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A new proof of Chew's theorem

Ken Mano
NTT Communication Science Labs.
2 Hikari-dai Seika-cho Soraku-gun
Kyoto 619-02 Japan
mano@cslab.kecl.ntt.jp

Mizuhito Ogawa
NTT Basic Research Labs.
3-1 Morinosato-Wakamiya Atsugi-shi
Kanagawa 243-01 Japan
mizuhito@ntt-20.ntt.jp

Abstract

We present a new proof of Chew's theorem, which states that normal forms are unique up to conversion in compatible term rewriting systems.

1 Introduction

A term rewriting system (TRS) R is *compatible* if for each pair of rules in R, there exist appropriate *linearizations* and they are *almost non-overlapping.* Chew's theorem [Che81] states that the unique normal form property (**UN**) holds in a compatible TRS, i.e., normal forms are unique up to conversion. The theorem is important since compatibility is a syntactic condition and the class partly contains non-left-linear non-terminating TRSs. However, there is a general feeling of doubt about the original proof in [Che81]. In fact, there is a gap in the proof of a key lemma[1].

There have been several attempts at a new proof, and partial answers have been obtained [dV90, Oga92, TO94]. De Vrijer showed that **UN** of a TRS R can be reduced to the Church-Rosser property (**CR**) of its conditional linearization, R^L [dV90]. In R^L, reductions are associated with *subproofs* that solve equivalence constraint. If R^L is non-overlapping, R^L is **CR** [BK86]. De Vrijer [dV90] showed that a particular compatible TRS CL-pc (combinatory logic with parallel-conditional) is **UN** by the following: (1) construct a model, (2) show CL-pc^L to be *semantically* non-overlapping, (3) CL-pc^L is thus **CR**. However, it is generally not easy to find such an appropriate model for a compatible TRS. Ogawa proved that **UN** holds for so-called weakly compatible TRSs [Oga92]. This class contains CL-pc, but is incomparable with the class of Chew's compatible TRSs. Toyama and Oyamaguchi [TO94] introduced a variant of conditional linearization and gave a sufficient condition of **UN** for *non-duplicating* TRSs. In this paper, we will give a new proof of the entire statement of Chew's theorem itself in a complete form.

Let us briefly outline the methodology of our proof. Given a compatible TRS R, we transform it into *conditional linearization* $\hat{R}$ *with extra variables* [TO94]. Similar to what de Vrijer observed, it is sufficient to prove that **CR** holds for $\hat{R}$ in order to conclude that R is **UN**. We will prove **CR** of $\hat{R}$ by a *peak elimination* process. Given a *proof* $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ in $\hat{R}$, the peak elimination replaces a peak $t_{i-1} \leftarrow t_i \rightarrow t_{i+1}$ in this proof with a conversion $t_{i-1} \leftrightarrow^* t_{i+1}$ in $\hat{R}$ according to the peak elimination rules. If all peaks are eliminated by applying the rules to the given proof repeatedly, i.e., if the peak elimination process eventually terminates, then we find a term s such that $t_1 \rightarrow^* s \leftarrow^* t_n$ as shown in figure 1. (Section 3)

We say a reduction is *in* a proof $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ not only for the reductions $t_i \rightarrow t_{i+1}$ (or $t_i \leftarrow t_{i+1}$) but also for the ones in the subproofs. When a proof A' is obtained by applying a peak elimination to a proof A, any reduction α' in A' can be regarded as a *descendant* of a reduction in A. Unfortunately, if a peak is made with overlapping reductions, the peak elimination may cause multiple descendants of a reduction. That is, if the reductions $\gamma_1 : t_{i-1} \leftarrow t_i$ and $\gamma_2 : t_i \rightarrow t_{i+1}$ are overlapping and a reduction α is in a subproof of γ_1 (or γ_2), then multiple descendants of α can be caused by eliminating the peak made with γ_1 and γ_2. In this case, α is said to be *duplicated* by γ_1 (γ_2). This makes it difficult to prove termination of peak elimination processes. However, we can estimate how many times eliminations of overlapping peak occur by examining which reduction

[1] See section 2.2 for details.

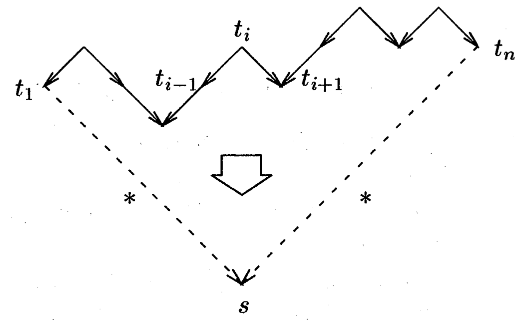

Figure 1: Peak elimination

(or descendants of it) can *not* go into a subproof of reductions making a overlapping peak during the process. Therefore, we introduce a binary relation *independence* on the reductions that satisfies the following properties:

1. Independence is preserved during a peak elimination process. (Theorem 4.1)
2. A reduction is not independent of its subproofs. (Lemma 4.4)
3. If two reductions are independent, their subproofs are also independent. (Lemma 4.5)
4. Two descendants of a reduction are independent of each other. (Lemma 4.6)

One of the candidates of independence is "*parallelness*" formally defined as $\perp\!\!\!\perp_1$ in this paper. In fact, if the conditional linearization $\hat{R}$ is non-overlapping, "parallelness" satisfies the above properties. However, for a compatible TRS this is not enough. Consider the following compatible TRS[2]:

$$R = \{f(x,a) \to x,\ f(a,y) \to y\}$$

Note that R is overlapping. Its conditional linearization $\hat{R}$ is:

$$\hat{R} = \{f(x',a) \to x \text{ if } x' = x \cdots (1),\ f(a,y') \to y \text{ if } y' = y \cdots (2)\}$$

Suppose that $t_1 \leftrightarrow^* a$ and $a \leftrightarrow^* t_2$ in $\hat{R}$, then there is a peak of the form:

$$\begin{array}{ccccccl} & t_1 \leftrightarrow^* a & & a \leftrightarrow^* t_2 & & : \text{subproofs} \\ t_1 & \longleftarrow_{(1)} & f(a,a) & \longrightarrow_{(2)} & t_2 & : \text{peak} \end{array}$$

where $t_1 \leftrightarrow^* a$ contracts with the first argument of $f(a,a)$, and $a \leftrightarrow^* t_2$ with the second argument, so they are "parallel". A peak elimination replaces the peak with:

$$t_1 \leftrightarrow^* a \leftrightarrow^* t_2$$

and "parallelness" is not preserved. However, in this case the term a is a normal form (since $\hat{R}$ contains only root-to-root overlap and a is a proper subterm of the LHS of a rule) and it "*splits*" the proof into $t_1 \leftrightarrow^* a$ and $a \leftrightarrow^* t_2$. By exploiting this observation, independence must be defined for the conditional linearizations of compatible TRSs. (Section 4)

We then introduce *initial labeling* and a *descendant forest* for a peak elimination process. Each reduction in the process is labeled an *initial label*, which indicates the ancestor of the reduction in the starting proof. For each reduction in the starting proof of the process, a *descendant tree* in the forest is associated; the reduction in the starting proof is the root vertex of the tree. Each path of the tree traces the descendants of the starting reduction, and non-leaf vertices represent applications of peak eliminations that duplications of descendants of the reduction occurred in. From the property of independence, if a reduction α is duplicated by another reduction labeled $[\gamma]$, then any descendants of α will never be duplicated by the reductions labeled $[\gamma]$ any more. This proves that the descendant forest is finite, which leads to the termination of the peak elimination process. Therefore, **CR** of $\hat{R}$ is obtained, and so we complete the proof of Chew's theorem. (Section 5)

[2]This is an example which is compatible butt not weakly compatible[Oga92].

2 Preliminaries

2.1 Abstract reduction systems and term rewriting systems

The definitions and terminologies of abstract reduction systems, terms, and term rewriting systems are taken from [Klo92].

Let $\rightarrow$ be an *abstract reduction system* that is a binary relation on some underlying domain. The symmetric closure, the reflexive transitive closure, and the reflexive transitive symmetric closure of $\rightarrow$ are written as $\leftrightarrow$, $\rightarrow^*$ and $\leftrightarrow^*$, respectively. If there is no a' such that $a \rightarrow a'$, then a is a *normal form* of the reduction system. A sequence $a_1 \leftrightarrow \cdots \leftrightarrow a_n$ is called a *proof.* A subsequence of the form $a' \leftarrow a \rightarrow a''$ is called a *peak.*

A reduction system $\rightarrow$ has the *unique normal form property* (**UN**) if $a \leftrightarrow^* a' \Rightarrow a \equiv a'$ for each pair of normal forms a, a'. We say $\rightarrow$ has the *Church-Rosser property* (**CR**) if, for any $a \leftrightarrow^* a'$, there exists b such that $a \rightarrow^* b$ and $a' \rightarrow^* b$.

Let F be a set of *function symbols*, and let V be a countably infinite set of *variables.* The set of all *terms* built from F and V is defined as usual. The set of variables occurring in a term t is denoted by $V(t)$.

Let $\Box$ be a fresh special constant symbol. A *context* $C[\,]$ is a term in $F \cup \Box$ and V. When $C[\,]$ is a context with n $\Box$'s and $t_1, \cdots, t_n$ are terms, $C[t_1, \cdots, t_n]$ denotes the term obtained by replacing all $\Box$ in $C[\,]$ with t_i in a left-to-right manner.

Let t be terms s.t. $t \equiv C[s]$ with a context $C[\,]$ and a non-variable term s. If s and t' are unifiable with a most general unifier θ, then $C[s\theta]$ is called a *superposition* of t and t'.

Positions of a term are encoded in the sequences of natural numbers. The set of positions of a term t is denoted by $P(t)$. For a position $p \in P(t)$, t/p is the subterm occurring at p in t. For terms t, s and a position $p \in P(t)$, $t[p \leftarrow s]$ is the term obtained by replacing the subterm at p in t with s.

For positions p_1, p_2, $p_1 \leq p_2$ if p_1 is a prefix of p_2. We write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. When neither $p_1 \leq p_2$ nor $p_2 \leq p_1$, p_1 and p_2 are said to be *parallel* (notation $p_1 \perp p_2$). The longest common prefix of p_1 and p_2 is denoted by $\wedge(p_1, p_2)$.

A *term rewriting system* (TRS) is a finite set R of *rewrite rules.* A rewrite rule is a pair of terms denoted by $l \rightarrow r$ satisfying (1) l is not a variable and (2) $V(l) \supseteq V(r)$.

The reduction system $\rightarrow_R$ on the set of terms is defined from a TRS R as follows:

$$\rightarrow_R = \{C[l\theta] \rightarrow_R C[r\theta] \mid C[\,] \text{ is a context, } \theta \text{ is a substitution, and } l \rightarrow r \in R\}.$$

A term $l\theta$ is called a *redex* of R if $l \rightarrow r \in R$. For a reduction $\alpha : C[l\theta] \rightarrow_R C[r\theta]$, the position of the redex $l\theta$ in $C[l\theta]$ is denoted by $p(\alpha)$.

When we think of a pair of rules S and S', we assume that S and S' are *standardized apart*, i.e., the variables in S and S' are renamed appropriately so that S and S' do not share variables.

Let $C[\,]$ be a context with n $\Box$'s, and let $t_i \leftrightarrow_R^* t_i'$ be proofs in R for $1 \leq i \leq n$. The *embedding* of the proofs into $C[\,]$ is the following:

$$C[t_1, t_2, \cdots, t_n] \leftrightarrow_R^* C[t_1', t_2, \cdots, t_n] \leftrightarrow_R^* C[t_1', t_2', \cdots, t_n] \leftrightarrow_R^* \cdots \leftrightarrow_R^* C[t_1', t_2', \cdots, t_n'],$$

which is denoted by $C[t_1, \cdots, t_n] \Leftrightarrow_R^* C[t_1', \cdots, t_n']$.

Rewrite rules S and S' are *overlay* if a superposition of l and l' exists only in a root-to-root case, i.e., the context $C[\,]$ in the definition of superposition is $\Box$. If S and S' are overlay and $r\sigma \equiv r'\sigma$ for all unifiers σ of l and l', then S and S' are *almost non-overlapping.*

Definition 2.1 A term $\bar{t}$ is a *linearization* of a term t if (1) $\bar{t}$ is linear, and (2) there is a substitution σ s.t. $\bar{t}\sigma = t$ and $x\sigma \in V$ for all $x \in V$. For a rewrite rule $l \rightarrow r$, $\bar{l} \rightarrow \bar{r}$ is called a linearization of $l \rightarrow r$, if the following properties hold:

- $\bar{l}$ is a linearization of l s.t. $\bar{l}\sigma = l$, and
- $\bar{r}\sigma = r$.

Definition 2.2 ([Che81, dV90]) Rewrite rules S and S' are said to be *compatible*[3] if there exist linearizations $\bar{S}$, $\bar{S'}$ of S, S' such that $\bar{S}$ and $\bar{S'}$ are almost non-overlapping. A TRS R is compatible if each pair of rules is compatible.

[3] De Vrijer's terminology [dV90] is used here. The corresponding notion in Chew's original paper is "strongly non-overlapping and compatible".

Example 2.1 Combinatory logic CL can be regarded as a TRS if the function application is expressed explicitly by a binary symbol, e.g., *Ap*. A compatible TRS CL-pc is the union of CL and the following *parallel-conditional* rules.

$$\overset{SKI}{\left\{\begin{array}{lcl} Sxyz & \to & xz(yz) \\ Kxy & \to & x \\ Ix & \to & x \end{array}\right\}} + \overset{\textit{parallel-conditional}}{\left\{\begin{array}{lcl} CTxy & \to & x \\ CFxy & \to & y \\ Czxx & \to & x \end{array}\right\}}$$

The aim of this paper is the proof of Chew's theorem [Che81].

Theorem 2.1 A compatible TRS is **UN**.

2.2 Scenario of Chew's original proof

Let R be a TRS, and let R' be the set of all linearizations of all rules in R. For example, if $g(h(x,x)) \to h(x,x) \in R$, then $g(h(x_1,x_2)) \to h(x_1,x_1)$, $g(h(x_1,x_2)) \to h(x_1,x_2)$, $g(h(x_1,x_2)) \to h(x_2,x_1)$ and $g(h(x_1,x_2)) \to h(x_2,x_2)$ are in R'. For a reduction $\to$, $t \overset{nr^*}{\to} t'$ denotes reduction sequence preserving the root symbol of t. In order to avoid difficulty caused by non-left-linearity, Chew introduced *closure* and *marker*[4].

The *closure* $\to_{\bar{R}}$ *of* $\to_R$ *with respect to* R' is (inductively) defined as the following conditional TRS obtained from R':

$$\left.\begin{array}{l} g(h(x_1,x_2)) \to h(x_1,x_1) \\ g(h(x_1,x_2)) \to h(x_1,x_2) \\ g(h(x_1,x_2)) \to h(x_2,x_1) \\ g(h(x_1,x_2)) \to h(x_2,x_2) \end{array}\right. \quad \text{if } \exists M \text{ s.t. } M \text{ is a redex of } R \text{ and } M \overset{nr^*}{\to}_{\bar{R}} g(h(x_1,x_2)) \ .$$

Two fresh symbols α and β called *markers* (corresponding to the right direction and the left direction respectively, as will become clear) are introduced to represent "all the possible choices of variables in the linearization" in one rewrite rule. For example, $g(h(x,x)) \to h(x,x)$ is transformed into the following rule using α:

$$g(h(x_1,x_2)) \to \alpha(h(x_1,x_1), h(x_1,x_2), h(x_2,x_1), h(x_2,x_2)).$$

The TRS obtained by such a transformation from R is denoted by αR. βR is defined similarly using the symbol β. To simulate $\to_R$, the following additional reductions are also introduced, i.e., copying reductions $\to_{\alpha+}$, $\to_{\beta+}$, selecting reductions $\to_{\alpha-}$, $\to_{\beta-}$, and distributing reductions $\to_{\alpha d}$, $\to_{\beta d}$. For instance,

$$\begin{array}{rcl} h(t_1,t_2) & \to_{\alpha+} & \alpha(h(t_1,t_2), h(t_1,t_2)), \\ \alpha(h(t_1,t_2), h(t_3,t_4)) & \to_{\alpha-} & h(t_1,t_2) \text{ or } h(t_3,t_4), \\ g(\alpha(h(t_1,t_2), h(t_3,t_4))) & \to_{\alpha d} & g(h(\alpha(t_1,t_3), \alpha(t_2,t_4)))). \end{array}$$

A reduction $\to_{\alpha R^c}$ ($\to_{\beta R^c}$, resp.) is the closure of R with respect to αR (βR) using $\overset{nr^*}{\to}_{\Re}$ in the conditional part, where $\to_{\Re} = \to_{\alpha R^c} \cup \to_{\beta R^c} \cup \to_{\alpha+} \cup \to_{\beta+} \cup \to_{\alpha-} \cup \to_{\beta-} \cup \to_{\alpha d} \cup \to_{\beta d}$. Let $\to_S = \to_{\alpha R^c} \cup \to_{\alpha d} \cup \to_{\beta+} \cup \to_{\beta-}$, and $\to_T = \to_{\beta R^c} \cup \to_{\beta d} \cup \to_{\alpha+} \cup \to_{\alpha-}$.

The outline of Chew's original proof is the following. At first, similar to what de Vrijer observed, **UN** of $\to_R$ is reduced to **CR** of $\to_{\bar{R}}$. Next, $\to_S$ and $\to_T$ are shown to be commutative. Finally, **CR** of $\to_{\bar{R}}$ is proved by the following steps: given a proof $t \leftrightarrow^*_{\bar{R}} t'$,

1. transform it into $t \leftrightarrow^*_R t'$ (since $\to_{\bar{R}}$ and $\to_R$ are the same in convertibility),
2. replace each $\to_R$ with $\to_{\alpha R^c} \cdot \leftarrow_{\alpha+}$ ($\in \to_S \cdot \leftarrow_T$) and replace each $\leftarrow_R$ with $\to_{\beta R^c} \cdot \leftarrow_{\beta+}$ ($\in \to_S \cdot \leftarrow_T$),
3. $t \to^*_T \cdot \leftarrow^*_S t'$ from the commutativity of $\to_S$ and $\to_T$,
4. $t \to^*_{\bar{R}} \cdot \leftarrow^*_{\bar{R}} t'$ by "stripping" α's and β's.

The key lemma 6.1 in [Che81] is necessary in the final step. It states that if A is a redex of $\to_{\alpha R^c}$ (by definition, this means that there exists a redex B of R such that $B \overset{nr^*}{\to}_{\Re} A$), then any $\to_{\alpha-} \cup \to_{\beta-}$-normal form

[4]Notations and definitions are slightly different from the original. We use $\to_R$, $\to_{\bar{R}}$, $\to_{\alpha R^c}$, $\to_{\beta R^c}$, $\to_{\Re}$ instead of the original notations $\to_G$, $\to_{\bar{G}}$, $\to_{\alpha G^c}$, $\to_{\beta G^c}$, $\to_R$.

$\bar{A}$ of A is a redex of $\rightarrow_{\bar{R}}$. The "proof" of the lemma is due to the induction on the length of $B \stackrel{nr*}{\rightarrow}_{\Re} A$. However, here is a gap which seems to be difficult to remedy.

The induction does not work for $\rightarrow_{\alpha d}$ [vO94]. Let us consider the following example:

$$B \rightarrow^*_{\Re} \underset{(\equiv B')}{g(\alpha(h(t_1,t_2),h(t_3,t_4)))} \rightarrow_{\alpha d} \underset{(\equiv A)}{g(h(\alpha(t_1,t_3),\alpha(t_2,t_4)))},$$

where t_i are arbitrary terms containing neither α nor β. Removing the markers by $\rightarrow_{\alpha -}$ and $\rightarrow_{\beta -}$, we obtain $C_{B'} = \{g(h(t_1,t_2)), g(h(t_3,t_4))\}$ from B' and $C_A = \{g(h(t_1,t_2)), g(h(t_1,t_4)), g(h(t_3,t_2)), g(h(t_3,t_4))\}$ from A. In the induction step, it must be shown that for each $s_A \in C_A$, there exists $s_{B'} \in C_{B'}$ such that $s_{B'} \rightarrow^*_{\bar{R}} s_A$; this is impossible in the case $s_A = g(h(t_1,t_4))$ or $g(h(t_3,t_2))$.

2.3 A property of compatible rewrite rules

In this section, we establish some properties of compatible systems used in the later sections.

Definition 2.3 The set of *non-common positions* $NC_{t,t'}$ of terms t and t' is the set of all minimal elements in $\{p \mid Root(t/p) \neq Root(t'/p)\}$ wrt $\leq$, where $Root(s)$ is the root symbol of the term s. The *common context* $C_{t,t'}[\,]$ of t and t' is $t[p \leftarrow \Box \mid p \in NC_{t,t'}]$ ($\equiv t'[p \leftarrow \Box \mid p \in NC_{t,t'}]$).

Definition 2.4 For terms t, t', a relation $\sim_{t,t'}$ is defined as follows:

$$s \sim_{t,t'} s' \text{ iff } s \equiv t/p \text{ and } s' \equiv t'/p \text{ for some } p \in NC_{t,t'}.$$

Lemma 2.1 Let t, t' be terms without shared variables. Assume $s \sim_{t,t'} C[u]$ and $u \sim_{t,t'} u'$. Then u is a ground term. ∎

Lemma 2.2 Let t, t' be linear terms without shared variables. Suppose that t and t' are unifiable. Then the substitution defined as below is a unifier of t and t':

$$\theta_{t,t'} = \{x := s' \mid x \sim_{t,t'} s'\} \cup \{x' := s \mid s \sim_{t,t'} x' \text{ and } s \notin V\}.$$

∎

Lemma 2.3 Let $S : l \rightarrow r$, $S' : l' \rightarrow r'$ be compatible rewrite rules with unifiable linearizations of left-hand sides, i.e., there exist linearizations $\bar{S} : \bar{l} \rightarrow \bar{r}$, $\bar{S}' : \bar{l}' \rightarrow \bar{r}'$ of S, S' respectively such that $\bar{l}$ and $\bar{l}'$ are unifiable and $\bar{r}\sigma \equiv \bar{r}'\sigma$ for each unifier σ of $\bar{l}$ and $\bar{l}'$. Then for all $q \in NC_{\bar{r},\bar{r}'}$, either of the following holds:

1. $\bar{r}/q \in V$, and there exist a context $C'_q[\,]$ with m' $\Box$'s ($m' \geq 0$), ground terms $g_1, \cdots, g_{m'}$ and variables $x'_1, \cdots, x'_{m'}$ in S' s.t.
   - $\bar{r}/q \sim_{\bar{l},\bar{l}'} C'_q[g_1, \cdots, g_{m'}]$,
   - $g_k \sim_{\bar{l},\bar{l}'} x'_k$ for all $k = 1, \ldots, m'$, and
   - $\bar{r}'/q \equiv C'_q[x'_1, \cdots, x'_{m'}]$.
2. $\bar{r}'/q \in V$, and there exist a context $C[\,]$ with n' $\Box$'s ($n' \geq 0$), ground terms $g'_1, \cdots, g'_{n'}$ and variables $x_1, \cdots, x_{n'}$ in S s.t.
   - $C_q[g'_1, \cdots, g'_{n'}] \sim_{\bar{l},\bar{l}'} \bar{r}'/q$,
   - $x_k \sim_{\bar{l},\bar{l}'} g'_k$ for all $k = 1, \ldots, n'$, and
   - $\bar{r}/q \equiv C_q[x_1, \cdots, x_{n'}]$.

Proof Since $\bar{r}$ and $\bar{r}'$ are unifiable, $\bar{r}/q \in V$ or $\bar{r}'/q \in V$. We only check the former case. The other case is treated similarly. Let $C'_q[\,] = \bar{r}/q\{x := \Box \mid x\theta_{l,l'} \neq x\}$, where $\theta_{l,l'}$ is the unifier defined in lemma 2.2. Since $\theta_{l,l'}$ is a unifier of $\bar{r}$ and $\bar{r}'$, there are terms $g_1, \cdots, g_{m'}$ and variables $x'_1, \cdots, x'_{m'}$ satisfying the three conditions. From lemma 2.1, g_k is a ground term for $k = 1, \ldots, m'$. ∎

3 Conditional linearization and peak elimination

3.1 Left-right separated CTRS and conditional linearization

Definition 3.1 A *left-right separated conditional term rewriting system* is a finite set of conditional rewrite rules with extra variables of the form $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ satisfying the following conditions:

1. l is left-linear, $V(l) = \{x_1, \cdots, x_n\}$,
2. $V(r) \subseteq \{y_1, \cdots, y_n\}$,
3. $\{x_1, \cdots, x_n\} \cap \{y_1, \cdots, y_n\} = \emptyset$, and
4. $x_i \not\equiv x_j$ if $i \neq j$.[5]

$l \to r$ is called the *unconditional part* and $x_1 = y_1, \cdots, x_n = y_n$ is called the *condition part* of $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$. For convenience:

1. A condition part is often abbreviated by Q, Q', etc.
2. Variables $x_1, \cdots, x_n$ are assumed to appear in the left-to-right order in l, e.g. $l = f(x_1, g(x_2, x_3))$.

Definition 3.2 Let $\hat{R}$ be a left-right separated CTRS. The reduction $\overset{\nabla}{\to}_{\hat{R}_i}$ is inductively defined as follows:

$$\begin{cases} \overset{\nabla}{\to}_{\hat{R}_0} & = \emptyset, \\ \overset{\nabla}{\to}_{\hat{R}_{i+1}} & = \{C[l\theta] \overset{\nabla}{\to}_{\hat{R}_{i+1}} C[r\theta] \mid \hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n \in \hat{R} \text{ and } x_j\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}_i} y_j\theta \text{ for } i = 1, \ldots, n\}. \end{cases}$$

Then, $\overset{\nabla}{\to}_{\hat{R}} = \cup_i \overset{\nabla}{\to}_{\hat{R}_i}$.

Proofs $x_j\theta \overset{\nabla *}{\leftrightarrow}_{R_i} y_j\theta$ are called *subproofs* associated with $C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{R}_{i+1}} C[\hat{r}\theta]$. Subproofs of an R_1 reduction are called trivial subproofs, and we eventually denote $\overset{\nabla}{\to}_{\hat{R}_1}$ as $\to_{\hat{R}_1}$.

When a reduction $t \overset{\nabla}{\to}_{\hat{R}} t'$ is done by a rewrite rule $\hat{S} \in \hat{R}$, it is also denoted by $t \overset{\nabla}{\to}_{\hat{S}} t'$. For a reduction $C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{R}} C[\hat{r}\theta]$, $\hat{l}\theta$ is called a *redex*.

Reductions are often treated as more than a relation; we assume a reduction in $\hat{R}$ is associated with the following "information" implicitly: the rule used, the position, and the subproofs.

Similarly, a rewrite proof $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ is regarded as a hierarchical object. Reductions $t_i \overset{\nabla}{\to}_{\hat{R}} t_{i+1}$ (or $t_i \overset{\nabla}{\leftarrow}_{\hat{R}} t_{i+1}$) themselves are top-level components, reductions in subproofs of them are second-level components, etc. A reduction α is *in* A when α is a component of the hierarchical object. Moreover, the top-level component is called the *top-level reduction.*

Definition 3.3 For a rewrite rule $S : l \to r$, a *conditional linearization* $\hat{S} : \hat{l} \to \hat{r} \Leftarrow Q$ is a left-right separated conditional rewrite rule constructed as follows:

1. $\hat{l}$ is a linearization of l s.t. $\hat{l}\sigma = l$ and $V(\hat{l}) \cap V(l) = \emptyset$,
2. $\hat{r} \equiv r$, and
3. add $x\sigma = x$ to the condition part Q for all $x \in V(l)$.

Note that conditional linearizations of S are unique up to renaming of variables in $\hat{l}$. In the rest of this paper, R denotes the TRS and $\hat{R}$ denotes the collection of conditional linearizations of all rules in R, called *the conditional linearization of R.*

Example 3.1 $\hat{R}$ is the conditional linearization of R.

$$R = \left\{ \begin{array}{lll} d(x,x) & \to & 0 \\ f(y) & \to & d(y, f(y)) \\ 1 & \to & f(1) \end{array} \right\}. \quad \hat{R} = \left\{ \begin{array}{lllll} d(x_1,x_2) & \to & 0 & \Leftarrow & x_1 = x, x_2 = x \\ f(y_1) & \to & d(y, f(y)) & \Leftarrow & y_1 = y \\ 1 & \to & f(1) & & \end{array} \right\}.$$

[5] $y_i \equiv y_j$ may hold for $i \neq j$.

The following theorem appeared in [TO94] with the condition of non-duplicating. The expansion to the general case is straightforward.

Theorem 3.1 ([TO94]) If $\hat{R}$ is CR, then R is UN. ∎

For left-right separated conditional rewrite rules $\hat{S}$, $\hat{S}'$, $\hat{S}$ and $\hat{S}'$ are said to be non-overlapping (almost non-overlapping, overlay) if their unconditional parts are overlapping (almost non-overlapping, overlay). A left-right separated CTRS $\hat{R}$ is non-overlapping (almost non-overlapping, overlay) when every pair of rules in $\hat{R}$ is non-overlapping (almost non-overlapping, overlay). A left-right separated CTRS $\hat{R}$ is compatible if there exists a compatible TRS R such that $\hat{R}$ is the conditional linearization of R.

Definition 3.4 A term t is a *head normal form* of $\hat{R}$ if s is not a redex of $\hat{R}$ for all s such that $t \overset{\nabla *}{\to}_{\hat{R}} s$. A term t is a *quasi-ground normal form* of $\hat{R}$ *wrt* $q \in P(t)$ if

1. for each $q' \leq q$, t/q' is a head normal form of $\hat{R}$, and
2. t/q is a ground normal form of $\hat{R}$.

Lemma 3.1 Let $\hat{l} \to \hat{r} \Leftarrow Q \in \hat{R}$. Suppose that $\hat{R}$ is compatible. Then for each non-variable proper subterm t of l and substitution θ, $t\theta$ is a head normal form of $\hat{R}$. ∎

3.2 Conditional peak elimination

In this section, the following notations will be established:

1. R is a compatible TRS.
2. $S : l \to r$, $S' : l' \to r' \in R$.
3. $\bar{S} : \bar{l} \to \bar{r}$, $\bar{S}' : \bar{l}' \to \bar{r}'$ are linearizations of S and S' s.t. $\bar{r}\sigma \equiv \bar{r}'\sigma$ for all unifiers σ of $\bar{l}$ and $\bar{l}'$.
4. $\hat{S} : \hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n$, $\hat{S}' : \hat{l}' \to \hat{r}' \Leftarrow x'_1 = y'_1, \cdots, x'_m = y'_m$ are conditional linearizations of S and S' s.t. $\hat{l} \equiv \bar{l}$ and $\hat{l}' \equiv \bar{l}'$.

Definition 3.5 Suppose that there is a peak of the form $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\to}_{\hat{S}'} C[\hat{r}'\theta]$. For $p \in NC_{\hat{l},\hat{l}'}$, the *left connecting proof* A_p of the peak is defined as follows:

$$A_p = \begin{cases} y_i\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} x_i\theta \equiv C'_p[x'_j\theta, \cdots, x'_{j+j'}\theta] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C'_p[y'_j\theta, \cdots, y'_{j+j'}\theta] & \text{if } \hat{l}/p \equiv x_i \text{ and } \hat{l}'/p = C'_p[x'_j, \cdots, x'_{j+j'}], \\ C_p[y_i\theta, \cdots, y_{i+i'}\theta] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C_p[x_i\theta, \cdots, x_{i+i'}\theta] \equiv x'_j\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} y'_j\theta & \text{if } l/p = C_p[x_i, \cdots, x_{i+i'}] \notin V \text{ and } l'/p \equiv x'_j, \end{cases}$$

where $x_k\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} y_k\theta$ ($x'_k\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} y'_k\theta$) are the subproofs of $C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{S}} C[\hat{r}\theta]$ ($C[\hat{l}'\theta] \overset{\nabla}{\to}_{\hat{S}'} C[\hat{r}'\theta]$), $V(\hat{l}/p) = \{x_i, \cdots, x_{i+i'}\}$ and $V(\hat{l}'/p) = \{x'_j, \cdots, x'_{j+j'}\}$.

Definition 3.6 For a rewrite rule $\hat{S} : \hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n$, $\mathcal{T}_{\hat{S}}$ is a substitution defined as follows:

$$\mathcal{T}_{\hat{S}} = \{x_1 := y_1, \cdots, x_n := y_n\}.$$

Lemma 3.2 Suppose that there is a peak of the form $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\to}_{\hat{S}'} C[\hat{r}'\theta]$. Assume $t \sim_{\hat{l},\hat{l}'} t'$. Then there exits $p \in NC_{\hat{l},\hat{l}'}$ such that $t\mathcal{T}_{\hat{S}}\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t'\mathcal{T}_{\hat{S}'}\theta$ is the left connecting proof A_p. ∎

Lemma 3.3 Suppose that there is a peak of the form $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\to}_{\hat{S}'} C[\hat{r}'\theta]$. For $q \in NC_{\hat{r},\hat{r}'}$, either of the following holds:

1. $\hat{r}/q \in V$, and there exist a context $C'_q[]$ with m' □'s ($m' \geq 0$), ground terms $g_1, \cdots, g_{m'}$ and variables $y'_{j_1}, \cdots, y'_{j_{m'}}$ s.t.
   - $\hat{r}/q\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\hat{S}'}\theta$ is a left connecting proof of the peak,
   - $g_k \overset{\nabla *}{\leftrightarrow}_{\hat{R}} y'_{j_k}\theta$ are left connecting proofs of the peak for all $k = 1, \ldots, m'$,

- $\hat{r}'/q\theta \equiv C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]\mathcal{T}_{\hat{S}'}\theta$, and
- $C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\hat{S}'}\theta$ is a quasi-ground normal form of $\hat{R}$ wrt q_k for each position q_k of $\square$ in $C'_q[]$.

2. $\hat{r}'/q \in V$, and there exist a context $C_q[]$ with n' $\square$'s ($n' \geq 0$), ground terms $g'_1, \cdots, g'_{n'}$ and variables $y_{i_1}, \cdots, y_{i_{n'}}$ s.t.

   - $C_q[g'_1, \cdots, g'_{n'}]\mathcal{T}_{\hat{S}}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} \hat{r}/q\theta$ is a left connecting proof of the peak,
   - $y_{i_k}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g'_k$ are left connecting proofs of the peak for all $k = 1, \ldots, n'$,
   - $\hat{r}/q\theta \equiv C_q[y_{i_1}, \cdots, y_{i_{n'}}]\mathcal{T}_{\hat{S}}\theta$, and
   - $C_q[g'_1, \cdots, g'_{n'}]\mathcal{T}_{\hat{S}}\theta$ is a quasi-ground normal form of $\hat{R}$ wrt q_k for each position q_k of $\square$ in $C_q[]$.

Proof We only check the former case. The other case is treated similarly. The first three conditions are satisfied by lemmas 2.3 and 3.2, $\hat{r} \equiv \bar{r}\mathcal{T}_{\hat{S}}$, and $\hat{r}' \equiv \bar{r}'\mathcal{T}_{\hat{S}'}$. The last condition follows from lemma 3.1 and the fact that $C'_q[g_1, \cdots, g_{m'}]$ is a proper subterm of $\hat{l}'$. ∎

Definition 3.7 Suppose that there is a peak of the form $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\rightarrow}_{\hat{S}'} C[\hat{r}'\theta]$. For $q \in NC_{\hat{r},\hat{r}'}$, the *right connecting proof* B_q of the peak is a proof connecting $\hat{r}/q\theta$ and $\hat{r}'/q\theta$ described in the previous lemma, i.e.,

$$B_q = \begin{cases} \hat{r}/q\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\hat{S}'}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]\mathcal{T}_{\hat{S}'}\theta & \text{if } \hat{r}/q \in V, \\ C_q[y_{i_1}, \cdots, y_{i_{n'}}]\mathcal{T}_{\hat{S}}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} C_q[g'_1, \cdots, g'_n]\mathcal{T}_{\hat{S}}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} \hat{r}'/q\theta & \text{if } \hat{r}/q \notin V \text{ and } \hat{r}'/q \in V. \end{cases}$$

Definition 3.8 For a proof $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ in $\hat{R}$, a *peak elimination* is a transformation of A where a peak in A, e.g., $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, is replaced with the sequence defined below. If A' is obtained from A by a conditional peak elimination of A, we write $A \mapsto A'$.

There are three *peak elimination rules* corresponding to the relative position of the reductions making the peak.

($P_\perp$) If two reductions making the peak occur at parallel positions, then the replacement sequence $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i \overset{\nabla}{\leftarrow}_{\hat{R}} t_{i+1}$ is obtained by exchanging the order of reductions making the peak.

($P_<$) Suppose that two reductions making the peak are nesting; e.g., $s \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'$ is a subproof of $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ occurs below the substitution part s of $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, and $t_{i-1} \equiv C[u] \overset{\nabla}{\leftarrow}_{\hat{R}} C[s] \equiv t_i$. The replacement sequence is $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ which has the same subproofs as $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ except for the modified subproof $u \overset{\nabla}{\leftarrow}_{\hat{R}} s \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'$. It is similar when $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ occurs above $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$.

(P_C) Suppose that two reductions making the peak overlap. Since $\hat{R}$ is compatible, the reductions making the peak occur at the same position in t_i. Assume the peak is of the form $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\rightarrow}_{\hat{S}'} C[\hat{r}'\theta]$. The replacement sequence is

$$t_{i-1} \equiv C[C_{\hat{r},\hat{r}'}[s_1, \cdots, s_k]] \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} C[C_{\hat{r},\hat{r}'}[s'_1, \cdots, s'_k]] \equiv t_{i+1},$$

where $s_j \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'_j$ are right connecting proofs of the peak.

Example 3.2 Let $\hat{R}$ be that of example 3.1. Suppose that $1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s$ and that $t \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s$. There is a peak of the form $d(f(1), t) \overset{\nabla}{\leftarrow}_{\hat{R}} d(1, t) \overset{\nabla}{\rightarrow}_{\hat{R}} 0$, where the left-oriented reduction is by the third rule and the right-oriented reduction by the first rule. By $P_<$, it is replaced with $d(f(1), t) \overset{\nabla}{\rightarrow}_{\hat{R}} 0$ as shown in figure 2.

Example 3.3 Let $\hat{R}$ be the following:

$$\hat{R} = \left\{ \begin{array}{l} \hat{S} : f(x_1, a) \to y_1 \Leftarrow x_1 = y_1, \\ \hat{S}' : f(g(x'_1, a, a), x'_2) \to g(y'_1, y'_2, y'_2) \Leftarrow x'_1 = y'_1, x'_2 = y'_2 \end{array} \right\}.$$

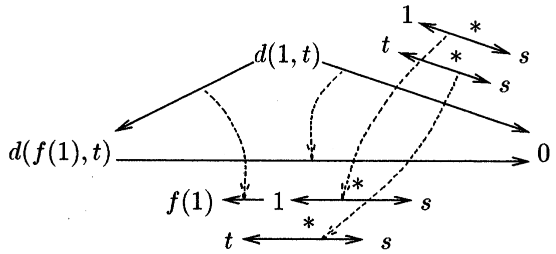

Figure 2: Rule $P_<$

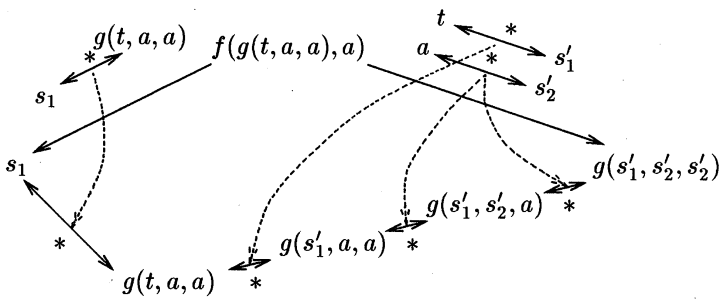

Figure 3: Rule P_C

Suppose that $t \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'_1$, $a \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'_2$ and that $g(t,a,a) \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_1$. Then there is a peak of the form $s_1 \overset{\nabla}{\leftarrow}_{\hat{S}} f(g(t,a,a),a) \overset{\nabla}{\rightarrow}_{\hat{S}'} g(s'_1, s'_2, s'_2)$. By P_C, it is replaced with $s_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(s'_1, s'_2, s'_2)$ (which itself is the right connecting proof of the peak) as shown in figure 3. Here, $s_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(t,a,a) \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(s'_1,a,a)$ and $a \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'_2$ are left connecting proofs. Note that $g(s'_1,a,a)$ is a quasi-ground normal form wrt p_1 and wrt p_2, where p_i are the positions of a^i in $g(t,a^1,a^2)$.

Definition 3.9 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof in $\hat{R}$. Suppose that $A \mapsto A'$, where a peak $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ is eliminated. For a reduction α' in A', the *ancestor* α in A of α' is defined as follows. We also say α' is a descendant of α if α is the ancestor if α'.

- If α' is not in the replacement sequence of the eliminated peak, then the ancestor is the same reduction in A as α'.
- If α' is in the replacement sequence of the eliminated peak, then we distinguish the following cases according to which peak elimination rule is applied.

($P_\perp$) Notations are the same as those used in the definition of $P_\perp$. If α' is $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i$, the ancestor is $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$. Since $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i$ and $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ have the same subproofs, if α' is in a subproof of $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i$, the ancestor is defined from the natural correspondence. It is similar when α' is $t'_i \overset{\nabla}{\leftarrow}_{\hat{R}} t_{i+1}$ or is in a subproof of it.

($P_<$) Notations are the same as those used in the definition of $P_<$. Here we consider the case $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ occurs below a substitution part s of $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$. The other cases are treated similarly. If α' is $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, then the ancestor is $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$. Since $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ and $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ have the same subproofs except for $u \overset{\nabla}{\leftarrow}_{\hat{R}} s$ -part, if α' is in a subproof of $t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i$ and not in $u \overset{\nabla}{\leftarrow}_{\hat{R}} s$, the ancestor is defined from the natural correspondence. If α is $u \overset{\nabla}{\leftarrow}_{\hat{R}} s$, the ancestor is $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$. Since $u \overset{\nabla}{\leftarrow}_{\hat{R}} s$ and $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ have the same subproofs, the ancestor is defined naturally if α' is in a subproof of $u \overset{\nabla}{\leftarrow}_{\hat{R}} s$. The dash lines in figure 2 illustrate the ancestor-descendant relation.

(P_C) Notations are the same as those used in the definition of P_C. In this case, the replacement sequence is an embedding of left connecting proofs. Moreover, each left connecting proof is a collection of right connecting proofs, and each right connecting proof is a collection of subproofs of the reductions making the peak as described in definition 3.7 and 3.5. Therefore, for all α' in the replacement sequence, there is a segment of the sequence such that α' is in the segment and the segment corresponds to a subproof of the reductions making the peak. The ancestor is defined from the natural correspondence. The dash lines in figure 3 illustrate the ancestor-descendant relation.

Moreover, for a peak elimination process $A_1 \mapsto \cdots \mapsto A_n$ and reductions α_i in A_i, we say α_1 is the ancestor of α_n and α_n is a descendant of α_1 if α_{i+1} is a descendant of α_i for each $1 \leq i < n$.

Note that, when P_C is applied, the top-level reductions making the eliminated peak have no descendant.

Definition 3.10 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof in $\hat{R}$. Suppose that $A \mapsto A'$, where a peak made with the reductions $\gamma : t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ and $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ is eliminated with P_C. Assume that a reduction α is in a subproof of γ_1 (γ_2, resp.). Then, α said to be *duplicated* by γ_1 (γ_2).

Lemma 3.4 Let $\hat{R}$ be compatible. If a peak elimination process $A_1 \mapsto A_2 \mapsto \cdots$ terminates for every proof A_1 in $\hat{R}$, then $\hat{R}$ is **CR**. ∎

4 Independence of reductions

4.1 Flattening and independence

In this section, the notion of independence is introduced. Independence is first defined for reductions in a proof in $\hat{R}_1$. It is then lifted up to any proof in $\hat{R}$ by flattening.

Lemma 4.1 For each non-$\hat{R}_1$ reduction $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$, there is a proof $t \equiv C[s_1, \cdots, s_m] \overset{\nabla^*}{\Leftrightarrow}_{\hat{R}} C[s'_1, \cdots, s'_m] \rightarrow_{\hat{R}_1} t'$ satisfying

1. $s_i \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} s'_i$ are the subproofs of $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$, and

2. in the reduction $C[s'_1, \cdots, s'_m] \rightarrow_{\hat{R}_1} t'$, the same rule is used at the same position as in $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$.

Proof Let $l \rightarrow r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ be the rewrite rule for the reduction $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$, $t \equiv C'[l\theta]$ and $t' \equiv C'[r\theta]$. Let $C''[\,]$ be a context such that $C''[x_1, \cdots, x_m] \equiv l$. The result follows by setting $C[\,] = C'[C''[\,]]$. ∎

Definition 4.1 For a non-$\hat{R}_1$ reduction $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$, the proof $t \equiv C[s_1, \cdots, s_m] \overset{\nabla^*}{\Leftrightarrow}_{\hat{R}} C[s'_1, \cdots, s'_m] \rightarrow_{\hat{R}_1} t'$ described in lemma 4.1 is called the *flattening* of $t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$. The flattening of a proof $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ at the i-th non-R_1 reduction is obtained by replacing $t_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1}$ with its flattening.

Lemma 4.2 When a flattening operation is regarded as a reduction on the set of proofs, there exists a unique normal form for each proof A. The normal form is called the *flat proof of* A and is denoted by $A^\flat$.

Proof Since a flattening operation is **WCR** and **SN**, it is **CR**. ∎

Note that $A^\flat$ contains only $\hat{R}_1$ reductions.

Definition 4.2 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof in $\hat{R}$ and A' be a flattening of A. Suppose A' is obtained by replacing $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ with $t_i \equiv C[s_1, \cdots, s_m] \overset{\nabla^*}{\Leftrightarrow}_{\hat{R}} C[s'_1, \cdots, s'_m] \rightarrow_{\hat{R}_1} t_{i+1}$. The mapping $flat$ is a bijection from reductions in A to ones in its flattening as follows.

1. If α is the top-level reduction $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, then $flat(\alpha)$ is $C[s'_1, \cdots, s'_m] \rightarrow_{\hat{R}_1} t_{i+1}$.

2. If α is in the i-th subproof $s_i \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} s'_i$, $flat(\alpha)$ is the corresponding reduction in $C[\cdots, s_i, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} C[\cdots, s'_i, \cdots]$.

3. Otherwise, $flat(\alpha)$ is the same reduction in A' as α.

When A' is obtained by replacing $t_i \overset{\nabla}{\leftarrow}_{\hat{R}} t_{i+1}$, $flat$ is defined similarly. For a reduction α in A, $\alpha^\flat$ in $A^\flat$ is obtained by repeated applications of $flat$.

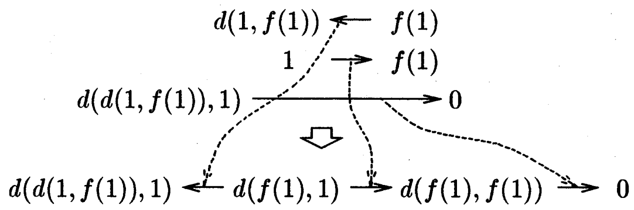

Figure 4: Flattening

Example 4.1 Let $\hat{R}$ be that in example 3.1. Let A be a one-step proof of the form $A : d(d(1, f(1)), 1) \overset{\nabla}{\rightarrow}_{\hat{R}} 0$ with subproofs $d(1, f(1)) \leftarrow_{\hat{R}_1} f(1)$ and $1 \rightarrow_{\hat{R}_1} f(1)$. Applying a flattening operation to A, we obtain $A^\flat$: $d(d(1, f(1)), 1) \leftarrow_{\hat{R}_1} d(f(1), 1) \rightarrow_{\hat{R}_1} d(f(1), f(1)) \rightarrow_{\hat{R}_1} 0$ as in figure 4. The dash-arrows illustrate the bijection *flat*.

Lemma 4.3 Let $A : t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$ be a one-step proof where the position of the reduction is p, and let $A^\flat : t \equiv t_1 \leftrightarrow_{\hat{R}_1} \cdots \leftrightarrow_{\hat{R}_1} t_n \rightarrow_{\hat{R}_1} t'$. For all $i \leq n$, there is a reduction $\alpha : t_i \overset{\nabla *}{\rightarrow}_{\hat{R}} t'$ satisfying $p(\alpha) = p$. ∎

Definition 4.3 Let $A_1 : t_1 \leftrightarrow_{\hat{R}_1} \cdots \leftrightarrow_{\hat{R}_1} t_n$ be a proof in $\hat{R}_1$, and let $\alpha_i : t_i \leftrightarrow_{\hat{R}_1} t_{i+1}$. Suppose that $i \leq j$. We say t_k is *between*

$$\begin{cases} t_i \textit{ and } t_j \text{ (or } t_j \textit{ and } t_i\text{)} & \text{when } i \leq k \leq j, \\ \alpha_i \textit{ and } t_j \text{ (or } t_j \textit{ and } \alpha_i\text{)} & \text{when } i+1 \leq k \leq j, \\ t_i \textit{ and } \alpha_j \text{ (or } \alpha_j \textit{ and } t_i\text{)} & \text{when } i \leq k \leq j, \\ \alpha_i \textit{ and } \alpha_j \text{ (or } \alpha_j \textit{ and } \alpha_i\text{)} & \text{when } i+1 \leq k \leq j. \end{cases}$$

A reduction $\alpha_k : t_k \rightarrow_{\hat{R}_1} t_{k+1}$ (or $\alpha_k : t_k \leftarrow_{\hat{R}_1} t_{k+1}$) is between terms (or a reduction and a term, or reductions) if both t_k and t_{k+1} are between terms (or a reduction and a term, or reductions).

Definition 4.4 Let A_1 be a proof in $\hat{R}_1$. Relations $\perp\!\!\!\perp$, $\perp\!\!\!\perp_1$, $\perp\!\!\!\perp_2$ and $\angle\!\!\angle_2$ on reductions α, β in A_1 are defined as follows:

- $\alpha \perp\!\!\!\perp \beta$ if $\alpha \perp\!\!\!\perp_1 \beta$ or $\alpha \perp\!\!\!\perp_2 \beta$.
- $\alpha \perp\!\!\!\perp_1 \beta$ if
  1. $p(\alpha) \perp p(\beta)$, and
  2. $p(\gamma) \not\leq \wedge(p(\alpha), p(\beta))$ for all reductions γ between α and β.
- $\alpha \perp\!\!\!\perp_2 \beta$ if either $\alpha \angle\!\!\angle_2 \beta$ or $\beta \angle\!\!\angle_2 \alpha$.
- $\alpha \angle\!\!\angle_2 \beta$ if there are a term t between α and β, positions $p \in P(t)$ and $q \in P(t/p)$ s.t.
  1. t/p is a quasi-ground normal form of $\hat{R}$ wrt q,
  2. $p(\alpha) \geq p \cdot q$,
  3. $p(\beta) \geq p$,
  4. $p(\gamma_1) \not< p \cdot q$ for all reductions γ_1 between α and t, and
  5. $p(\gamma_2) \not< p$ for all reductions γ_2 between t and β.

For a proof A in $\hat{R}$ and reductions α, β in A, we also write $\alpha \perp\!\!\!\perp \beta$, $\alpha \perp\!\!\!\perp_1 \beta$, $\alpha \perp\!\!\!\perp_2 \beta$ and $\alpha \angle\!\!\angle_2 \beta$ if $\alpha^\flat \perp\!\!\!\perp \beta^\flat$, $\alpha^\flat \perp\!\!\!\perp_1 \beta^\flat$, $\alpha^\flat \perp\!\!\!\perp_2 \beta^\flat$ and $\alpha^\flat \angle\!\!\angle_2 \beta^\flat$, respectively.

Reductions α and β are *independent* if $\alpha \perp\!\!\!\perp \beta$. If $\alpha \perp\!\!\!\perp_2 \beta$, the term t in the definition of $\angle\!\!\angle_2$ is called a *split of α and β*. Moreover, the term in the definition of $\angle\!\!\angle_2$ is called the *body* of the split t.

Example 4.2 Let $\hat{R}$ be that in example 3.3. Consider the following proof in $\hat{R}_1$:

$$\underline{f(g(f(a,a),a,a),a)} \overset{\alpha_1}{\rightarrow}_{\hat{S}'} g(\underline{f(a,a)},a,a) \overset{\alpha_2}{\rightarrow}_{\hat{S}} g(a,a,a) \overset{\alpha_3}{\leftarrow}_{\hat{S}} g(a,\underline{f(a,a)},a) \overset{\alpha_4}{\leftarrow}_{\hat{S}} g(a,f(a,a),\underline{f(a,a)}),$$

where underlines indicate the redexes contracted. Then, $\alpha_2 \perp\!\!\!\perp_1 \alpha_3$, $\alpha_3 \perp\!\!\!\perp_1 \alpha_4$, and $\alpha_4 \perp\!\!\!\perp_1 \alpha_2$. Furthermore, $\alpha_3 \not\perp\!\!\!\perp_2 \alpha_1$ and $\alpha_4 \not\perp\!\!\!\perp_2 \alpha_1$ since $g(f(a,a),a,a)$ is a quasi-ground normal form wrt p_1 and p_2, where p_i are the positions of a^i in $g(f(a,a),a^1,a^2)$.

Suppose that $A \mapsto A'$. For any reduction α' in A', there is a corresponding reduction α in A. If α is in the replacement sequence, then we can find α in the peak that α' originates from (indicated by dash-arrows in figure 2 and figure 3); otherwise, α is the same reduction as α'. In this case, α' is called a *descendant* of α. Moreover, for a peak elimination process $A_1 \mapsto \cdots \mapsto A_n$ and reductions α_i in A_i, we say α_n is a descendant of α_1 if α_{i+1} is a descendant of α_i for each $1 \le i < n$.

4.2 Properties of independence

Theorem 4.1 Let $\hat{R}$ be compatible and let A, A' be proofs in $\hat{R}$. Suppose that $A \mapsto A'$ and that reductions α', β' in A' are descendants of α, β in A, respectively. Then, $\alpha \perp\!\!\!\perp \beta$ implies $\alpha' \perp\!\!\!\perp \beta'$.

Proof From lemmas A.6, A.7, and A.9 in Appendix. ∎

Lemma 4.4 Let A be a proof in $\hat{R}$ with reductions α and β. Suppose that α is in a subproof of β. Then, $\alpha \not\perp\!\!\!\perp \beta$.

Proof Since $p(\alpha^b) > p(\beta^b)$, $\alpha \not\perp\!\!\!\perp_1 \beta$. Suppose that there is a split t of α^b and β^b, where the body t/p is a quasi-ground normal form wrt q. Then, $p(\beta^b) \perp p \cdot q$ from lemma 4.3 and A.1. Also, $p(\alpha^b) \perp p \cdot q$ since $p(\alpha^b) \ge p(\beta^b)$. This contradicts the definition of $\perp\!\!\!\perp_2$. ∎

Lemma 4.5 Let A be a proof in $\hat{R}$ with reductions α and β. If $\alpha \perp\!\!\!\perp \beta$ and β' is in a subproof of β, then $\alpha \perp\!\!\!\perp \beta'$.

Proof Suppose that β^b is between α^b and β'^b. It is obvious that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha \perp\!\!\!\perp_1 \beta'$. If $\alpha \perp\!\!\!\perp_2 \beta$, then any split t of α^b and β^b is also a split of α^b and β'^b.

Suppose that β'^b is between α^b and β^b. It is obvious that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha \perp\!\!\!\perp_1 \beta'$. Suppose that $\alpha \perp\!\!\!\perp_2 \beta$ and that t is a split of α and β, where the body t/p is a quasi-ground normal form wrt q. The result is obvious when t is between α^b and β'^b. If t is between β^b and β, $t/p(\beta^b)$ is a redex from lemma 4.3. Thus, $p(\beta^b) \perp p \cdot q$ from lemma A.1 in Appendix. Hence, $p(\alpha^b) \ge p \cdot q$ so $\alpha \perp\!\!\!\perp_1 \beta$. Therefore, $\alpha \perp\!\!\!\perp_1 \beta'$. ∎

Lemma 4.6 Let A, A' be proofs such that $A \mapsto A'$. Suppose that $\alpha'_1, \cdots, \alpha'_m$ in A' are descendants of α in A. Then $m_1 \neq m_2 \Rightarrow \alpha'_{m_1} \perp\!\!\!\perp \alpha'_{m_2}$.

Proof Notations are the same as those used in definition 3.5 or lemma 3.3. It is clear that α has multiple descendants only when α is duplicated, i.e., P_C is applied to a peak $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{S}} C[\hat{l}\theta] \equiv C[\hat{l}'\theta] \overset{\nabla}{\rightarrow}_{\hat{S}'} C[\hat{r}'\theta]$ in A and when α is in a subproof of either reduction making the peak.

The replacement sequence for the peak is a collection of right connecting proofs, B_q. If α'_{m_1} is in B_q-part and α'_{m_2} is in B'_q-part such that $q \neq q'$, then $\alpha'_{m_1} \perp\!\!\!\perp_1 \alpha'_{m_2}$. Suppose that $q = q'$ and assume $\hat{r}/q \equiv y_i \in V$. B_q is as follows:

$$y_i\theta \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\hat{S}'}\theta \overset{\nabla^*}{\Leftrightarrow}_{\hat{R}} C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]\mathcal{T}_{\hat{S}'}\theta,$$

where $A_p : y_i\theta \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} C'_q[g_1, \cdots, g_{m'}]$ and $A_{p_k} : g_k \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} y'_{j_k}\theta$ are left connecting proofs of the peak. Note that p_k is the position of x'_{j_k} in $\hat{l}'$. Since $\hat{l}/p_k = g_k$ are ground terms, A_{p_k} themselves are also subproofs of $C[\hat{l}'\theta] \overset{\nabla}{\rightarrow}_{\hat{S}'} C[\hat{r}'\theta]$ for $1 \le k \le m'$.

The other left connecting proof A_p is rewritten as follows:

$$y_i\theta \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} x_i\theta \equiv C'_p[x'_j\theta, \cdots, x'_{j+j'}\theta] \overset{\nabla^*}{\Leftrightarrow}_{\hat{R}} C'_p[y'_j\theta, \cdots, y'_{j+j'}\theta].$$

Then, A_{p_k} and $x'_{j+k'}\theta \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} y'_{j+k'}\theta$ can not originate from the same subproof for any k, k'. For $p_{x'_{j+k'}} \ge p$ but $p_k \perp p$, where $p_{x'_{j+k'}}$ is the position of $x'_{j+k'}$ in $\hat{l}'$.

It is clear that subproofs $x'_{j+k'}\theta \overset{\nabla^*}{\leftrightarrow}_{\hat{R}} y'_{j+k'}\theta$ originated from different subproofs. Hence, only the following case is possible: α'_{m_1} is in A_{p_k}-part and α'_{m_2} is in $A_{p_{k'}}$-part such that $k \neq k'$. Thus, $\alpha'_{m_1} \perp\!\!\!\perp_1 \alpha'_{m_2}$. The proof is similar for $\hat{r}/q \notin V$. ∎

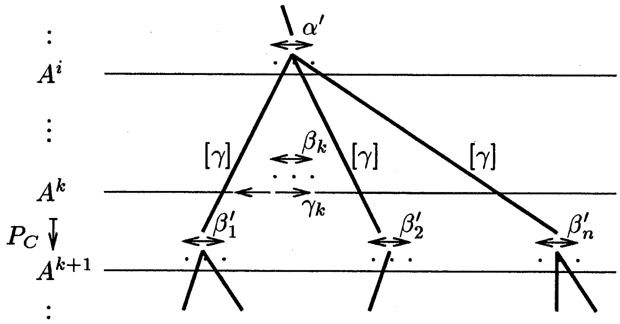

Figure 5: Descendant tree

5 Church-Rosser property of $\hat{R}$

Let R be a compatible TRS, and let $\hat{R}$ be the conditional linearization. Assume that $A^1 : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ is an arbitrary proof in $\hat{R}$ and that $A^1 \mapsto A^2 \mapsto \cdots$ is an arbitrary peak elimination process. The following section will show that the process $A^1 \mapsto A^2 \mapsto \cdots$ terminates. This implies that **CR** holds for $\hat{R}$ by lemma 3.4.

Definition 5.1 The *initial labeling* on each reduction in A^i for $i = 1, 2, \cdots$ is defined as follows:

1. The set of *initial labels* is $\{[\alpha] \mid \alpha \text{ is in } A^1\}$.

2. Each reduction α in A^1 is labeled $[\alpha]$.

3. For each reduction β in A^i for $i \geq 2$, β is labeled $[\alpha]$ if β is a descendant of α in A^1.

Definition 5.2 Let α be a reduction in A^1. The *descendant tree* $T_{[\alpha]}$ *associated with* α is an edge-labeled tree defined as follows:

1. The root vertex is the reduction α in A^1.

2. Let α' in A^i be a vertex of $T_{[\alpha]}$. Suppose that there are $k > i$, satisfying the following conditions:

   (a) In $A^j \mapsto A^{j+1}$, the descendant β_j of α' in A_j is not duplicated for $j = i+1, \ldots, k-1$.

   (b) In $A^k \mapsto A^{k+1}$, the descendant β_k of α' in A_k is duplicated.

   Suppose that β_k is duplicated by γ_k. Then all the descendants $\beta'_1, \cdots, \beta'_n$ in A^{k+1} of β_k are the child vertices of α'. The label of the edges (α', β'_j) is the initial label of γ_k, e.g. $[\gamma]$ (figure 5).

The set of all descendant trees associated with reductions in A_1 in the peak elimination process is called the *descendant forest* of the peak elimination process.

Note that all vertices in $T_{[\alpha]}$ are descendants of α in A_1.
We classify P_C into the following:

(P_C^1) The replacement sequence is empty.

(P_C^2) The replacement sequence is not empty.

Lemma 5.1 Suppose that P_C^2 is applied in $A^i \mapsto A^{i+1}$. Then, there are a reduction β in A^{i+1} and a descendant tree $T_{[\alpha]}$ such that β is a vertex of $T_{[\alpha]}$. ∎

A path of $T_{[\alpha]}$ is a sequence of edges starting from the root. A *label path* is the sequence of labels of edges in a path. The set of all label paths of $T_{[\alpha]}$ is denoted by $Lpath_{T_{[\alpha]}}$.

Lemma 5.2 Let $[\gamma_1], [\gamma_2], \cdots \in Lpath_{T_{[\alpha]}}$. Then, $[\gamma_i] \neq [\gamma_j]$ for all $i \neq j$.

Proof Suppose that $[\gamma_i] = [\gamma_j] = [\beta]$ for some $i \neq j$. There exist descendants α_1, α_2 of α and descendants β_1, β_2, β_3 of β as shown in figure 6, where α_2 (β_2) is a descendant of α_1 (β_3).

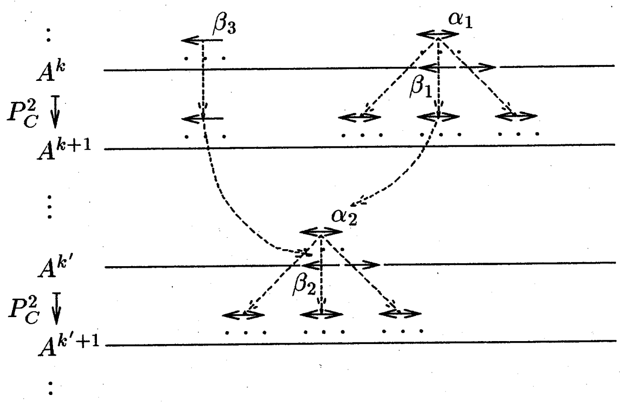

Figure 6: Proof of lemma 5.5

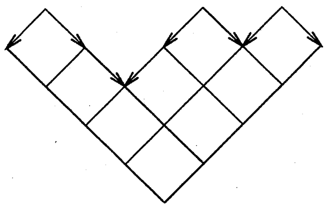

Figure 7: Mass

Since β_1 and β_3 are descendants of the same reduction, $\beta_1 \perp\!\!\!\perp \beta_3$ from lemma 4.6 and theorem 4.1. Since α_1 is in a subproof of β_1, $\alpha_1 \perp\!\!\!\perp \beta_3$ from lemma 4.5. Hence, $\alpha_2 \perp\!\!\!\perp \beta_2$ from theorem 4.1. However, $\alpha_2 \not\perp\!\!\!\perp \beta_2$ by lemma 4.4. This leads to a contradiction. ∎

Lemma 5.3 For each initial label $[\alpha]$, the descendant tree $T_{[\alpha]}$ is finite. Therefore, the descendant forest of the peak elimination process is finite.

Proof From lemma 5.2, each path of $T_{[\alpha]}$ has finite length (bounded by the number of reductions in A). Since $T_{[\alpha]}$ is obviously finitely branching, *König*'s lemma shows that $T_{[\alpha]}$ is finite. ∎

Lemma 5.4 In the peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$, only finitely many peak eliminations occur with P_C^2.

Proof From lemma 5.3 and 5.1. ∎

Definition 5.3 Let $B : t_0 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof and let $\gamma_i : t_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1}$. A reduction γ_i is right-oriented (left-oriented) if $\gamma_i : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ ($\gamma_i : t_i \overset{\nabla}{\leftarrow}_{\hat{R}} t_{i+1}$). The *height* of γ_i is defined as follows:

$$height(\gamma_i) = \#\{\gamma_j \mid \gamma_j \text{ is left-oriented and } j < i\}.$$

The *mass* of B is defined as

$$mass(B) = \sum_{\text{right-oriented } \gamma_i} height(\gamma_i).$$

That is, the mass is the number of tiles as shown in figure 7.

Lemma 5.5 Let B, B' be proofs such that $B \mapsto B'$ with $P_\perp$, $P_<$ or P_C^1. Then, $mass(B) > mass(B')$. ∎

Corollary 5.1 Let $B_1 \overset{P_1}{\mapsto} B_2 \overset{P_2}{\mapsto} B_3 \overset{P_3}{\mapsto} \cdots$ be a peak elimination process starting from B_1. If each P_i is any of $P_\perp$, $P_<$ or P_C^1, then the length of the process is finite. ∎

Theorem 5.1 Any peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$ terminates.

Proof From corollary 5.4 and corollary 5.1. ∎

Corollary 5.2 Let R be a compatible TRS, and let $\hat{R}$ be the conditional linearization of R. Then, $\hat{R}$ is **CR**. Therefore, R is **UN**. ∎

6 Conclusion and Future Work

We have presented a complete proof of Chew's theorem which states that compatible term rewriting systems enjoy the unique normal form property. We exploited a technique introduced in [TO94] and partly extended to apply it to duplicating systems by introducing the notion of independence.

There are many interesting non-linear term rewriting systems that have (or believed to have) the unique normal form property, for example, the system of combinatory logic with surjective pairing[KdV89], non-ω-overlapping term rewriting systems[Oga92], etc. Despite of the importance, Chew's theorem is not powerful enough to infer the unique normal form properties of these systems. Therefore, we would like to relax the condition of compatibility. We also interested in extending the result to higher order rewriting systems.

Acknowledgements

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A Appendix

In the following, it is assumed that $\hat{R}$ is a compatible left-right separated CTRS.

A.1 Innocent swap

Lemma A.1 Let t be a term such that t/p is a quasi-ground normal form wrt q. If $p' \not< p$ and t/p' is a redex, then $p' \perp p \cdot q$. ∎

Lemma A.2 Let t be a term such that t/p is a quasi-ground normal form wrt q. If there is a reduction $\alpha : t \overset{\nabla}{\rightarrow}_{\hat{R}} t'$ such that $p(\alpha) \not< p$, then t'/p is also a quasi-ground normal form wrt q. ∎

Lemma A.3 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof with a reduction $\gamma : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, and let $t_i \equiv t_i^1 \leftrightarrow_{\hat{R}_1} \cdots \leftrightarrow_{\hat{R}_1} t_i^m \rightarrow_{\hat{R}_1} t_{i+1}$ be the flat proof of $t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$. Suppose that there exist reductions α, β in A satisfying (1) $\alpha^b \perp\!\!\!\perp_2 \beta^b$, (2) both t_i and t_{i+1} in A^b are between α^b and β^b, and (3) there exists j s.t. t_i^j is a split of α^b and β^b. Then t_{i+1} is also a split of α^b and β^b.

Proof Let t_i^j/p be the body of t_i^j. Since γ^b is between α^b and β^b, $p(\gamma^b) \not< p$. From lemmas 4.3, A.1, and A.2, the result follows. ∎

Lemma A.4 Let A be a proof in $\hat{R}$ and let α, β be reductions in A. Suppose that $\alpha \perp\!\!\!\perp_2 \beta$ and t is a split of α^b and β^b, where the body t/p is a quasi-ground normal form wrt q. Assume there is a position $p' \geq p$ satisfying (1) t/p' is a redex, (2) for each reduction γ between t and β^b, $p(\gamma) \not< p'$, and (3) $p(\beta^b) \geq p'$. Then, $\alpha \perp\!\!\!\perp_1 \beta$.

Proof Since t/p' is a redex, $p' \perp p \cdot q$ from lemma A.1, so $p(\beta) \perp p \cdot q$. Thus, $\alpha \not\perp\!\!\!\perp_2 \beta$. Hence, $p(\alpha^b) \geq p \cdot q$, and $p(\gamma') \not< p \cdot q$ for each reduction γ' between α^b and t. Moreover, $p(\beta^b) \geq p'$, and $p(\gamma) \not< p'$ for each reduction γ between t and β^b. Therefore, $\alpha \perp\!\!\!\perp_1 \beta$. ∎

Definition A.1 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ be a proof, and let $\gamma_1 : t_{i-1} \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_i$ and $\gamma_2 : t_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1}$ be reductions such that $p(\gamma_1) \perp p(\gamma_2)$. Suppose that either $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ or $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$ holds.

When $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, the *innocent swap* of γ_1 and γ_2 is a transformation that changes the order of γ_1 and γ_2 in A, i.e., A is transformed to

$$A' : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1} \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n,$$

where $\gamma'_1 : t'_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1}$ ($\gamma'_2 : t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t'_i$) is a reduction with the same rule, position, and subproofs as γ_1 (γ_2). In the case $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$, an innocent swap is similarly defined. For a reduction α in A, the *descendant* α' in A' is defined in the same way as that of peak eliminations by $P_\perp$.

Lemma A.5 Let α, β be reductions in a proof A. Suppose that A' is obtained by applying an innocent swap to A and that α' and β' are the descendants of α and β, respectively. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$. Assume that the innocent swap is applied to $\gamma_1 : t_{i-1} \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_i$, $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$. Let $p_1 = p(\gamma_1)$ and $p_2 = p(\gamma_2)$. Let $C[] \equiv t_{i-1}[p_1 \leftarrow \square, p_2 \leftarrow \square]$, $t_{i-1} \equiv C[s_1, s_2]$, $t_i \equiv C[s'_1, s_2]$, and $t_{i+1} \equiv C[s_1, s'_2]$. We divide A and A' into the following proofs:

- $A_1 : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_{i-1}$, $A_2 : t_{i+1} \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$,
- $B_1 : (t_{i-1} \equiv) C[s_1, s_2] \overset{\nabla}{\leftrightarrow}_{\hat{R}} C[s'_1, s_2] (\equiv t_i)$, $B_2 : (t_i \equiv) C[s'_1, s_2] \overset{\nabla}{\rightarrow}_{\hat{R}} C[s'_1, s'_2] (\equiv t_{i+1})$,
- $B'_2 : (t_{i-1} \equiv) C[s_1, s_2] \overset{\nabla}{\rightarrow}_{\hat{R}} C[s_1, s'_2] (\equiv t'_i)$, $B'_1 : (t'_i \equiv) C[s_1, s'_2] \overset{\nabla}{\leftrightarrow}_{\hat{R}} C[s'_1, s'_2] (\equiv t_{i+1})$.

where A (A') is the concatenation of A, B_1, B_2 and A_2 (A, B'_2, B'_1 and A_2).

Since an innocent swap preserves the positions of reductions, it is obvious that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha' \perp\!\!\!\perp_1 \beta'$.

We will now prove that $\alpha \perp\!\!\!\perp_2 \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Without loss of generality, it can be assumed that α^b is on the "left-hand side" of β^b in A^b. Let t be a split of α^b and β^b in A^b, where the body t/p is a quasi-ground normal form wrt q. Then the following cases exist:

1. Both α and β are in any of A_1, A_2, B_1 or B_2.
2. α is in A_1, β is in B_1.
3. α is in A_1, β is in B_2.
4. α is in A_1, β is in A_2.
5. α is in B_1, β is in B_2.
6. α is in B_1, β is in A_2.
7. α is in B_2, β is in A_2.

Case 1. The result is obvious.
Case 2. If the split t is in $A_1^\flat$, then it is obvious. If the split t is in $B_1^\flat$, then $t' \equiv t[p_2 \leftarrow s_2']$ in ${B_1'}^\flat$ is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$ from lemma A.2. Thus, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 3. If the split t is in $A_1^\flat$, then it is obvious. If the split t is in either $B_1^\flat$ or $B_2^\flat$, t/p_2 is a redex by lemma 4.3. For all reductions γ between t and $\beta^\flat$, $p(\gamma) \not< p_2$ since γ is in either $B_1^\flat$ or $B_2^\flat$. Suppose that $p_2 \geq p$. Then, $\alpha \perp\!\!\!\perp_1 \beta$ from lemma A.4, so $\alpha' \perp\!\!\!\perp_1 \beta'$. Next, suppose that $p_2 \not\geq p$. Since $p_2 \leq p(\beta^\flat)$ and $p \leq p(\beta^\flat)$, $p_2 < p$. Hence, $t[p_1 \leftarrow s_1]$ in B_2' is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 4. If the split t is in either $A_1^\flat$ or $A_2^\flat$, then it is obvious. If the split t is in $B_1^\flat$, then $t[p_2 \leftarrow s_2']$ in ${B_1'}^\flat$ is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$ from lemma A.2. If the split t is in ${B_2}^\flat$, then t_{i+1} is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma A.3. Thus, t_{i+1} is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 5. Since $p(\alpha^\flat) \geq p_1$, $p(\beta^\flat) \geq p_2$ and $p_1 \perp p_2$, $\alpha' \perp\!\!\!\perp_1 \beta'$.
Case 6. If the split t is in $A_2^\flat$, then it is obvious. Assume that the split t is in $B_1^\flat$. Since $p \leq p(\alpha^\flat)$ and $p_1 \leq p(\alpha^\flat)$, $p \not\perp p_1$. Thus, $p_2 \not< p$ from the assumption $p_1 \perp p_2$. Hence, $t[p_2 \leftarrow s_2']$ in ${B_1'}^\flat$ is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$ from lemma A.2. Next, assume the split t is in $B_2^\flat$. Then t_{i+1} is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma A.3. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 7. If the split t is in $A_2^\flat$, then it is obvious. If the split t is in $B_2^\flat$, then t_{i+1} in $A_2^\flat$ is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma A.3. Thus, t_{i+1} is a split of ${\alpha'}^\flat$ and ${\beta'}^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$. ∎

A.2 Proof of theorem 4.1

Lemma A.6 Let A, A' be proofs in $\hat{R}$ such that $A \overset{P_\perp}{\mapsto} A'$. Let reductions α', β' in A' be descendants of α, β in A. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof From lemma A.5. ∎

Lemma A.7 Let A, A' be proofs in $\hat{R}$ such that $A \overset{P_<}{\mapsto} A'$. Let reductions α', β' in A' be descendants of α, β in A. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ be the peak that $P_<$ is applied to in $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$. Let $\gamma_1 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i-1}$ and $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ and suppose $p(\gamma_1) < p(\gamma_2)$.

Let $l \rightarrow r \ \Leftarrow\ x_1 = y_1, \cdots, x_m = y_m$ be the rule for the reduction γ_2, where $t_i \equiv C[l\theta]$ and $t_{i+1} \equiv C[r\theta]$. Suppose that γ_1 occurs below the j-th substitution part of γ_2 and that $\gamma_2' : t_{i-1} \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ is the replacement sequence for the peak. Then, the flattening of A at γ_2 is

$$fA : \cdots \underline{t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i} \equiv t_i^0 \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^{j-1} \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^j \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^m \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1} \cdots,$$

where $t_i^{k-1} \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^k$ corresponds to the subproof $x_k\theta \overset{\nabla *}{\leftrightarrow}_{\hat{R}} y_k\theta$ of γ_2, and the flattening of A' at γ_2' is

$$fA' : \cdots t_{i-1} \equiv t_{i-1}^0 \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \underline{t_{i-1}^{j-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i^{j-1}} \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^j \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^m \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1} \cdots,$$

where $t_{i-1}^k \equiv t_i^k[p(\gamma_1) \leftarrow t_{i-1}/p(\gamma_1)]$.

Thus, fA' is obtained from fA by repeated applications of innocent swaps to $flat(\gamma_1)$, γ_1^1 (a descendant of $flat(\gamma_1)$), γ_1^2 (a descendant of γ_1^1), $\cdots$ with their right adjacent reductions since $p(\gamma) \perp p(flat(\gamma_1))$ for each reduction γ in $t_i^0 \overset{\nabla *}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_i^{j-1}$. From lemma 4.2 and lemma A.5, independence is preserved.

The proof is similar when $p(\gamma_1) > p(\gamma_2)$. ∎

Lemma A.8 Let $t_1 \overset{\nabla}{\leftarrow}_{\hat{R}} t \overset{\nabla}{\rightarrow}_{\hat{R}} t_2$ be a critical peak and let A_p be a left connecting proof of the peak. Suppose that α, β are reductions in subproofs of either reduction making the peak such that the corresponding reductions, denoted by α_p, β_p, are in A_p. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha_p \perp\!\!\!\perp \beta_p$.

Proof Suppose that A_p is of the form $s' \overset{\nabla *}{\leftrightarrow}_{\hat{R}} s \equiv C_p[u_1, \cdots, u_n] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C_p[u_1', \cdots, u_n']$, where $s \overset{\nabla *}{\leftrightarrow}_{\hat{R}} s$ and $u_i \overset{\nabla *}{\leftrightarrow}_{\hat{R}} u_i'$ are subproofs of the reductions making the peak. The following cases exist:

1. Both α_p and β_p are in either $s' \overset{\nabla *}{\leftrightarrow}_{\hat{R}} s$ or $C_p[\cdots, u_i, \cdots] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C_p[\cdots, u_i', \cdots]$ for some $i = 1, \ldots, n$.

2. α_p is in $C_p[\cdots, u_i, \cdots] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C_p[\cdots, u_i', \cdots]$, β_p is in $C_p[\cdots, u_j, \cdots] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C_p[\cdots, u_j', \cdots]$, and $i \neq j$.

3. α_p is in $s' \overset{\nabla *}{\leftrightarrow}_{\hat{R}} s$ and β_p is in $C_p[\cdots, u_i, \cdots] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C_p[\cdots, u_i', \cdots]$ (or vice versa).

In case 1, it is obvious. In case 2, $\alpha_p \perp\!\!\!\perp_1 \beta_p$. Let us consider case 3. The flat proofs of reductions making the peak can be written as follows.

$$t_1 \leftarrow_{\hat{R}_1} \cdots \leftrightarrow^*_{\hat{R}_1} C_1[s'] \leftrightarrow^*_{\hat{R}_1} C_1[s] \leftrightarrow^*_{\hat{R}_1} t \qquad \text{: the flat proof of } t_1 \overset{\nabla}{\leftarrow}_{\hat{R}} t$$
$$t \leftrightarrow^*_{\hat{R}_1} C_2[C_p[u_1,\cdots,u_n]] \Leftrightarrow^*_{\hat{R}_1} C_2[C_p[u'_1,\cdots,u'_n]] \leftrightarrow^*_{\hat{R}_1} \cdots \rightarrow_{\hat{R}_1} t_2 \quad \text{: the flat proof of } t \overset{\nabla}{\rightarrow}_{\hat{R}} t_2$$

where the position of $\square$ both in $C_1[]$ and in $C_2[]$ is p. Let p' be the position of reductions making the peak. Then, for all reductions γ in $C_1[s] \leftrightarrow^*_{\hat{R}_1} t$ or $t \leftrightarrow^*_{\hat{R}_1} C_2[C_p[u_1,\cdots,u_n]]$, $p(\gamma) \perp p'{\cdot}p$ from the definition of flattening. Therefore, the result follows. The proof is similar when A_p is of the form $C'_p[s'_1,\cdots,s'_n] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C'_p[s_1,\cdots,s_n] \equiv u \overset{\nabla *}{\leftrightarrow}_{\hat{R}} u'$. ∎

Lemma A.9 Let A, A' be a proof in $\hat{R}$ such that $A \overset{PC}{\mapsto} A'$. If reductions α, β in A have descendants α', β' in A', then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ and $t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ be the critical peak eliminated in $A \mapsto A'$. Let $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$, $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$ and $p' = p(\gamma_1) = p(\gamma_2)$.

Without loss of generality, it can be assumed that α^b is on the "left-hand side" of β^b in A^b. We divide A into the following proofs: $A_1 : t_1 \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_{i-1}$,q $A_2 : t_{i+1} \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_n$, $B_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_i$, and $B_2 : t_i \overset{\nabla}{\rightarrow}_{\hat{R}} t_{i+1}$, where A (A') is the concatenation of A_1, B_1 and A_2 (A, B_2, A_2).

Let $B : t_{i-1} \overset{\nabla *}{\leftrightarrow}_{\hat{R}} t_{i+1}$ be the replacement sequence for the critical peak. Note that for each reduction γ in B, $p(\gamma) \geq p'$. Then the following cases exist:

1. Both α and β are either A_1 or A_2.
2. α is in A_1 and β is in A_2.
3. α and β are in either B_1 or B_2.
4. α is in A_1 and β is in either B_1 or B_2 (or, α is in either B_1 or B_2 and β is in A_2).

Case 1. The result is obvious.
Case 2. Since $p(\gamma) \geq p'$ for each reduction γ in B, it follows that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha' \perp\!\!\!\perp_1 \beta'$. Assume that $\alpha \perp\!\!\!\perp_2 \beta$ and that t is a split of α^b and β^b. If t is in B_1^b, then t_{i-1} is also a split from lemma A.3. If t is in B_2^b, t_{i+1} is also a split from lemma A.3. Thus, we can assume that t is in either A_1^b or A_2^b. Since $p(\gamma) \geq p'$ for each reduction γ in B^b, we have $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 3. Recall that B is a collection of the right connecting proofs B_q of the peak. Suppose that α', β' are in B_q-part, $B_{q'}$-part of B, respectively. If $q \neq q'$, then $\alpha' \perp\!\!\!\perp_1 \beta'$. Hence, suppose that $q = q'$. Recall that B_q is a collection of left connecting proofs. Suppose that B_q is as follows:

$$s \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C[g_1,\cdots,g_m] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C[u_1,\cdots,u_m],$$

where $A_s : s \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C[g_1,\cdots,g_m]$ and $A_{u_i} : C[\cdots,g_i,\cdots] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} C[\cdots,u_i,\cdots]$ are left connecting proofs. If α' (or β') is in A_s-part and β' (or α') is in A_{u_i}-part, then $\alpha' \perp\!\!\!\perp_2 \beta'$ by lemma 3.3. If α', β' are in A_{u_i}-part, A_{u_j}-part respectively such that $i \neq j$, then $\alpha' \perp\!\!\!\perp_1 \beta'$. The remaining case is both α' and β' are in either A_s-part or A_{u_i}-part, and the result follows from lemma A.8. The proof is similar when B_q is of the form $C[s_1,\cdots,s_n] \overset{\nabla *}{\Leftrightarrow}_{\hat{R}} C[g_1,\cdots,g_n] \overset{\nabla *}{\leftrightarrow}_{\hat{R}} u$.
Case 4. From symmetry, we can assume that α is in A_1 and that β is in either B_1 or B_2. If β is either γ_1 or γ_2, β' does not exist. Thus, β is in a subproof of γ_1 or γ_2. Suppose that $\alpha \perp\!\!\!\perp_1 \beta$. Then $p(\gamma) \not\leq \wedge(p(\alpha^b), p')$ for each reduction γ between α^b and t_{i-1} since γ_1^b is between α^b and β^b, and $p' \leq p(\beta^b)$. Since $p(\gamma') \geq p'$ for each reduction γ' in B'^b, we have $\alpha' \perp\!\!\!\perp_1 \beta'$.

Assume that $\alpha \perp\!\!\!\perp_2 \beta$ and that t is a split of α^b and β^b, where the body t/p is a quasi-ground normal form wrt q. If t is in A_1^b, then $\alpha' \perp\!\!\!\perp_2 \beta'$ since $p(\gamma') \geq p'$ for each reduction γ' in B_1^b. Suppose that t is in either B_1^b or B_2^b. Since γ_1^b is between α^b and t, $p' \not< p$. Also, $p' \not\perp p$ since $p' \leq p(\beta^b)$ and $p \leq p(\beta^b)$. Hence, $p' \geq p$, so $\alpha \perp\!\!\!\perp_1 \beta$ from lemma A.4. Thus, $\alpha' \perp\!\!\!\perp_1 \beta'$. ∎