \perp_2, \ll_2$ and $\triangleleft_2$ on reductions $\alpha, \beta$ in $A_1$ are defined as follows:

- $\alpha \perp_1 \beta$ if $\alpha \perp_1 \beta$ or $\alpha \perp_2 \beta$.
- $\alpha \perp_2 \beta$ if
  
  1. $p(\alpha) \perp p(\beta)$, and
  2. $p(\gamma) \not\leq \wedge(p(\alpha), p(\beta))$ for all reductions $\gamma$ between $\alpha$ and $\beta$.
- $\alpha \ll_2 \beta$ if either $\alpha \ll_2 \beta$ or $\beta \ll_2 \alpha$.
- $\alpha \ll_2 \beta$ if there are a term $t$ between $\alpha$ and $\beta$, positions $p \in P(t)$ and $q \in P(t/p)$ s.t.
  
  1. $t/p$ is a quasi-ground normal form of $\hat{R}$ wrt $q$,
  2. $p(\alpha) \geq p \cdot q$,
  3. $p(\beta) \geq p$,
  4. $p(\gamma_1) \not\leq p \cdot q$ for all reductions $\gamma_1$ between $\alpha$ and $t$, and
  5. $p(\gamma_2) \not\leq p$ for all reductions $\gamma_2$ between $t$ and $\beta$.

For a proof $A$ in $\hat{R}$ and reductions $\alpha, \beta$ in $A$, we also write $\alpha \ll_1 \beta$, $\alpha \perp_1 \beta$ and $\alpha \ll_2 \beta$ if $\alpha \ll_1 \beta$, $\alpha \perp_1 \beta$, $\alpha \ll_2 \beta$ and $\alpha \ll_2 \beta$, respectively.

Reductions $\alpha$ and $\beta$ are independent if $\alpha \perp_1 \beta$. If $\alpha \ll_2 \beta$, the term $t$ in the definition of $\ll_2$ is called a split of $\alpha$ and $\beta$. Moreover, the term in the definition of $\ll_2$ is called the body of the split $t$.

Example 4.2 Let $\hat{R}$ be that in example 3.3. Consider the following proof in $\hat{R}_1$:

\[
f(g(f(a, a), a, a), a)^{a_1} g(f(a, a), a, a)^{a_2} g(a, f(a, a), a)^{a_3} g(a, f(a, a), a)^{a_4}.
\]
where underlines indicate the redexes contracted. Then, $\alpha_2 \not\vdash_1 \alpha_3, \alpha_3 \not\vdash_1 \alpha_4,$ and $\alpha_4 \not\vdash_1 \alpha_2.$ Furthermore, $\alpha_3 \not\perp_{2} \alpha_1$ and $\alpha_4 \not\perp_{2} \alpha_1$ since $g(f(a,a),a,a)$ is a quasi-ground normal form wrt $p_1$ and $p_2,$ where $p_i$ are the positions of $a^i$ in $g(f(a,a),a^1,a^2)$.

Suppose that $A \mapsto A'.$ For any reduction $\alpha'$ in $A',$ there is a corresponding reduction $\alpha$ in $A.$ If $\alpha$ is in the replacement sequence, then we can find $\alpha$ in the peak that $\alpha'$ originates from (indicated by dash-arrows in figure 2 and figure 3); otherwise, $\alpha$ is the same reduction as $\alpha'.$ In this case, $\alpha'$ is called a descendant of $\alpha.$ Moreover, for a peak elimination process $A_1 \mapsto \cdots \mapsto A_n$ and reductions $\alpha_i$ in $A_i,$ we say $\alpha_n$ is a descendant of $\alpha_1$ if $\alpha_{i+1}$ is a descendant of $\alpha_i$ for each $1 \leq i < n.$

4.2 Properties of independence

Theorem 4.1  Let $\hat{R}$ be compatible and let $A, A'$ be proofs in $\hat{R}.$ Suppose that $A \mapsto A'$ and that reductions $\alpha', \beta'$ in $A'$ are descendants of $\alpha, \beta$ in $A,$ respectively. Then, $\alpha \not\vdash \beta$ implies $\alpha' \not\vdash \beta'.$

Proof From lemmas A.6, A.7, and A.9 in Appendix.

Lemma 4.4  Let $A$ be a proof in $\hat{R}$ with reductions $\alpha$ and $\beta.$ Suppose that $\alpha$ is in a subproof of $\beta.$ Then, $\alpha \not\not\vdash \beta.$

Proof  Since $p(\alpha^s) > p(\beta^s),$ $\alpha \not\vdash_1 \beta.$ Suppose that there is a split $t$ of $\alpha^s$ and $\beta^s,$ where the body $t/p$ is a quasi-ground normal form wrt $q.$ Then, $p(\beta^s) \not\vdash \alpha \cdot q$ from lemma 4.3 and A.1. Also, $p(\alpha^s) \not\vdash \beta \cdot q$ since $p(\alpha^s) > p(\beta^s).$ This contradicts the definition of $\not\vdash_2.$

Lemma 4.5  Let $A$ be a proof in $\hat{R}$ with reductions $\alpha$ and $\beta.$ If $\alpha \not\vdash \beta$ and $\beta'$ is in a subproof of $\beta,$ then $\alpha \not\vdash \beta'.$

Proof  Suppose that $\beta'$ is a split of $\alpha^s$ and $\beta^s.$ It is obvious that $\alpha \not\vdash_1 \beta \Rightarrow \alpha \not\vdash_1 \beta'.$ If $\alpha \not\vdash_2 \beta,$ then any split $t$ of $\alpha^s$ and $\beta^s$ is also a split of $\alpha^s$ and $\beta^s.$

Suppose that $\beta^s$ is between $\alpha^s$ and $\beta^s.$ It is obvious that $\alpha \not\vdash_1 \beta \Rightarrow \alpha \not\vdash_1 \beta'.$ Suppose that $\alpha \not\vdash_2 \beta$ and that $t$ is a split of $\alpha$ and $\beta,$ where the body $t/p$ is a quasi-ground normal form wrt $q.$ The result is obvious when $t$ is between $\alpha^s$ and $\beta^s.$ If $t$ is between $\beta^s$ and $\beta,$ $t/p(\beta^s)$ is a redex from lemma 4.3. Thus, $p(\beta^s) \not\vdash \alpha \cdot q$ from lemma A.1 in Appendix. Hence, $p(\alpha^s) > p \cdot q$ so $\alpha \not\vdash \beta'.$ Therefore, $\alpha \not\vdash \beta'.$

Lemma 4.6  Let $A, A'$ be proofs such that $A \mapsto A'.$ Suppose that $\alpha'_1, \cdots, \alpha'_m$ in $A'$ are descendants of $\alpha$ in $A.$ Then $m_1 \neq m_2 \Rightarrow \alpha'_m \not\vdash \alpha'_m.$

Proof  Notations are the same as those used in definition 3.5 or lemma 3.3. It is clear that $\alpha$ has multiple descendants only when $\alpha$ is duplicated, i.e., $P_{\alpha}$ is applied to a peak $C[\tilde{r}\theta] \Sigma^* \tilde{C}[\tilde{t}\theta] \equiv C[\tilde{r}\theta] \Sigma^* \tilde{C}[\tilde{t}\theta]$ in $A$ and when $\alpha$ is in a subproof of either making the peak.

The replacement sequence for the peak is a collection of right connecting proofs, $B_q.$ If $\alpha'_1$ is in $B_q$-part and $\alpha'_m$ is in $B_q$-part such that $q \neq q',$ then $\alpha'_m \not\vdash \alpha'_m.$ Suppose that $q = q'$ and assume $r/q \equiv y_i \in V.$ $B_q$ is as follows:

$$y_i \theta \Sigma^* \hat{R} C'_i[g_1, \cdots, g_m]T_{\tilde{r}\theta} \Sigma^* \hat{R} C'_i[y'_j, \cdots, y'_{m'}]T_{\tilde{r}\theta},$$

where $A_p : y_i \theta \Sigma^* \hat{R} C'_i[g_1, \cdots, g_m]$ and $A_{p_k} : g_k \Sigma^* \hat{R} y'_{j_k} \theta$ are left connecting proofs of the peak. Note that $p_k$ is the position of $x'_{j_k}$ in $\tilde{l}.$ Since $\tilde{l}/p_k = g_k$ are ground terms, $A_{p_k}$ themselves are also subproofs of $C[\tilde{r}\theta] \Sigma^* \tilde{C}[\tilde{t}\theta]$ for $1 \leq k \leq m'.$

The other left connecting proof $A_{p}$ is rewritten as follows:

$$y_i \theta \Sigma^* \hat{R} x_i \theta \equiv C'_p[x'_j \theta, \cdots, x'_{j+k} \theta] \Sigma^* \hat{R} C'_p[y'_{j'}, \cdots, y'_{j+k'} \theta].$$

Then, $A_{p_k}$ and $x'_{j+k'} \theta \Sigma^* \hat{R} y'_{j+k'} \theta$ can not originate from the same subproof for any $k, k'.$ For $p_{x'_{j+k'}} \geq p$ but $p_k \not\vdash \perp p,$ where $p_{x'_{j+k'}}$ is the position of $x'_{j+k'}$ in $\tilde{l}.$

It is clear that subproofs $x'_{j+k'} \theta \Sigma^* \hat{R} y'_{j+k'} \theta$ originated from different subproofs. Hence, only the following case is possible: $\alpha'_m$ is in $A_{p_k}$-part and $\alpha'_m$ is in $A_{p_{k'}}$-part such that $k \neq k'.$ Thus, $\alpha'_m \not\vdash \alpha'_m.$ The proof is similar for $r/q \not\in V.$
5 Church-Rosser property of $\hat{R}$

Let $R$ be a compatible TRS, and let $\hat{R}$ be the conditional linearization. Assume that $A^1 : t_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_n} t_n$ is an arbitrary proof in $\hat{R}$ and that $A^1 \Rightarrow A^2 \Rightarrow \cdots$ is an arbitrary peak elimination process. The following section will show that the process $A^1 \Rightarrow A^2 \Rightarrow \cdots$ terminates. This implies that CR holds for $\hat{R}$ by lemma 3.4.

**Definition 5.1** The initial labeling on each reduction in $A^i$ for $i = 1, 2, \cdots$ is defined as follows:

1. The set of initial labels is $\{[\alpha] | \alpha$ is in $A^1\}$.
2. Each reduction $\alpha$ in $A^1$ is labeled $[\alpha]$.
3. For each reduction $\beta$ in $A^i$ for $i \geq 2$, $\beta$ is labeled $[\alpha]$ if $\beta$ is a descendant of $\alpha$ in $A^1$.

**Definition 5.2** Let $\alpha$ be a reduction in $A^1$. The descendant tree $T_{[\alpha]}$ associated with $\alpha$ is an edge-labeled tree defined as follows:

1. The root vertex is the reduction $\alpha$ in $A^1$.
2. Let $\alpha'$ in $A^i$ be a vertex of $T_{[\alpha]}$. Suppose that there are $k > i$, satisfying the following conditions:
   
   (a) In $A^j \Rightarrow A^{j+1}$, the descendant $\beta_j$ of $\alpha'$ in $A_j$ is not duplicated for $j = i + 1, \ldots, k - 1$.
   
   (b) In $A^k \Rightarrow A^{k+1}$, the descendant $\beta_k$ of $\alpha'$ in $A_k$ is duplicated.

   Suppose that $\beta_k$ is duplicated by $\gamma_k$. Then all the descendants $\beta'_1, \ldots, \beta'_n$ in $A^{k+1}$ of $\beta_k$ are the child vertices of $\alpha'$. The label of the edges $(\alpha', \beta'_j)$ is the initial label of $\gamma_k$, e.g. $[\gamma]$ (figure 5).

The set of all descendant trees associated with reductions in $A_1$ in the peak elimination process is called the descendant forest of the peak elimination process.

Note that all vertices in $T_{[\alpha]}$ are descendants of $\alpha$ in $A_1$.

We classify $P_C$ into the following:

$(P_C^1)$ The replacement sequence is empty.

$(P_C^2)$ The replacement sequence is not empty.

**Lemma 5.1** Suppose that $(P_C^2)$ is applied in $A^i \Rightarrow A^{i+1}$. Then, there are a reduction $\beta$ in $A^{i+1}$ and a descendant tree $T_{[\alpha]}$ such that $\beta$ is a vertex of $T_{[\alpha]}$.

A path of $T_{[\alpha]}$ is a sequence of edges starting from the root. A label path is the sequence of labels of edges in a path. The set of all label paths of $T_{[\alpha]}$ is denoted by $Lpath_{T_{[\alpha]}}$.

**Lemma 5.2** Let $[\gamma_1], [\gamma_2], \cdots \in Lpath_{T_{[\alpha]}}$. Then, $[\gamma_i] \neq [\gamma_j]$ for all $i \neq j$.

**Proof** Suppose that $[\gamma_i] = [\gamma_j] = [\beta]$ for some $i \neq j$. There exist descendants $\alpha_1, \alpha_2$ of $\alpha$ and descendants $\beta_1, \beta_2, \beta_3$ of $\beta$ as shown in figure 6, where $\alpha_2 (\beta_2)$ is a descendant of $\alpha_1 (\beta_3)$.
Since $\beta_1$ and $\beta_2$ are descendants of the same reduction, $\beta_1 \perp \beta_2$ from lemma 4.6 and theorem 4.1. Since $\alpha_1$ is in a subproof of $\beta_1, \alpha_1 \perp \beta_2$ from lemma 4.5. Hence, $\alpha_2 \perp \beta_2$ from theorem 4.1. However, $\alpha_2 \perp \beta_2$ by lemma 4.4. This leads to a contradiction.

**Lemma 5.3** For each initial label $[\alpha]$, the descendant tree $T_{[\alpha]}$ is finite. Therefore, the descendant forest of the peak elimination process is finite.

**Proof** From lemma 5.2, each path of $T_{[\alpha]}$ has finite length (bounded by the number of reductions in $A$). Since $T_{[\alpha]}$ is obviously finitely branching, König's lemma shows that $T_{[\alpha]}$ is finite.

**Lemma 5.4** In the peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$, only finitely many peak eliminations occur with $P_C^2$.

**Proof** From lemma 5.3 and 5.1.

**Definition 5.3** Let $B : t_0 \triangleright_R \cdots \triangleright_R t_n$ be a proof and let $\gamma_i : t_i \triangleright_R t_{i+1}$. A reduction $\gamma_i$ is right-oriented (left-oriented) if $\gamma_i : t_i \triangleright_R t_{i+1}$ ($\gamma_i : t_i \triangleright_R t_{i+1}$). The height of $\gamma_i$ is defined as follows:

$$\text{height}(\gamma_i) = \#\{\gamma_j \mid \gamma_j \text{ is left-oriented and } j < i\}.$$ 

The mass of $B$ is defined as

$$\text{mass}(B) = \sum_{\text{right-oriented } \gamma_i} \text{height}(\gamma_i).$$

That is, the mass is the number of tiles as shown in figure 7.

**Lemma 5.5** Let $B, B'$ be proofs such that $B \mapsto B'$ with $P_\perp, P_\lt$ or $P_C^1$. Then, $\text{mass}(B) > \text{mass}(B')$.

**Corollary 5.1** Let $B_1 \xrightarrow{P_\perp} B_2 \xrightarrow{P_\lt} B_3 \xrightarrow{P_C^1} \cdots$ be a peak elimination process starting from $B_1$. If each $P_i$ is any of $P_\perp, P_\lt$ or $P_C^1$, then the length of the process is finite.

**Theorem 5.1** Any peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$ terminates.

**Proof** From corollary 5.4 and corollary 5.1.
Corollary 5.2 Let \( R \) be a compatible TRS, and let \( \hat{R} \) be the conditional linearization of \( R \). Then, \( \hat{R} \) is \( \text{CR} \). Therefore, \( \hat{R} \) is \( \text{UN} \).

6 Conclusion and Future Work

We have presented a complete proof of Chew’s theorem which states that compatible term rewriting systems enjoy the unique normal form property. We exploited a technique introduced in [TO94] and partly extended to apply it to duplicating systems by introducing the notion of independence.

There are many interesting non-linear term rewriting systems that have (or believed to have) the unique normal form property, for example, the system of combinatory logic with surjective pairing [KdV89], non-\( \omega \)-overlapping term rewriting systems [Oga92], etc. Despite of the importance, Chew’s theorem is not powerful enough to infer the unique normal form properties of these systems. Therefore, we would like to relax the condition of compatibility. We also interested in extending the result to higher order rewriting systems.

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References


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A Appendix

In the following, it is assumed that \( \hat{R} \) is a compatible left-right separated CTRS.

A.1 Innocent swap

Lemma A.1 Let \( t \) be a term such that \( t/p \) is a quasi-ground normal form wrt \( q \). If \( p' \not\subseteq p \) and \( t/p' \) is a redex, then \( p' \perp \bot \).
Lemma A.3  Let $A : t_1 \mathcal{R} \cdots \mathcal{R} t_n$ be a proof with a reduction $\gamma : t_i \mathcal{R} t_{i+1}$, and let $t_i \equiv t^1_i \mathcal{R} t^1_{i+1}$ be the flat proof of $t_i \mathcal{R} t_{i+1}$. Suppose that there exist reductions $\alpha, \beta$ in $A$ satisfying
1. $\alpha^\perp \mathcal{R}_i \beta^\perp$, 2. both $t_i$ and $t_{i+1}$ in $A^\beta$ are between $\alpha^\perp$ and $\beta^\perp$, and 3. there exists $j$ s.t. $t^1_i$ is a split of $\alpha^\perp$ and $\beta^\perp$. Then $t_{i+1}$ is also a split of $\alpha^\perp$ and $\beta^\perp$.

Proof  Let $t^1_i/p$ be the body of $t^1_i$. Since $\gamma^\perp$ is between $\alpha^\perp$ and $\beta^\perp$, $p(\gamma^\perp) \not\equiv p$. From lemmas 4.3, A.1, and A.2, the result follows.

Lemma A.4  Let $A$ be a proof in $\mathcal{R}$ and let $\alpha, \beta$ be reductions in $A$. Suppose that $\alpha \perp \mathcal{R}_i \beta$ and $t$ is a split of $\alpha^\perp$ and $\beta^\perp$, where the body $t/p$ is a quasi-ground normal form wrt $q$. Assume there is a position $p' \geq p$ satisfying
1. $t/p'$ is a redex, 2. for each reduction $\gamma$ between $t$ and $\beta^\perp$, $p(\gamma) \not\equiv p'$, and 3. $p(\beta^\perp) \geq p'$. Then, $\alpha \perp \mathcal{R}_i \beta$.

Proof  Since $t/p'$ is a redex, $p' \perp p \cdot q$ from lemma A.1, so $p(\beta) \perp p \cdot q$. Thus, $\alpha \not\perp \mathcal{R}_i \beta$. Hence, $p(\alpha^\perp) \geq p \cdot q$, and $p(\gamma^\perp) \not\equiv p \cdot q$ for each reduction $\gamma^\perp$ between $\alpha^\perp$ and $t$. Moreover, $p(\beta^\perp) \geq p'$ and $p(\gamma^\perp) \not\equiv p'$ for each reduction $\gamma$ between $t$ and $\beta^\perp$. Therefore, $\alpha \perp \mathcal{R}_i \beta$.

Definition A.1  Let $A : t_1 \mathcal{R} \cdots \mathcal{R} t_n$ be a proof, and let $\gamma_1 : t_i-1 \mathcal{R} t_i$ and $\gamma_2 : t_i \mathcal{R} t_{i+1}$ be reductions such that $p(\gamma_1) \perp p(\gamma_2)$. Suppose that either $\gamma_1 : t_i \mathcal{R} t_{i+1}$ or $\gamma_2 : t_i \mathcal{R} t_{i+1}$ holds.

When $\gamma_2 : t_i \mathcal{R} t_{i+1}$, the innocent swap of $\gamma_1$ and $\gamma_2$ is a transformation that changes the order of $\gamma_1$ and $\gamma_2$ in $A$, i.e., $A$ is transformed to

$$A' : t_1 \mathcal{R} \cdots \mathcal{R} t_{i-1} \mathcal{R} t_i \mathcal{R} t_{i+1} \mathcal{R} \cdots \mathcal{R} t_n,$$

where $\gamma'_1 : t_i \mathcal{R} t_{i+1}$ is a reduction with the same rules, position, and subproofs as $\gamma_1$ ($\gamma_2$). In the case $\gamma_1 : t_i \mathcal{R} t_{i+1}$, an innocent swap is similarly defined. For a reduction $\alpha$ in $A$, the descendant $\alpha'$ in $A'$ is defined in the same way as that of peak eliminations by $P_1$.

Lemma A.5  Let $\alpha, \beta$ be reductions in a proof $A$. Suppose that $A'$ is obtained by applying an innocent swap to $A$ and that $\alpha'$ and $\beta'$ are the descendants of $\alpha$ and $\beta$, respectively. Then $\alpha \perp \mathcal{R}_i \beta \Rightarrow \alpha' \perp \mathcal{R}_i \beta'$.

Proof  Let $A : t_1 \mathcal{R} \cdots \mathcal{R} t_n$. Assume that the innocent swap is applied to $\gamma_1 : t_i-1 \mathcal{R} t_i$, $\gamma_2 : t_i \mathcal{R} t_{i+1}$.

Let $p_1 = p(\gamma_1)$ and $p_2 = p(\gamma_2)$. Let $C[] \equiv t_i-1 \equiv \square, p_2 \equiv \square$, $t_{i-1} \equiv C[s_1, s_2]$, $t_i \equiv C[s_1', s_2]$, and $t_{i+1} \equiv C[s_1, s_2']$. We divide $A$ and $A'$ into the following proofs:

- $A_1 : t_1 \mathcal{R} \cdots \mathcal{R} t_{i-1}$, $A_2 : t_{i+1} \mathcal{R} \cdots \mathcal{R} t_n$,
- $B_1 : (t_{i-1} \equiv C[s_1, s_2] \mathcal{R} C[s_1', s_2] \equiv t_i)$, $B_2 : (t_i \equiv C[s_1, s_2] \mathcal{R} C[s_1', s_2] \equiv t_{i+1})$,
- $B'_1 : (t_{i-1} \equiv C[s_1, s_2] \mathcal{R} C[s_1', s_2] \equiv t_i)$, $B'_2 : (t_i \equiv C[s_1, s_2] \mathcal{R} C[s_1', s_2] \equiv t_{i+1})$.

where $A (A')$ is the concatenation of $A_1, B_1$ and $B_2, A_2 (A, B'_2, B'_1$, and $A_2$).

Since an innocent swap preserves the positions of reductions, it is obvious that $\alpha \perp \mathcal{R}_i \beta \Rightarrow \alpha' \perp \mathcal{R}_i \beta'$.

We will now prove that $\alpha \perp \mathcal{R}_i \beta \Rightarrow \alpha' \perp \mathcal{R}_i \beta'$.

Without loss of generality, it can be assumed that $\alpha^\perp$ is on the "left-hand side" of $\beta^\perp$ in $A'$. Let $t$ be a split of $\alpha^\perp$ and $\beta^\perp$ in $A^\beta$, where the body $t/p'$ is a quasi-ground normal form wrt $q$. Then the following cases exist:

1. Both $\alpha$ and $\beta$ are in any of $A_1, A_2, B_1$ or $B_2$.
2. $\alpha$ is in $A_1$, $\beta$ is in $B_1$.
3. $\alpha$ is in $A_1$, $\beta$ is in $B_2$.
4. $\alpha$ is in $A_2$, $\beta$ is in $A_2$.
5. $\alpha$ is in $B_1$, $\beta$ is in $B_2$.
6. $\alpha$ is in $B_1$, $\beta$ is in $A_2$.
7. $\alpha$ is in $B_2$, $\beta$ is in $A_2$. 
Case 1. The result is obvious.

Case 2. If the split $t$ is in $A_1^j$, then it is obvious. If the split $t$ is in $B_1^j$, then $t'' \equiv t[p_2 \leftarrow s'_2]$ in $B_1^b$ is a split of $\alpha^b$ and $\beta^b$ from lemma A.2. Thus, $\alpha' \parallel_2 \beta'$.

Case 3. If the split $t$ is in $A_1^j$, then it is obvious. If the split $t$ is in either $B_1^j$ or $B_2^j$, $t/p_2$ is a redex by lemma 4.3. For all reductions $\gamma$ between $t$ and $\beta^b$, $\gamma(t) \neq p_2$ since $\gamma$ is in either $B_1^j$ or $B_2^j$. Suppose that $p_2 \geq p$. Then, $\alpha \parallel_1 \beta$ from lemma A.4, so $\alpha' \parallel_1 \beta'$. Next, suppose that $p_2 \leq p$. Since $p_2 \leq p(\beta^b)$ and $p \leq p(\beta^b)$, $p_2 < p$. Hence, $t[p_2 \leftarrow s'_1]$ in $B_2^b$ is a split of $\alpha^b$ and $\beta^b$. Therefore, $\alpha' \parallel_2 \beta'$.

Case 4. If the split $t$ is in $A_1^j$ or $A_2^j$, then it is obvious. If the split $t$ is in $B_3^j$, then $t[p_2 \leftarrow s'_2]$ in $B_3^b$ is a split of $\alpha^b$ and $\beta^b$ from lemma A.2. Thus, $t_{i+1}$ is also a split of $\alpha^b$ and $\beta^b$. Therefore, $\alpha' \parallel_2 \beta'$.

A.2 Proof of theorem 4.1

Lemma A.6 Let $A, A'$ be proofs in $\hat{R}$ such that $A \vdash A'$. Let reductions $\alpha', \beta'$ in $A'$ be descendants of $\alpha, \beta$ in $A$. Then $\alpha \parallel \beta \Rightarrow \alpha' \parallel \beta'$.

Proof From lemma A.5.

Lemma A.7 Let $A, A'$ be proofs in $\hat{R}$ such that $A \vdash A'$. Let reductions $\alpha', \beta'$ in $A'$ be descendants of $\alpha, \beta$ in $A$. Then $\alpha \parallel \beta \Rightarrow \alpha' \parallel \beta'$.

Proof Let $t_{i-1} \overset{\gamma_R}{\rightarrow} t_i \overset{\gamma_R}{\rightarrow} t_{i+1}$ be the peak that $P_\prec$ is applied to in $A : t_1 \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_n$. Let $\gamma_1 : t_i \overset{\gamma_R}{\rightarrow} t_{i-1}$ and $\gamma_2 : t_i \overset{\gamma_R}{\rightarrow} t_{i+1}$ and suppose $p(\gamma_1) < p(\gamma_2)$.

Let $l \leftarrow r \equiv x_1 = y_1, \ldots, x_m = y_m$ be the rule for the reduction $\gamma_2$, where $t_i \equiv C[r\theta]$ and $t_{i+1} \equiv C[r\theta]$. Suppose that $\gamma_1$ occurs below the $j$-th substitution part of $\gamma_2$ and that $\gamma_2 : t_{i-1} \overset{\gamma_R}{\rightarrow} t_{i+1}$ is the replacement sequence for the peak. Then, the flattening of $A$ at $\gamma_2$ is

$$fA : \cdots : t_{i-1} \overset{\gamma_R}{\rightarrow} t_i \overset{\gamma_R}{\rightarrow} t_{i+1} \equiv t_i^{(0)} \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_{i-1}^{(1)} \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_{i+1}$$

where $t_{i-1}^{(k)} \overset{\gamma_R}{\rightarrow} t_{i+1}^{(k+1)}$ corresponds to the subproof $x_k \theta \overset{\gamma_R}{\rightarrow} y_k \theta$ of $\gamma_2$, and the flattening of $A'$ at $\gamma_2'$ is

$$fA' : \cdots : t_{i-1} \overset{\gamma_R}{\rightarrow} t_i \overset{\gamma_R}{\rightarrow} t_{i+1} \equiv t_i'^{(0)} \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_{i-1}'^{(1)} \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_{i+1}'$$

where $t_{i-1}'^{(k)} \overset{\gamma_R}{\rightarrow} t_{i+1}'^{(k+1)}$ corresponds to the peak $t_i' \vdash t_{i+1}'$.

Thus, $fA'$ is obtained from $fA$ by repeated applications of innocent swaps to $flat(\gamma_1)$, $\gamma_1'$ (a descendant of $flat(\gamma_1)$), $\gamma_2'$ (a descendant of $\gamma_1$), $\gamma_2'$ with their right adjacent reductions since $p(\gamma) \perp p(flat(\gamma_1))$ for each reduction $\gamma$ in $t_i \overset{\gamma_R}{\rightarrow} \cdots \overset{\gamma_R}{\rightarrow} t_{i+1}$. From lemma 4.2 and lemma A.5, independence is preserved.

The proof is similar when $p(\gamma_1) > p(\gamma_2)$.

Lemma A.8 Let $t_1 \overset{\gamma_R}{\rightarrow} t_2$ be a critical peak and let $A_p$ be a left connecting proof of the peak. Suppose that $\alpha, \beta$ are reductions in subproofs of either reduction making the peak such that the corresponding reductions, denoted by $\alpha_p, \beta_p$, are in $A_p$. Then $\alpha \parallel \beta \Rightarrow \alpha_p \parallel \beta_p$.

Proof Suppose that $A_p$ is of the form $s' \overset{\gamma_R}{\rightarrow} s \equiv C_p[u_1, \ldots, u_n] \overset{\gamma_R}{\rightarrow} C_p[u_1', \ldots, u_n']$, where $s' \overset{\gamma_R}{\rightarrow} s$ and $u_i \overset{\gamma_R}{\rightarrow} u_i'$ are subproofs of the reductions making the peak. The following cases exist:

1. Both $\alpha_p$ and $\beta_p$ are in either $s' \overset{\gamma_R}{\rightarrow} s$ or $C_p[u_1, \ldots, u_i, \ldots] \overset{\gamma_R}{\rightarrow} C_p[u_1', \ldots, u_i', \ldots]$ for some $i = 1, \ldots, n$.
2. $\alpha_p$ is in $C_p[u_1, \ldots, u_i, \ldots] \overset{\gamma_R}{\rightarrow} C_p[u_1', \ldots, u_i', \ldots]$, $\beta_p$ is in $C_p[u_j, \ldots, u_i, \ldots] \overset{\gamma_R}{\rightarrow} C_p[u_j', \ldots, u_i', \ldots]$, and $i \neq j$.
3. $\alpha_p$ is in $s' \overset{\gamma_R}{\rightarrow} s$ and $\beta_p$ is in $C_p[u_1, \ldots, u_i, \ldots] \overset{\gamma_R}{\rightarrow} C_p[u_1', \ldots, u_i', \ldots]$ (or vice versa).
In case 1, it is obvious. In case 2, \( \alpha_p \perp_1 \beta_p \). Let us consider case 3. The flat proofs of reductions making the peak can be written as follows.

\[
\begin{align*}
t_1 \leftarrow_{\hat{R}_1} & \cdots \leftarrow_{\hat{R}_1} C_1[s'] \leftarrow_{\hat{R}_1} C_1[s] \leftarrow_{\hat{R}_1} t & : \text{the flat proof of } t_1 \leftarrow_{\hat{R}} t \\
t \leftarrow_{\hat{R}_1} & C_2[C_p[u_1', \ldots, u_m'] \equiv_{\hat{R}_1} C_2[C_p[u_1', \ldots, u_m']] \leftarrow_{\hat{R}_1} \cdots \rightarrow_{\hat{R}_1} t_2 & : \text{the flat proof of } t \leftarrow_{\hat{R}} t_2
\end{align*}
\]

where the position of \( \Box \) both in \( C_1[s] \) and in \( C_2 \) is \( p \). Let \( p' \) be the position of reductions making the peak. Then, for all reductions \( \gamma \) in \( C_1[s] \leftarrow_{\hat{R}_1} t \) or \( t \leftarrow_{\hat{R}_1} C_2[C_p[u_1', \ldots, u_m']] \), \( p(\gamma) \perp p' \cdot p \) from the definition of flattening.

Therefore, the result follows. The proof is similar when \( A_p \) is of the form \( C_p'[s_1', \ldots, s_n'] \equiv_{\hat{R}} C_p'[s_1, \ldots, s_n] \equiv u \leftarrow_{\hat{R}} u' \).

**Lemma A.9** Let \( A, A' \) be a proof in \( \hat{R} \) such that \( A \perp A' \). If reductions \( \alpha, \beta \) in \( A \) have descendants \( \alpha', \beta' \) in \( A' \), then \( \alpha \perp \beta \Rightarrow \alpha' \perp \beta' \).

**Proof** Let \( A : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n \) and \( A' : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n \) be the critical peak eliminated in \( A \leftrightarrow A' \). Let \( \gamma_1 : t_1 \leftarrow_{\hat{R}} t_2, \gamma_2 : t_i \leftarrow_{\hat{R}} t_{i+1} \) and \( p' = p(\gamma_1) = p(\gamma_2) \).

Without loss of generality, it can be assumed that \( \alpha' \) is the “left-hand side” of \( \beta' \) in \( A' \). We divide \( A \) into the following proofs: \( A_1 : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n, A_2 : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n, B_1 : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n, \) and \( B_2 : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_n \), where \( A(A') \) is the concatenation of \( A_1, B_1, \) and \( A_2 \) (\( A_1, B_2, A_2 \)). Let \( B : t_1 \leftarrow_{\hat{R}} \cdots \leftarrow_{\hat{R}} t_{i+1} \) be the replacement sequence for the critical peak. Note that for each reduction \( \gamma \) in \( B \), \( p(\gamma) \geq p' \). Then the following cases exist:

1. Both \( \alpha \) and \( \beta \) are either \( A_1 \) or \( A_2 \).
2. \( \alpha \) is in \( A_1 \) and \( \beta \) is in \( A_2 \).
3. \( \alpha \) and \( \beta \) are in either \( B_1 \) or \( B_2 \).
4. \( \alpha \) is in \( A_1 \) and \( \beta \) is in either \( B_1 \) or \( B_2 \) (or, \( \alpha \) is in either \( B_1 \) or \( B_2 \) and \( \beta \) is in \( A_2 \)).

**Case 1.** The result is obvious.

**Case 2.** Since \( p(\gamma) \geq p' \) for each reduction \( \gamma \) in \( B \), it follows that \( \alpha \perp \beta \Rightarrow \alpha' \perp \beta' \). Assume that \( \alpha \perp \beta \) and \( t \) is a split of \( \alpha \) and \( \beta \). If \( t \) is in \( B_1 \), then \( t_{i-1} \) is also a split from lemma A.3. If \( t \) is in \( B_2 \), \( t_{i+1} \) is also a split from lemma A.3. Thus, we can assume that \( t \) is in either \( A_1' \) or \( A_2' \). Since \( p(\gamma) \geq p' \) for each reduction \( \gamma \) in \( B' \), we have \( \alpha' \perp \beta' \).

**Case 3.** Recall that \( B \) is a collection of the right connecting proofs \( B_q \) of the peak. Suppose that \( \alpha', \beta' \) are in \( B_q \)-part, \( B_{q'} \)-part of \( B \), respectively. If \( q \neq q' \), then \( \alpha_1 \perp \beta_1 \). Hence, suppose that \( q = q' \). Recall that \( B_q \) is a collection of left connecting proofs. Suppose that \( B_q \) is as follows:

\[
s \leftarrow_{\hat{R}_1} C[g_1, \ldots, g_m] \equiv_{\hat{R}_1} C[u_1, \ldots, u_m],
\]

where \( A_s : s \leftarrow_{\hat{R}_1} C[g_1, \ldots, g_m] \) and \( A_{u_i} : C[u_1, \ldots, u_i \cdot \cdot \cdot] \equiv_{\hat{R}_1} C[u_1, \ldots, u_i \cdot \cdot \cdot] \) are left connecting proofs. If \( \alpha' \) (or \( \beta' \)) is in \( A_{u_i} \)-part and \( \beta' \) (or \( \alpha' \)) is in \( A_{u_j} \)-part, then \( \alpha' \perp \beta' \) by lemma 3.3. If \( \alpha' \) and \( \beta' \) are in \( A_{u_i} \)-part and \( A_{u_j} \)-part respectively such that \( i \neq j \), then \( \alpha' \perp \beta' \). The remaining case is both \( \alpha' \) and \( \beta' \) are in either \( A_{s} \)-part or \( A_{u_i} \)-part, and the result follows from lemma A.8. The proof is similar when \( B_q \) is of the form \( C[s_1, \ldots, s_n] \equiv_{\hat{R}_1} C[u_1, \ldots, u_m] \).

**Case 4.** From symmetry, we can assume that \( \alpha \) is in \( A_1 \) and that \( \beta \) is in either \( B_1 \) or \( B_2 \). If \( \beta \) is either \( \gamma_1 \) or \( \gamma_2 \), \( \beta \) does not exist. Thus, \( \beta \) is in a subproof of \( \gamma_1 \) or \( \gamma_2 \). Suppose that \( \alpha \perp_1 \beta \). Then \( p(\gamma) \not\leq \wedge(p(\alpha^\triangleright), p') \) for each reduction \( \gamma \) between \( \alpha \) and \( t_{i-1} \) since \( \gamma_i \) is between \( \alpha \) and \( \beta \), and \( p' \leq p(\beta) \). Since \( p(\gamma) \geq p' \) for each reduction \( \gamma \) in \( B' \), we have \( \alpha_1 \perp_1 \beta' \).

Assume that \( \alpha \perp_1 \beta \) and that \( t \) is a split of \( \alpha \) and \( \beta \), where the body \( t/p \) is a quasi-ground normal form wrt \( q \). If \( t \) is in \( A_1 \), then \( \alpha_1 \perp_1 \beta \) since \( p(\gamma) \geq p' \) for each reduction \( \gamma \) in \( B_1 \). Suppose that \( t \) is in either \( B_1 \) or \( B_2 \). Since \( \gamma_i \) is between \( \alpha \) and \( t \), \( p' \not\leq p(\beta) \). Also, \( p' \not\leq p \) since \( p' \leq p(\beta) \) and \( p \leq p(\beta) \). Hence, \( p' \geq p \), so \( \alpha_1 \perp_1 \beta \) from lemma A.4. Thus, \( \alpha_1 \perp_1 \beta \).