Some Results on the CR property of non-E-overlapping and depth-preserving TRS's (Theory of Rewriting Systems and Its Applications)

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Some Results on the CR property of non-E-overlapping and depth-preserving TRS's

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Abstract

A term rewriting system (TRS) is said to be depth-preserving if for any rewrite rule and any variable appering in the both sides, the maximal depth of the variable occurrences in left-hand-side is greater than or equal to that of the variable occurrences in the right-hand-side, and to be strongly depth-preserving if it is depth-preserving and for any rewrite rule and any variable appering in the left-hand-side, all the depths of the variable occurrences in the left-hand-side are the same. This paper shows that there exists non-E-overlapping and depth-preserving TRS's which do not satisfy the Church-Rosser property, but all the non-E-overlapping and strongly depth-preserving TRS's satisfy the Church-Rosser property.

1 Introduction

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules). A TRS is Church-Rosser (CR) if any two interconvertible terms reduce to some common term by applications of the rewrite rules. Church-Rosser is an important property in various applications of TRS's and has received much attention so far [1-5,8-15]. Although the CR property is undecidable for general TRS's, many sufficient conditions for ensuring this property have been obtained [1,3,5,8-15]. For example, for noetherian (i.e. terminating) TRS's, the CR property is decidable and reduces to joinability of the critical pairs [5], and for nonterminating and linear TRS's, some sufficient conditions (e.g., nonoverlapping) have been given [3, 11].

On the other hand, for nonlinear and nonterminating TRS's, only a few results on the CR property have been obtained. Our previous paper [9,10,13] may be pioneer ones which have first given nontrivial conditions for the CR property. In [10], it was shown that if TRS's are non-E-overlapping (stronger than nonoverlapping) and right-ground, then they are CR. Here, a TRS is right-ground if no variables occur in the right-hand-side of a rewrite rule. This result is compared with an example given by G.Huet [3], i.e., a nonoverlapping, right-ground and non-CR TRS with the three rules: \( f(x, x) \rightarrow a, f(x, g(x)) \rightarrow b, c \rightarrow g(c) \). Here, \( f, g, a, b, c \) are function symbols and \( x \) is a variable. The above result was extended in [9,13,14,15] and it was shown that if TRS's are non-E-overlapping and simple-right-linear, then they are CR. Here, a TRS is simple-right-linear if for any rewrite rule, the right-hand-side is linear (i.e., any variable occurs at most once in the term) and no variables occurring more than once in the left-hand-side occur in the right-hand-side. Moreover, it was shown that even if simple-right-linear TRS's are E-overlapping, some additional conditions ensure that they are CR [9,13,15].

However, these results were restricted to those on the CR property of subclasses of right-linear TRS's. On the other hand, if we omit the right-linearity condition, then it has been shown that
only the non-E-overlapping condition is insufficient for ensuring the CR property of TRS's. For example, the following non-E-overlapping TRS \( R_1 \) is not CR: \( R_1 = \{ f(x, x) \rightarrow a, g(x) \rightarrow f(x, g(x)), c \rightarrow g(c) \} \) given by Barendregt and Klop. Here, \( f, g, a, c \) are function symbols and \( x \) is a variable.

In this paper, we consider the CR property of nonlinear, nonterminating and depth-preserving TRS's. Here, a TRS is depth-preserving if for each rule \( \alpha \rightarrow \beta \) and any variable \( x \) appearing in both \( \alpha \) and \( \beta \), the maximal depth of the \( x \) occurrences in \( \alpha \) is greater than or equal to that of the \( x \) occurrences in \( \beta([6]) \). For example, TRS \( R_2 = \{ f(x, g(x)) \rightarrow h(k(x), x) \} \), where \( x \) is a variable, is depth-preserving, since the maximal depths of the \( x \) occurrences of the left-hand-side and of the right-hand-side are 2 and 2, respectively.

We first show that only the non-E-overlapping and depth-preserving properties are insufficient for ensuring the CR property. That is, the following TRS \( R_3 \) is not CR: \( R_3 = \{ f(x, x) \rightarrow a, c \rightarrow h(c, g(c)), h(x, g(x)) \rightarrow f(x, h(x, g(c))) \} \) where \( x \) is a variable. Note that \( R_3 \) is non-E-overlapping and depth-preserving, but \( R_3 \) is not CR, since \( c \rightarrow h(c, g(c)) \rightarrow^* a \) and \( c \rightarrow^* h(a, g(a)) \), but \( a \) and \( h(a, g(a)) \) are not joinable. Note that \( R_3 \) is also non-duplicating, since for each rule the number of \( x \) occurrences of the left-hand side \( \geq \) that of the right-hand side. Thus, non-E-overlapping, non-duplicating and depth-preserving conditions do not necessarily ensure CR.

Next, we introduce the notion of strongly depth-preserving property (stronger than the depth-preserving one). A TRS \( R \) is strongly depth-preserving if \( R \) is depth-preserving and for each \( \alpha \rightarrow \beta \) and for any variable \( x \) appearing in \( \alpha \), all the depths of the \( x \) occurrences in \( \alpha \) are the same. For example, TRS \( R_4 = \{ h(g(x), g(x)) \rightarrow f(x, h(x, g(c))) \} \) is strongly depth-preserving, since \( R_4 \) is depth-preserving and all the depths of \( x \) occurrences of the left-hand side are 2.

In this paper, we prove that non-E-overlapping and strongly depth-preserving TRS's are CR. For example, the following three TRS's \( R_1', R_3' \) and \( R_5 \) are ensured to be CR:

\[
R_1' = \{ f(x, x) \rightarrow a, c \rightarrow g(c), g(x) \rightarrow f(x, x) \}
\]

\[
R_3' = \{ f(x, x) \rightarrow a, c \rightarrow h(c, g(c)), h(g(x), g(x)) \rightarrow f(x, h(x, g(c))) \}
\]

\[
R_5 = \{ f(x, x) \rightarrow h(x, z, z) \}
\]

This paper is organized as follows. Section 2 is devoted to definitions. In Section 3, we explain how to prove the above main theorem. In Section 4, we make concluding remarks about the strongly depth-preserving property.

## 2 Definitions

The following definitions and notations are similar to those in [3, 10]. Let \( X \) be a set of variables, \( F \) be a finite set of operation symbols and \( T \) be the set of terms constructed from \( X \) and \( F \).

**Definitions of** \(< O(M), M/u, M[u \leftarrow N], V(M), O_x(M) >\)

For a term \( M \), we use \( O(M) \) to denote the set of occurrences (positions) of \( M \), and \( M/u \) to denote the subterm of \( M \) at occurrence \( u \), and \( M[u \leftarrow N] \) to denote the term obtained form \( M \) by replacing the subterm \( M/u \) by term \( N \), \( V(M) \) to denote the set of variables in \( M \), \( O_x(M) \) to denote the set of occurrences of variable \( x \in V(M) \).

**Definitions of** \(< \bar{O}(M) >\)

\( \bar{O}(M) \) is the set of non-variable occurrences, i.e.,

\( \bar{O}(M) = O(M) - \cup_{x \in V(M)} O_x(M) \)
Definition of $h(M)$ — height of $M$

For a term $M$, $h(M) = \max\{|u| \mid u \in O(M)\}$. $h(M)$ is called "height of $M".

Example.

$h(f(g(x))) = 2$, $h(a) = 0$, $h(g(x)) = 1$.

Definition of $\text{TRS}$

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules).

Definition of $\text{depth-preserving TRS } R$

TRS $R$ is depth-preserving if

$\forall \alpha \rightarrow \beta \in R \forall x \in V(\alpha) \quad \max\{|v| \mid v \in O_x(\beta)\} \leq \max\{|u| \mid u \in O_x(\alpha)\}$

Note

TRS $R$ is depth-preserving if and only if $R$ is locally increasing, i.e., $\exists l \geq 0$ such that $\forall \alpha \rightarrow \beta \in R \forall \sigma \in \mathcal{O}(\alpha)$: $h(\sigma(\alpha)) < h(\sigma(\beta))$ then $h(\sigma(\alpha)) \leq l$

Definition of $\text{strongly depth-preserving TRS } R$

TRS $R$ is strongly depth-preserving if $R$ is depth-preserving and satisfies that $\forall \alpha \rightarrow \beta \in R \forall x \in V(\alpha) \forall \sigma \in \mathcal{O}(\alpha)$:

$|u| = |v|$ hold.

Definition of $\text{parallel-one-step } \leftrightarrow$

$M \leftrightarrow N$ if

$\exists U \subseteq O(M)$ s.t.

$\forall u, v \in U \quad u \neq v \Rightarrow u \nmid v \text{ (disjoint)}$

$\forall u \in U \quad M/u \nmid N/u$

$N = M[u \leftarrow N/u, u \in U]$ where $M/u \nmid N/u$ is one step reduction between $\{M/u, N/u\} = \{\sigma(\alpha), \sigma(\beta)\}$ for some $\alpha \rightarrow \beta \in R$ and $\sigma : X \rightarrow T$.

In this case, let $R(M \leftrightarrow N) = U$.

(Note. $U = \emptyset$ is allowed.)

Example.

Let $R = \{a \rightarrow c\}$, then $f(c, g(a)) \leftrightarrow f(a, g(c))$.

Definition of $R(\gamma)$, $\text{MR}(\gamma)$, $u$-invariant

$R(\gamma) = \{u_i \mid u_i \in R(M_i \leftrightarrow M_{i+1}) (0 \leq i \leq n)\}$

$\text{MR}(\gamma)$ is the set of minimal occurrences in $R(\gamma)$.

For $u \in O(M_0)$, if there exists no $v \in R(\gamma)$ such that $v \leq u$, then $\gamma$ is said to be $u$-invariant.

Definition of $\text{composition, cut of reduction sequence}$

Let $\delta : N_0 \leftrightarrow N_1 \leftrightarrow \cdots \leftrightarrow N_k$. If $M_n = N_0$, then the composition of $\gamma$ and $\delta$, i.e.,

$M_0 \leftrightarrow M_1 \leftrightarrow \cdots \leftrightarrow M_n(=N_0) \leftrightarrow N_1 \leftrightarrow \cdots \leftrightarrow N_k$ is denoted by $(\gamma; \delta)$.

Let $\gamma$ be $u$-invariant, then the cut sequence of $\gamma$ at $u$ is $\gamma/u = (M_0/u \leftrightarrow M_1/u \leftrightarrow \cdots \leftrightarrow M_n/u)$. 

Definition of \( H(\gamma) \) — the height of reduction sequence

\[
H(\gamma) = \text{Max}\{h(M_i) \mid 0 \leq i \leq n\}
\]

Example.
Let \( \gamma : f(c) \to f(g(c)) \to a \), then \( H(\gamma) = h(f(g(c))) = 2 \).

Definition of \( |\gamma|_{p} \) — the number of parallel reduction steps of \( \gamma \)

\( |\gamma|_{p} = n \)

Note.
If \( \delta : M \leftrightarrow M \), then \( |\delta|_{p} = 1 \).

Example.
Let \( \gamma : f(c) \to f(g(c)) \to a \to a \), then \( |\gamma|_{p} = 2 \).

Definition of \( \text{net}(\gamma) \)

\( \text{net}(\gamma) \) is the sequence obtained from \( \gamma \) by removing all \( M_i \to M_{i+1} \) satisfying \( M_i = M_{i+1} \), \( 0 \leq i < n \).

Example.
Let \( \gamma : f(c) \to f(g(c)) \to a \to a \), then \( \text{net}(\gamma) : f(c) \to f(g(c)) \to a \).

Definition of \( |\gamma|_{np} \)

\( |\gamma|_{np} = |\text{net}(\gamma)|_{p} \)

Definitions of \( \text{left}(\gamma, h), \text{right}(\gamma, h), \text{width}(\gamma, h), \text{ldis}(\gamma, h), \text{rdis}(\gamma, h) \)

\[
\text{left}(\gamma, h) \downarrow \overset{\text{def}}{=} \text{left}(\gamma, h) \neq \perp \quad \text{if } \exists i \ (0 \leq i \leq n) \text{ s.t. } h(M_i) = h \text{ and } \forall j (0 \leq j < i) \ h(M_j) < h \\
\text{otherwise}
\]

\[
\text{right}(\gamma, h) \downarrow \overset{\text{def}}{=} \text{right}(\gamma, h) \neq \perp \quad \text{if } \exists i \ (0 \leq i \leq n) \text{ s.t. } h(M_i) = h \text{ and } \forall j \ (i < j \leq n) \ h(M_j) < h \\
\text{otherwise}
\]

\[
\text{left}(\gamma, h) \uparrow \overset{\text{def}}{=} \text{left}(\gamma, h) = \perp
\]

\[
\text{right}(\gamma, h) \uparrow \overset{\text{def}}{=} \text{right}(\gamma, h) = \perp
\]

\[
\text{width}(\gamma, h) = \text{right}(\gamma, h) - \text{left}(\gamma, h)
\]

\[
\text{width}(\gamma, h) = \text{right}(\gamma, h) - \text{left}(\gamma, h')
\]

\[
\text{left}(\gamma, h) \downarrow \text{right}(\gamma, h) \downarrow \text{if } \text{left}(\gamma, h) \downarrow \land \text{right}(\gamma, h) \downarrow \\
\text{left}(\gamma, h) \uparrow \text{right}(\gamma, h) \downarrow \text{if } \text{left}(\gamma, h) \uparrow \land \text{right}(\gamma, h) \downarrow \\
h' = \text{Min}\{h' \mid h' > h \land \text{left}(\gamma, h') \downarrow \}
\text{if } \text{left}(\gamma, h) \downarrow \land \text{right}(\gamma, h) \uparrow \\
\text{left}(\gamma, h') \downarrow \text{right}(\gamma, h) \downarrow \text{if } \text{left}(\gamma, h) \downarrow \land \text{right}(\gamma, h') \downarrow \\
\text{otherwise}
\]
\begin{align*}
\text{ldis}(\gamma, h) & = n - \text{left}(\gamma, h) & \text{if } \text{left}(\gamma, h) \downarrow \\
& = \bot & \text{otherwise} \\
\text{rdis}(\gamma, h) & = \text{right}(\gamma, h) & \text{if } \text{right}(\gamma, h) \downarrow \\
& = \bot & \text{otherwise} \\
\text{ldis}(\gamma, h) \downarrow & \overset{\text{def}}{=} \text{ldis}(\gamma, h) \neq \bot \\
\text{rdis}(\gamma, h) \downarrow & \overset{\text{def}}{=} \text{rdis}(\gamma, h) \neq \bot \\
\text{ldis}(\gamma, h) \uparrow & \overset{\text{def}}{=} \text{ldis}(\gamma, h) = \bot \\
\text{rdis}(\gamma, h) \uparrow & \overset{\text{def}}{=} \text{rdis}(\gamma, h) = \bot \\
\end{align*}

In Fig. 1, we illustrate \textit{width}, \textit{ldis} and \textit{rdis} with examples.

\begin{figure}
\centering
\begin{tikzpicture}
\draw[->] (0,0) -- (5,0) node[below] {height};
\draw[->] (0,0) -- (0,5) node[left] {\text{ldis}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{rdis}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{width}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{left}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{right}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{width}(\gamma, h)};
\draw[->] (0,0) -- (0,5) node[above] {\text{ldis}(\gamma, h)};
\end{tikzpicture}
\caption{Definitions of \textit{ldis}, \textit{rdis}, \textit{width}.}
\end{figure}

Example.

Let \( \gamma : f(c) \leftarrow f(g(g(c))) \leftarrow f(g(c)) \leftarrow f(f(g(g(c)))) \leftarrow f(f(c)) \leftarrow g(c) \). Then \( \text{left}(\gamma, 1) = 0, \text{left}(\gamma, 2) \uparrow, \text{ldis}(\gamma, 1) = 5, \text{ldis}(\gamma, 2) \uparrow, \text{right}(\gamma, 1) = 5, \text{right}(\gamma, 3) \uparrow, \text{right}(\gamma, 0) \uparrow, \text{rdis}(\gamma, 1) = 5, \text{rdis}(\gamma, 3) \uparrow, \text{width}(\gamma, 1) = \text{right}(\gamma, 1) - \text{left}(\gamma, 1) = 5, \text{width}(\gamma, 2) = 3, \text{width}(\gamma, 3) = 2, \text{width}(\gamma, 4) = 0 \)

\textbf{Definition of } \langle K(\gamma), W(\gamma) \rangle

\begin{align*}
K(\gamma) & = \{(h, \text{ldis}(\gamma, h)) \mid \text{ldis}(\gamma, h) \downarrow\} \\
W(\gamma) & = \{(h, \text{width}(\gamma, h)) \mid \text{width}(\gamma, h) \downarrow\}
\end{align*}

\textbf{Notation}

We denote by \( \gamma[\delta'/\delta] \) the sequence obtained from reduction sequence \( \gamma \) by replacing the subsequence or cut sequence \( \delta \) of \( \gamma \) by sequence \( \delta' \).
3 Assertions

In this section, we explain how to prove that non-E-overlapping and strongly depth-preserving TRS $R$ is CR. For this purpose, we need the following five assertions $S(k), S'(k), P(k), Q(k), Q'(k)$ for $k \geq 0$.

**Assertion $S(k)$**

Let $\gamma : M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_k$ where $|\gamma|_p = k, M_0 = \sigma(\beta), M_1 = \sigma(\alpha), M_{k-1} = \sigma'(\alpha), M_k = \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta \in R$ and mappings $\sigma, \sigma'$ and $\gamma : M_1 \rightarrow \ldots \rightarrow M_{k-1}$ is $\epsilon$-invariant.

Then $\exists \delta : \sigma(\beta) \dashvarrow^{*} \sigma'(\beta)$ such that

(i) $|\delta|_p \leq k - 2$

(ii) If $\beta$ is a variable, then $H(\delta) < H(\gamma)$.

Otherwise, $\delta$ is $\epsilon$-invariant and $H(\delta) \leq H(\gamma)$.

(iii) $\forall h \geq 0$ if $ldis(\delta, h) \downarrow$, then

$\exists h' \geq h$ such that $ldis(\gamma, h') \downarrow$ and $ldis(\delta, h) < ldis(\gamma, h')$.

**Assertion $S'(k)$**

Let $\gamma : M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_k$

where $|\gamma|_p = k, M_0 = \sigma(\beta), M_1 = \sigma(\alpha), M_{k-1} = \sigma'(\alpha), M_k = \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta \in R$ and mappings $\sigma, \sigma'$ and $\gamma : M_1(= \sigma(\alpha)) \rightarrow \ldots \rightarrow M_{k-1}(= \sigma'(\alpha))$ is $\epsilon$-invariant.

Then $\exists \delta : \sigma(\beta) \dashvarrow^{*} \sigma'(\beta)$ such that

(i) $|\delta|_p = |\gamma|_p, |\delta|_{np} \leq |\gamma|_{np} - 2$

(ii) If $\beta$ is a variable, then $H(\delta) < H(\gamma)$.

Otherwise, $\delta$ is $\epsilon$-invariant and $H(\delta) \leq H(\gamma)$.

(iii) $\forall h \geq 0$ if $left(\delta, h) \downarrow$, then

$\exists h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\delta, h) \leq left(\gamma, h')$.

**Assertion $P(k)$**

Let $\gamma : \sigma(\beta) \rightarrow \sigma(\alpha) \rightarrow M$ for some rule $\alpha \rightarrow \beta \in R$ and mapping $\sigma$ where $H(\gamma) = k$ and $\gamma : \sigma(\alpha) \rightarrow \ldots \rightarrow M$ is $\epsilon$-invariant.

Then, if $\beta$ is not a variable, then

$\exists \delta : \sigma(\beta) \dashvarrow^{*} N \dashvarrow^{*} M$ for some $N$ such that

$H(\delta) \leq k, M \dashvarrow^{*} N$ and $\delta' : \sigma(\beta) \dashvarrow^{*} N$ is $\epsilon$-invariant.

If $\beta$ is a variable, then $\exists \delta : \sigma(\beta) \dashvarrow^{*} N \dashvarrow^{*} M$ for some $N$ such that

$H(\delta) \leq k, M \dashvarrow^{*} N$ and $H(\delta') < k$ for $\delta' : \sigma(\beta) \dashvarrow^{*} N$

**Assertion $Q(k)$**

Let $\gamma : M \dashvarrow^{*} N$ where $H(\gamma) \leq k$.

Then, $\exists \delta : M \dashvarrow^{*} L \dashvarrow^{*} N$ such that $H(\delta) \leq k, M \dashvarrow^{*} L$ and $N \dashvarrow^{*} L$.

**Assertion $Q'(k)$**
Let $\gamma_i : M \leftarrow \ast M_i$, where $H(\gamma_i) \leq k$, $1 \leq i \leq n$. Then, $\exists \delta : M \leftarrow \ast N$ such that $H(\delta) \leq k$ and $\forall i$ $(1 \leq i \leq n)$ $M_i \leftarrow \ast N$.

The assertions $S(k)$ and $S'(k)$ are similar to the Elimination lemma in [7]. For any reduction sequence $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftarrow \ast \sigma'(\alpha) \rightarrow \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta$ and mappings $\sigma, \sigma'$ where $\tilde{\gamma} : \sigma(\alpha) \leftarrow \ast \sigma'(\alpha)$ is $\varepsilon$-invariant, $S(k)$ ensures that there exists $\delta : \sigma(\beta) \leftarrow \ast \sigma'(\beta)$ such that $|\delta|_p \leq |\gamma|_p - 2$, $H(\delta) \leq H(\gamma)$ (where $\delta$ is $\varepsilon$-invariant or $H(\delta) < H(\gamma)$) and $K(\delta) \ll K(\gamma)$. Here, $\ll$ is the multiset ordering of a lexicographic ordering $\prec$. And $S'(k)$ ensures that there exists $\delta^* : \sigma(\beta) \leftarrow \ast \sigma'(\beta)$ such that $|\delta|_p = |\gamma|_p, |\delta|_{np} \leq |\gamma|_{np} - 2$, $H(\delta) \leq H(\gamma)$ (where $\delta$ is $\varepsilon$-invariant or $H(\delta) < H(\gamma)$) and $W(\delta) \preceq W(\gamma)$. Here, $\preceq$ is $\ll$ or $\geq$.

To prove these assertions, we use the following properties for left, right, width.

**Property 1**

Let $\gamma : M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_k$,
$\delta : N_0 \leftarrow N_1 \leftarrow \cdots \leftarrow N_k$.

1. Assume that for $h > 0$, $left(\delta, h) \downarrow$ and there exists $j$ such that $j \leq left(\delta, h)$ and $h(M_j) \geq h$.

Then, there exists $h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq left(\delta, h)$.

2. Assume that for $h > 0$, $right(\delta, h) \downarrow$ and there exists $j$ such that $right(\delta, h) \leq j$ and $h(M_j) \geq h$.

Then, there exists $h' \geq h$ such that $right(\gamma, h') \downarrow$ and $right(\gamma, h') \geq right(\delta, h)$.

**Property 2**

If $H(\gamma) > H(\delta)$, then $K(\gamma) \gg K(\delta)$ and $W(\gamma) \gg W(\delta)$.

Here, $\gg$ is the multiset ordering of a lexicographic ordering $\succ$.

These proofs are obvious by the definitions of left, right and width, etc.

We first prove $S(k)$ and $S'(k)$ by induction on $k \geq 0$, where $k$ is the number of parallel reduction steps of $\gamma$. In the case of $k > 2$, we prove $S(k)$ and $S'(k)$ by induction on $weight(\gamma)$ which is defined as follows:

$$weight(\gamma) = \sum_{\gamma_i \in \Gamma} |\gamma_i|_{np}$$

where $\Gamma = \{ \gamma_i \mid \gamma_i = \gamma/u_i \text{ for some } u_i \in MR(\gamma) \cap \tilde{O}(\alpha) \}$,
$\gamma : \sigma(\alpha) \leftarrow \ast \sigma'(\alpha)$.

1. Basis, i.e., the case of $weight(\gamma) = 0$

   The proof is straightforward.

2. Induction step, i.e., the case of $weight(\gamma) > 0$

   Let $\gamma_1 = \gamma/u_1 : L_1 \leftarrow L_2 \cdots \leftarrow L_{k-1}$ where $\gamma_1 \in \Gamma$ and $L_i = M_i/u_1$, $1 \leq i \leq k - 1$.

   Then, there exist $i, j$ such that $1 \leq i < j < k - 1$ and $\delta_1 : L_i \leftarrow \ast L_{i+1} \cdots \leftarrow L_j \leftarrow L_{j+1}$

   where $L_i = \theta(\beta'), L_{i+1} = \theta(\alpha'), L_j = \theta'(\alpha'), L_{j+1} = \theta'(\beta')$ for some rule $\alpha' \rightarrow \beta'$ and mappings $\theta, \theta'$.
By the induction hypothesis $S(k')$, where $k' = |\delta_1|_p$, there exists $\eta_1 : L_i \longleftarrow L_{i+1}$ satisfying the conditions (i), (ii) and (iii). Let $\eta'_1 = ((L_i \longleftarrow L_i \cdots \longleftarrow L_i); \eta_1)$ where $|\eta'_1|_p = |\delta_1|_p$.

Let $\gamma' = \gamma[\eta'_1/\delta_1]$. Then, obviously weight($\gamma$) > weight($\gamma'$) holds. Hence, by the induction hypothesis that $S(k)$ holds for $\gamma'$, it follows that $S(k)$ holds for $\gamma$.

The proof of $S'(k)$ is similar to that of $S(k)$.

We then prove that $Q(k) \Rightarrow Q'(k)$ for all $k \geq 0$. Using these results, we can prove $P(k) \land Q(k)$ by induction on $k \geq 0$.

Outline of the proof of $P(k) \land Q(k)$.

We first prove $P(k)$. Basis: $k = 0$. The proof is obvious.

Induction step: Let $\gamma : M_0 \longleftarrow M_1 \longleftarrow M_2 \cdots \longleftarrow M_n$ where $H(\gamma) = k$, $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$ and $M_n = M$. Let $\gamma : M_1 \longleftarrow M_2 \cdots \longleftarrow M_n$. We prove $P(k)$ by induction on the following weight($\gamma$).

$$\text{weight}(\gamma) = \bigcup_{\gamma_i \in \Gamma} K(\text{net}(\gamma_i^R))$$

where $\Gamma = \{\gamma_i | \gamma_i = \gamma/u; \text{for some } u \in MR(\gamma) \cap \bar{O}(\alpha)\}.$

Here, $\gamma_i^R$ is the reverse sequence of $\gamma_i$.

Note that if $\Gamma = \phi$, then weight($\gamma$) = $\phi$.

1. Basis: the case of weight($\gamma$) = $\phi$, i.e., all the reductions of $\gamma$ occur in the variable parts of $\sigma(\alpha)$.

We can prove $P(k)$ by using the induction hypothesis $Q(k - 1)$ and the strongly depth-preserving property.

2. Induction step: the case of weight($\gamma$) $\gg$ $\phi$ i.e., some reduction occurs in the non variable part.

By the definition of $\gamma_i^R$, then there exists an $\epsilon$-reduction.

Let $\delta = \text{net}(\gamma_i^R) : (L_0 \longleftarrow L_1 \cdots \longleftarrow L_m)$ where $m \leq n$, $L_0 = M_n/u_1$, $L_m = M_1/u_1$.

There are two cases depending on whether there exists $\xi : L_i(= \sigma'(\beta')) \longleftarrow \cdots L_i+1(= \sigma'(\alpha')) \longleftarrow \cdots L_j(= \sigma''(\alpha')) \longleftarrow \cdots L_{i+1}(= \sigma''(\beta'))$ for some $i, j$ ($1 \leq i < j < m$), where $L_{i+1} \longleftarrow \cdots L_j$ is $\epsilon$-invariant.

(a) The case in which $\delta$ includes $\xi$.

By $S([\xi']_p)$, there exists $\xi' : L_i \longleftarrow \cdots L_{i+1}$ satisfying the conditions (i), (ii), (iii).

Let $\delta' = \delta[\xi'/\xi]$ and $\gamma' = \gamma[\eta_1'/\delta_1]$ where $\text{net}(\gamma_1^R) = \delta'$ and $\text{net}(\gamma_i^R) = \delta$.

By weight($\gamma$) $\gg$ weight($\gamma'$), the induction hypothesis for $\gamma'$ ensures that $P(k)$ holds for $\gamma$.

(b) The case in which $\delta$ does not include such $\xi$.

In this case, $\delta$ includes $\epsilon$-reductions, but the direction of the $\epsilon$-reductions is left-to-right by the non-$E$-overlapping property.

Using a finite number of the induction hypothesis $P(k'), k' < k$, we can prove that there exists $\eta : L_0 \longleftarrow \cdots \longleftarrow L_i$ for some term $N$ and $i$ ($0 < i \leq m$) such that $H(\eta) \leq H(\delta), L_0 \longleftarrow \cdots N$ and either $i = m$ and $\eta : N \longleftarrow \cdots L_i$ is $\epsilon$-invariant or $H(\eta) < H(\delta_i)$ holds where $\eta : N \longleftarrow \cdots L_i$ and $\delta_i : L_0 \longleftarrow \cdots L_i$. 
Let $\delta = \delta[\eta'/\delta_1]$. Then, $\delta$ is $\varepsilon$-invariant or $K(\delta) \gg K(\delta)$ holds. Let $\gamma' = \gamma[\gamma_1'/\gamma_1]$ where $\delta = \text{net}(\gamma_1^{R})$ and $\delta = \text{net}(\gamma_1^{R})$. Then, $\text{weight}(\gamma) \gg \text{weight}(\gamma')$ holds, so that the induction hypothesis $P(k)$ for $\gamma'$ ensures that $P(k)$ holds for $\gamma$.

Next, we prove $Q(k)$ by induction on $(H(\gamma), W(\gamma), \varepsilon(\gamma))$, where $\varepsilon(\gamma)$ is the number of $\varepsilon$-reductions in $\gamma$ and $W(\gamma) = \{ (h, \text{width}(\gamma, h)) \mid \text{width}(\gamma, h) \}$. If $H(\gamma) \leq k - 1$ or $\gamma$ has no $\varepsilon$-reductions, then the proof can be reduced to that of $Q(k - 1)$. So, let $H(\gamma) = k$ and $\gamma$ has $\varepsilon$-reductions.

There are two cases depending on whether there exists a subsequence

$$\gamma_1 : N_1 \leftarrow^{\varepsilon} N_2 \rightarrow^{*} N_3 \rightarrow^{\varepsilon} N_4$$

of $\gamma$ for some $N_i, 1 \leq i \leq 4$, where $N_2 \rightarrow^{*} N_3$ is $\varepsilon$-invariant.

1. The case in which $\gamma$ includes such $\gamma_1$.

   In this case, we apply $S(|\gamma_1|)$ or $S'(|\gamma_1|)$ and obtain $\delta_1 : N_1 \rightarrow^{*} N_4$ satisfying the conditions (i),(ii) and (iii).

   Let $\gamma' = \gamma[\delta_1/\gamma_1]$. Then, either $W(\gamma) \gg W(\gamma')$ or $W(\gamma) = W(\gamma')$ and $\delta_1$ is $\varepsilon$-invariant. In either case, the induction hypothesis for $\gamma'$ ensures that $Q(k)$ holds for $\gamma$.

2. The case in which $\gamma$ does not include such $\gamma_1$.

   We can prove this case by using $P(k)$ and $Q(k - 1)$. But, the details are omitted.

Since $Q(k), k > 0$, ensures that TRS $R$ is CR, we have the following our main theorem.

**Main Theorem**

A TRS $R$ is CR if $R$ is non-$E$-overlapping and strongly depth-preserving.

Matsuura et al.[6] showed that if a TRS $R$ is non-$\omega$-overlapping and depth-preserving, then $R$ is non-$E$-overlapping, so that we have the following corollary.

**Corollary**

A TRS $R$ is CR if $R$ is non-$\omega$-overlapping and strongly depth-preserving.

**Note**

Whether $R$ is non-$\omega$-overlapping or not can be checked efficiently.

4 Concluding Remarks

In this paper, we have shown that there exists a non-$E$-overlapping and depth-preserving TRS which is not CR, but all the non-$E$-overlapping and strongly depth-preserving TRS's satisfy the CR property.

Finally, we make a comment on the strongly depth-preserving property. This property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\alpha)$, all the depths of the $x$ occurrences in $\alpha$ are the same. By replacing the restriction on $\alpha$ by that on $\beta$, we can define an analogous property. That is, this new property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\beta)$, all the depths of the $x$ occurrences in $\beta$ are the same. However, this new property and non-$E$-overlapping do not necessarily ensure CR. For example, TRS $R_0 = \{ f(g(x), x) \rightarrow a, c \rightarrow h(c, g(c)), h(x, g(x)) \rightarrow f(g(x), h(x, g(c))) \}$ is non-$E$-overlapping and satisfies this new condition, but $R_0$ is not CR.

It will be a next step following the work of this paper to study the CR property of $E$-overlapping and strongly depth-preserving TRS, that is, to find restriction conditions that $E$-critical pairs must satisfy for ensuring the CR property of strongly depth-preserving TRS's.
References


