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<th>Simple gap termination for term graph rewriting systems (Theory of Rewriting Systems and Its Applications)</th>
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<tr>
<td>Author(s)</td>
<td>Ogawa, Mizuhito</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1995), 918: 99-108</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1995-08</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/59673">http://hdl.handle.net/2433/59673</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Simple gap termination for term graph rewriting systems

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Abstract

This paper proves the extension of Kruskal-Friedman theorem, which is an extension of the ordinary Kruskal's theorem with gap-condition, on $\omega$-trees (Main theorem 1 in section 3). Based on the theorem, a new termination criteria for cyclic term graph rewriting systems, named simple gap termination (Main theorem 2 in section 4), is proposed where the naive extension of simple termination (based on [Lav78]) does not work well.

1 Introduction

A term graph rewriting system (TGRS) has been commonly used from efficiency reasons in implementations of a term rewriting system (TRS), such as CLEAN\(^1\). A TGRS can be regarded as a TRS with addresses - i.e., a variable in a rule of a TRS is regarded as an address in a TGRS. Thus, subterms will be shared in each reduction step of a TGRS, whereas each reduction step of a TRS simply copies. Theoretical basis for a TGRS has been extensively worked [MSvE94], but the most works has been devoted to an acyclic TGRS. For a cyclic TGRS which can simulate infinite reductions on infinite terms, only few works have been started [AK94, JKdV94, Blo95].

This paper investigates a new termination criteria simple gap termination for a cyclic TGRS. First, we prove the extension of Kruskal-Friedman theorem, which is an extension of the ordinary Kruskal's theorem with gap-condition, on $\omega$-trees (Main theorem 1 in section 3). The proof consists of four steps similar to the proof in [Lav78] with an extension inspired by [Sim85b].

Second, based on the theorem, a new termination criteria for cyclic TGRSs - named simple gap termination (Main theorem 2 in section 4) - is proposed, where the naive extension of simple termination (based on [Lav78]) does not work well. Unfortunately, a feasible construction of an ordering for simple gap termination (like recursive path ordering, etc.) is a future issue.

2 Better-Quasi-Order

For infinite objects such as $\omega$-trees, Well-Quasi-Order (WQO) does not close under the embedability construction. Instead, we need an extension of WQO, called Better-Quasi-Order (BQO). Note that (1) if $(Q, \leq)$ is a well order then $(Q, \leq)$ is a BQO, and if $(Q, \leq)$ is a BQO then $(Q, \leq)$ is a WQO, and (2) if $Q$ is finite then $(Q, \leq)$ is BQO for any $QQ \leq [Lav78]$.

Definition 2.1 Let $\omega$ be the least countable ordinal (i.e., set of natural numbers). If $s, t \subseteq \omega$, then $s \leq t$ ($s < t$) means that $s$ is a (proper) initial segment of $t$. Define $s \prec t$ to hold if there is an $n > 0$ and $i_0 < \cdots < i_n < \omega$ s.t. for some $m < n$, $s = \{i_0, \cdots, i_m\}$ and $t = \{i_1, \cdots, i_n\}$. (Thus, e.g., $\{3\} \prec \{5\}, \{3, 5, 6\} \prec \{5, 6, 8, 9\}, \{3, 5, 6\} \not\prec \{5, 6\}$.)

\(^1\)Try http://www.cs.kun.nl:80/ clean/
Definition 2.2  For an infinite set $X \subseteq \omega$, a barrier $B$ on $X$ is a set of finite sets of $X$ s.t. $\phi \notin B$ and

1. for every infinite set $Y \subseteq \omega$ there is an $s \in B$ s.t. $s < Y$.
2. if $s, t \in B$ and $s \neq t$ then $s \not\triangleleft t$.

Theorem 2.1  If $B$ is a barrier and $B = \bigcup_{i \leq n} B_i$ for some $n < \omega$, then some $B_i$ contains a barrier (on $\bigcup_{b \in B_i} b$).

Definition 2.3  Let $\leq$ be a transitive binary relation on a set $Q$. Then,

- if $\leq$ is reflexive, $R$ is called a quasi-order (QO).
- if $\leq$ is antisymmetric, $R$ is called a partial order (or, simply order).
- if each pair of different elements in $Q$ is comparable by $\leq$, $\leq$ is said to be total.

A strict part of $\leq$ is $\leq - \geq$ and denoted as $<$. We also say a strict (quasi) order $<$ if it is a strict part of a (quasi) order $\leq$. When $\leq$ is a QO, we will sometimes use $\leq$ (resp. $<$) instead of $\leq$ (resp. $<$), for clarity.

Definition 2.4  Let $\leq$ be a QO on $Q$. If $B$ is a barrier, $f : B \to Q$ is good if there are $s, t \in B$ s.t. $s \prec t$ and $f(s) \leq f(t)$, and $f$ is bad otherwise. $f$ is perfect if for all $s, t \in B$, if $s \prec t$ then $f(s) \leq f(t)$. $Q$ is better-quasi-ordered (bqo) if for every barrier $B$ and every $f : B \to Q$, $f$ is good.

Remark 2.1  If we restrict the BQO definition s.t. $B$ runs only barriers of singleton sets (i.e., $B = \{1, 2, \cdots\}$), then we get the familiar well-quasi-order (WQO) definition.

A (possibly infinite) tree is a set of $T$ on which a strict partial order $\prec_T$ is defined s.t. for every $t \in T$, $\{s \in T \mid s \prec_T t\}$ is well ordered under $\prec_T$. Thus $T = \bigcup_{\alpha} T_\alpha$ where $\alpha$ runs on ordinals and $T_\alpha$, the $\alpha$-th level of $T$, is the set of all $t \in T$ s.t. $\{s \mid s \prec_T t\}$ has type $\alpha$. The height of $T$ is the least $\alpha$ with $T_\alpha = \emptyset$. A path in $T$ is a linearly ordered downward closed subset of $T$. If $x \in T$ (resp. a path $P$ in $T$), let $S(x)$ (resp. $S(P)$) be the set of immediate successors of $x$ (resp. $P$). A path is maximal in $T$ if $S(P) = \emptyset$. Let $br_T(x)$ (or simply $br(x)$ if unambiguous) be $\{y \in T \mid x \leq_T y\}$, the branch above $x$. An $\omega$-tree is a (possibly infinitely branching) tree of the height at most $\omega$.

Definition 2.5  Let $T$ be a set of trees which satisfies

1. For each $T \in T$, $T$ has a root (minimum element),
2. For each $T \in T$, if $P$ is a path in $T$ with no largest element then $\text{Card}(S(P)) \leq 1$. A Q-tree

$T_Q$ is a pair $(T, l)$ where $T \in T$ and $l : T \to Q$.

If $T \in T$, $s, t \in T$, there is a greatest lower bound of $s$ and $t$ in $T$, denoted by $s \wedge t$.

Definition 2.6  Let $Q$ be a QO set and $(T_1, l_1), (T_2, l_2) \in T_Q$. $(T_1, l_1)$ is embeddable to $(T_2, l_2)$ (and denoted $(T_1, l_1) \leq (T_2, l_2)$, or simply $T_1 \leq T_2$) if there exists $\psi : T_1 \to T_2$ s.t.

1. For $s, t \in T_1$, $\psi(s \wedge t) = \psi(s) \wedge \psi(t)$,
2. For $t \in T_1$, $l_1(t) \leq l_2(\psi(t))$.

$^2$Corollary 1.5 in [Lav78]. The proof is due to Galvin-Prikry. See Theorem 9.9 in [Sim85a].
Theorem 2.2 [Lav78, NW65] If $Q$ is BQO, $\mathcal{M}_Q$ is BQO wrt the embedability $\leq$.

Remark 2.2 WQO is not enough for Kruskal-type theorem for infinite objects. For instance, consider $Q = \{(i, j) \mid i < j < \omega\}$ ordered by $(i, j) \leq (k, l)$ if and only if either $i = k$ wedge $j \leq k$ or $j < k$. Then $Q$ is WQO, but a set $Q^\omega$ of infinite sequence on $Q$ is not WQO, namely,

$$
\begin{align*}
    f_1 &= \langle(0, 1), (1, 2), (1, 3), (1, 4), \cdots\rangle, \\
    f_2 &= \langle(0, 1), (1, 2), (2, 3), (2, 4), \cdots\rangle, \\
    \vdots
    &\ldots \\
    f_i &= \langle(0, 1), \cdots, (i, i + 1), (i, i + 2), (i, i + 3), \cdots\rangle, \\
    \vdots
\end{align*}
$$

The main techniques to prove Kruskal-type theorems are (1) Ramsey-like theorem and (2) the existence of the minimal bad sequence (MBS). For (1), theorem 2.1 works. For (2), we first prepare some definitions (See [Lav78]).

Definition 2.7 Suppose $Q$ is quasi-ordered by $\leq$. A partial ranking on $Q$ is a well-founded (irreflexive) partial order $<'$ on $Q$ s.t. $q <' r$ implies $q < r$. Let $B, C$ be barriers. Then $B \subseteq C$ if

1. $\cup C \subseteq \cup B$, and
2. for each $c \in C$ there is a $b \in B$ with $b \leq c$.

$B \subset C$ if $B \subseteq C$ and there are $b \in B$, $c \in C$ with $b < c$. For $f : B \rightarrow Q$, $g : C \rightarrow Q$ and a partial ranking $<'$ on $Q$, $f \subseteq g$ ($f \subset g$) wrt $<'$ if $B \subseteq C$ ($B \subset C$) and

1. $g(a) = f(a)$ for $a \in B \cap C$, 
2. $g(c) <' f(b)$ for $b \in B$, $c \in C$ s.t. $b < c$.

Definition 2.8 Suppose $<'$ is a partial ranking on $Q$. For a barrier $C$, $g : C \rightarrow Q$ is minimal bad if $g$ is bad and there is no bad $h$ with $g \subset h$.

Theorem 2.3 3 Let $Q$ be quasi-ordered by $\leq$, $<'$ a partial ranking on $Q$. Then for any bad $f$ on $Q$ there is minimal bad $g$ s.t. $f \subseteq g$.

Thus, the proof of Kruskal-type theorem on infinite objects is reduced to find some appropriate partial ranking $<'$.

3 Kruskal-type theorems with gap-condition on infinite trees

Kruskal's theorem with gap-condition for finite trees have been proposed for finite ordinals[Sim85a]. The aim of this section is to prove main theorem 1, which extends Kruskal's theorem with gap-condition to $\omega$-trees. Main theorem 1 is obtained as a corollary to the the stronger statement theorem 3.2). The scenario of its proof is similar to those that in [Lav78] and its extension is inspired by [Sim85b].

Definition 3.1 For $n < \omega$, let $\mathcal{M}_n$ be a set of $\omega$-trees with labels in $n = \{0, 1, \cdots, n - 1\}$, and $(T_1, l_1), (T_2, l_2) \in \mathcal{M}_n$. $(T_1, l_1) \leq (T_2, l_2)$ if there exists $\psi : T_1 \rightarrow T_2$ s.t.

3Theorem 1.9 in [Lav78], or equivalently theorem 9.17 in [Sim85a].

4There are two variants of its extensions for infinite ordinals[K89, Gor90].
1. $T_1 \leq T_2$,

2. For each $t \in T_1$, $l_1(t) = l_2(\psi(t))$,

3. For $t \in T_1$, if there is $t' \in T_1$ s.t. $t \in S(t')$ then $l_2(s) \geq l_1(t)$ for each $s$ s.t. $\psi(t') <_{T_2} s <_{T_2} \psi(t)$,

4. For the root $t$ of $T_1$, $l_2(s) \geq l_1(t)$ for each $s$ s.t. $s <_{T_2} \psi(t)$.

**Theorem 3.1** [Sim85b] For $n < \omega$, let $T(n)$ be the set of all finite trees with labels less-than-equal $n$. Then $\leq_G$ is a WQO on the set $T(n)$.

The next theorem is the extension of Kruskal-Friedman theorem to $\omega$-trees.

**Main theorem 1** For $n < \omega$, let $M_n$ be a set of $\omega$-trees with labels in $n(= \{0, 1, \ldots, n-1\})$. Then $M_n$ is BQO wrt $\leq_G$.

To show the theorem, we will prove the slightly stronger statement.

**Definition 3.2** For $n < \omega$, let $Q$ be a QO and $q : Q \to n(= \{0, 1, \ldots, n-1\})$. $M_n(Q)$ is a set of $\omega$-trees satisfying: for $(T, l) \in M_n(Q)$, $l(t) \in n$ if $t \in T$ is not maximal wrt $<_T$ and $l(t) \in n \cup Q$ if $t \in T$ is maximal wrt $<_T$. Let $(T_1, l_1), (T_2, l_2) \in M_n(Q)$, $(T_1, l_1) \leq_G (T_2, l_2)$ if there exists $\psi : T_1 \to T_2$ s.t.

1. $T_1 \leq T_2$,

2. For each interior vertex $t \in T_1$, $\psi(t)$ is an interior vertex of $T_2$ and $l_1(t) = l_2(\psi(t))$,

3. For each end vertex $t \in T_1$, $\psi(t)$ is an end vertex of $T_2$ and either $l_1(t) = l_2(\psi(t)) \in n$ or $l_1(t) \leq l_2(\psi(t)) \in Q$.

4. For each interior vertex $t \in T_1$, $t' \in S(t)$ and $s \in T_2$ with $\psi(t) <_{T_2} s <_{T_2} \psi(t')$, $l_2(s) \geq l_1(\psi(t'))$ when $l_1(\psi(t')) \in n$ and $l_2(s) \geq q(l_1(\psi(t'))) \in Q$.

5. For the root $t$ of $T_1$ and $s \in T_2$ with $s <_{T_2} \psi(t)$, $l_2(s) \geq l_1(\psi(t))$ when $l_1(\psi(t)) \in n$ and $l_2(s) \geq q(l_1(\psi(t)))$ when $l_1(\psi(t)) \in Q$.

We will denote $(T_1, l_1) \equiv (T_2, l_2)$ if $(T_1, l_1) \leq_G (T_2, l_2)$ and $(T_1, l_1) \geq_G (T_2, l_2)$

**Theorem 3.2** Let $n < \omega$, $Q$ be a BQO and $q : Q \to n (= \{0, 1, \ldots, n-1\})$. Let $M_n(Q)$ be the set of all $\omega$-trees with labels in $n$ for non-maximal vertices and with labels in $n \cup Q$ for maximal vertices. Then $M_n(Q)$ is BQO wrt $\leq_G$.

**Definition 3.3** Let $n < \omega$. Let $Q$ be a QO and $q : Q \to n$. $W_n(Q), S_n(Q), F_n(Q)(\subseteq M_n(Q))$ are defined to be:

1. $W_n(Q)$ is a set of $\omega$-words in $M_n(Q)$.

2. $S_n(Q)$ is a set of scattered $\omega$-trees in $M_n(Q)$. (i.e., for each $(S, l) \in S_n(Q)$, $\eta \not\leq S$ where $\eta$ is a complete binary $\omega$-tree $(2^\omega)$.)

3. $F_n(Q)$ is a set of descensionally finite trees. (i.e., For $(T, l) \in F_n(Q)$, there is no infinite sequence $x_0 <_T x_1 <_T \cdots$ with $(br(x_0), l) >_G (br(x_1), l) >_G \cdots$)
The scenario of the proof of theorem 3.2 consists of four steps: First, \( W_n(Q) \), which is a set of \( \omega \)-words, is shown to be a BQO wrt \( \leq_G \) (theorem 3.3). Second, \( S_n(Q) \), which is a set of scattered \( \omega \)-trees, is shown to be a BQO wrt \( \leq_G \) (theorem 3.4). During this step, the principle tool is a recursive definition of \( S_n(Q) \) which (a) starts with one-point or empty trees and (b) constructs the next stage using an element in \( W_n(Q) \) as a spine.

\[(T,l) \in M_n(Q) \) is a countable union of scattered \( \omega \)-trees, i.e., \( T = \cup_i S_i \) with \( (S_i,l) \in S_n(Q) \). Using this decomposition, thirdly \( F_n(Q) \), which is a set of descensionally finite \( \omega \)-trees, is shown to be a BQO wrt \( \leq_G \) (theorem 3.5). Again using this decomposition, lastly \( M_n(Q) = F_n(Q) \) is shown (theorem 3.6).

**Theorem 3.3** Let \( n < \omega \). For a barrier \( D, g : D \to W_n(Q) \) is bad wrt \( \leq_G \), then there is a barrier \( E \) and \( g \subseteq j \) s.t. \( j : E \to Q \) is bad.

**Proof** Assume \( g \) is minimal bad wrt a partial ranking \( \prec \) on \( W_n(Q) \) where \( J \prec K \) if and only if \( J \leq_G K \) and \( dom(J) < dom(K) \). From theorem 2.1, we can assume \( \forall d \in D \) s.t. either (1) \( \text{dom}(g(d)) = 1 \), (2) \( \text{dom}(g(d)) < \omega \), or (3) \( \text{dom}(g(d)) = \omega \).

For (1), there exists a barrier \( E(\subseteq D) \) s.t. \( g\{e\} \subseteq Q \) for \( e \in E \). By taking \( j = g|E \), theorem is proved.

For (2), we will prove by induction on \( n \). Again by theorem 2.1, we can assume \( \forall d \in D \) s.t. either (2-a) \( g(d) \) does not contain 0, (2-b) the first element of \( g(d) \) is 0, or (2-c) \( g(d) \) contains 0 and the first element of \( g(d) \) is not 0. For (2-a), by subtracting 1 from each label of \( g(d) \), it is reduced to the induction hypothesis. For (2-b), let \( g'(d) \) be obtained from \( g(d) \) by taking the first element. Then, \( g'(d) \) is bad and this contradicts to the minimal bad assumption of \( g \). For (2-c), let \( g(d) = (g_1(d), g_2(d)) \). Since \( g_1(d) \) and \( g_2(d) \) are good from the minimal bad assumption of \( g \), there is a barrier \( E \) s.t. \( g_1(d) \) and \( g_2(d) \) are perfect. This implies that \( g(d) \) is good.

For (3), if \( g(d_1) \not\leq_G g(d_2) \) with \( d_1 \neq d_2 \), there exists an initial segment \( J \) s.t. \( J \not\leq_G g(d_2) \). Let \( h : D(2) \to (n)^{\omega} \) by \( h(d_1 \cup d_2) = J \). Then \( g \subseteq h \) and this contradicts to the minimal bad assumption on \( g \).

**Definition 3.4** Let \( T \in T, P \) a path in \( T, z \in P \). Then let \( \tilde{P}(z) = \{br(y) \mid y \in S(z) \text{ and } y \not\in P \} \).

**Lemma 3.1** (lemma 2.1 in [Lav78]) Let \( n < \omega \) and \( Q \) be a QO. Let \( \alpha \) be an ordinal and \( \lambda \) be a limit ordinal. Let

\[
\begin{align*}
S^0(Q) & = \{\text{the empty tree}\} \cup n \cup Q \\
S^\alpha+1(Q) & = \left\{ T \mid \begin{array}{l}
\text{there is a maximal path } P \in W_n(Q) \text{ in } T \\
\text{s.t. } \tilde{P}(z) \subseteq S^\alpha(Q) \text{ for all } z \in P
\end{array} \right\} \\
S^\alpha(Q) & = \bigcup_{\alpha<\lambda} S^\alpha.
\end{align*}
\]

by regarding \( n, Q \) as one point trees. Then \( S_n(Q) = \cup_\alpha S^\alpha(Q) \). We say \( \text{rank}(T) \) for \( T \in S_n(Q) \) be the least \( \alpha \) s.t. \( T \in S^\alpha(Q) \).

**Theorem 3.4** Let \( n < \omega \). For a barrier \( C, g : C \to S_n(Q) \) is bad wrt \( \leq_G \), then there is a barrier \( E \) and \( g \subseteq j \) s.t. \( j : E \to Q \) is bad.

**Proof** Let a partial ranking \( \prec \) on \( S_n(Q) \) be \( (T_1,l_1) \prec (T_2,l_2) \) if \( (T_1,l_1) \leq_G (T_2,l_2) \) and \( \text{rank}(T_1) < \text{rank}(T_2) \). Assume \( g \) is minimal bad wrt a partial ranking \( \prec \) on \( S_n(Q) \). From theorem 2.1, we can assume \( \forall d \in C \) s.t. either (1) \( \text{card}(g(d)) = 1 \) or (2) \( \text{card}(g(d)) > 1 \). For (1), there exists a barrier \( E(\subseteq C) \) s.t. \( g\{e\} \subseteq Q \) for \( e \in E \). By taking \( j = g|E \), theorem is proved.

For (2), let \( c \in C \). Let \( P_c \) be a maximal path in \( T_c \) where \( g(c) = (T_c,l_c) \in S_n(Q) \) s.t. for each \( x \in P_c \) and each \( T' \in \tilde{P}_c(x) \) \( \text{rank}(T') < \text{rank}(T_c) \). Let \( J_c : P_c \to W_{n+1}(Q) \times \mathcal{P}(S_n(Q)) \) be defined by

\[
J_c = (I_c(x), \tilde{P}_c(x))
\]
where $I_c(x)$ is the sequence which is obtained by adding $n + 1$ as the maximal element (wrt $<_{T_c}$) to the path from the root of $T_c$ to $x$. By regarding $J_c$ as a sequence, $J_c \leq J_d$ (embedability without gap-condition) implies $(T_c, I_c) \leq (T_d, I_d)$ for $c, d \in C$. From theorem 1.10 in [Lav78], if $g$ is bad, there is a barrier $D$ and $g : D \to \mathcal{W}_{n+1}(Q) \times \mathcal{P}(S_n(Q))$ s.t. $g \subseteq g$ and $g$ is bad (by identifying an element as a sequence of the length 1). From theorem 3.3 and theorem 1.11 in [Lav78] (with $\leq_1$ on $\mathcal{P}(S_n(Q))$, which is an one-to-one embedability on sets), there exists a barrier $E$ and $j : E \to \mathcal{W}_{n+1}(Q) \times S_n(Q)$ s.t. $D \subseteq E$ and $j$ is bad. For $j(e) = (I_c(x), T')$ where $x \in P_e \subseteq T_c$ and each $T' \in \mathcal{P}(x)$ for $c \subseteq e$, let $j'(e)$ be a tree obtained by replacing the last element of $I_c(x)$ (whose label is $n + 1$) with $T'$. $g \subseteq g'$ and rank$(j'(e)) <$ rank$(T_c)$ (since rank$(T') <$ rank$(T_c)$ and adding a label to the root of $T'$ does not change its rank). This contradicts to the minimal bad assumption of $g$.

Adding (possibly infinite numbers of) finite trees to $(S, l) \in S_n(Q)$ does not exceed the class of $S_n(Q)$. Thus without loss of generality, for each $(T, l) \in M_n(Q)$ we can assume the decomposition $T = \bigcup_{i}T_i$ with $(T_i, l) \in S_n(Q)$ satisfies that if $x$ is maximal wrt $<_{T_i}$ then either br$(x)$ does not contain 0 or $l(x) = 0$.

**Definition 3.5** Let $(T, l) \in \mathcal{F}_n(Q) (\subseteq M_n(Q))$ and $T = \bigcup_{i}T_i$ with $(T_i, l) \in S_n(Q)$ s.t. if $x \in T_i$ is maximal wrt $<_{T_i}$ then either br$(x)$ does not contain 0 or $l(x) = 0$. If $T$ does not contain a vertex labeled 0, sum$(T, l) \in \mathcal{F}_{n-1}(Q)$ is $(T', l')$ where $l'(x) = l(x) - 1$ for each $x \in T$. With a fresh symbol $\Omega$, let $Q^+ = Q \cup \{\Omega\}$ with $q(\Omega) = 0$.

We denote $\mathcal{F}_n(Q)^{(T, l)} = \{(U, m) \in \mathcal{F}_n(Q) | (U, m) <_{\mathcal{G}} (T, l)\}$

Define $A(T, l) = \{(T_i, l) \in S_{n+1}(Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{(T', l'))} | (U, m) \in \mathcal{F}_n(Q), (U, m) \in \mathcal{F}_n(Q)^{(T, l)}, A((U, m))(i) \in A((U, m)) \}$ s.t. $A(T, l) \subseteq A((U, m))(i) \subseteq A((U, m))(i)$.

**Lemma 3.2** For $(T, l), (U, m) \in \mathcal{F}_n(Q)$, $A((T, l)) \subseteq A((U, m))$ implies $(T, l) \leq_{\mathcal{G}} (U, m)$.

**Proof** We will construct an embedding $H : (T, l) \to (U, m)$ (with gap-condition) in $\omega$ steps. The induction hypothesis is:

If $x \in T_i$ is maximal wrt $<_{T_i}$, there is a 1-1 function $J_i$ s.t.

1. if (br$(y)$, $l$) does not contain 0 then (br$(y)$, $l$) $<_{\mathcal{G}}$ (br$(J_i(y))$, $m$),
2. if $l(y) = 0$ and (br$(y)$, $l$) $<_{\mathcal{G}}$ (T, $l$) then $m(J_i(y)) = 0$ and (br$(y)$, $l$) $<_{\mathcal{G}}$ (br$(J_i(y))$, $m$),
3. if $l(y) = 0$ and (br$(y)$, $l$) $\equiv$ (T, $l$) then $m(J_i(y)) = 0$ and (br$(J_i(y))$, $m$) $\equiv$ (U, $m$).

Since $A((T, l)) \leq A((U, m))$, there exists $A((U, m))(j) \in A((U, m))$ s.t. $A((T, l))(0) = (T_0, l) <_{\mathcal{G}} A((U, m))(j) = (U_j, m)$. Then set $H_0$ by the embedding $T_0 \to U_j$.

Suppose that $H_i$ has been defined, $y \in T_i$ is maximal. If either (1) (br$(y)$, $l$) does not contain 0 or (2) $l(y) = 0$ and (br$(y)$, $l$) $<_{\mathcal{G}}$ (T, $l$) then (br$(y)$, $l$) $<_{\mathcal{G}}$ (br$(J_i(y))$, $m$). Thus extend $H_i$ with an embedding of br$(y)$ into br$(J_i(y))$.

---

5 If $Q$ is a BQO, $Q^+$ is also a BQO.
Suppose that (3) \( l(y) = 0 \) and \( (br(y), l) \equiv \mathcal{G}(T, l) \) then there exists an embedding \( L: (U, m) \to (br(J_t(y)), m) \). Since \( A((T, l)) \leq A((U, m)) \), there exists \( A((U, m))(j) \in A((U, m)) \) s.t. \( A_{n}(i + 1) = (T+i), l \leq_G A_{m}(j) = (U_j, m) \). Let \( K: (T_i+1, l) \to (U_j, m) \subseteq (U, m) \) be an induced embedding. Thus extend \( H_t \) on \( br(y) \cap T_{i+1} \) with \( LK \). Since \( L \) isomorphically embeds \( (U, m) \) into \( (br(J_t(y)), m) \), the induction hypothesis is satisfied to the next stage.

**Theorem 3.5** Let \( n < \omega \). For a barrier \( B, f: B \to \mathcal{F}_n(Q) \) is bad wrt \( \leq_G \), then there is a barrier \( E \) and \( f \subseteq e \) s.t. \( E \to Q \) is bad. Thus if \( Q \) is a BQO then \( \mathcal{F}_n(Q) \) is a BQO (wrt \( \leq_G \)).

**Proof** We will prove by induction on \( n \). For \( n = 0, \leq_G \) and \( \leq \) (without gap-condition) are equivalent (see lemma 2 in theorem 2.4 of [Lav78]). Assume the theorem has been proved until \( n - 1 \).

Define a partial ranking \( <' \) by: \((U, m) <' (T, l) \) if and only if for some \( x \in T (U, m) = (br(x), l) <_G (T, l) \). By theorem 2.3, we can assume \( f: B \to \mathcal{F}_n(Q) \) is minimal bad. Let \( f(b) = (T_b, l_b) \) for \( b \in B \) and let \( f(b) = A((T_b, l_b)) \). From lemma 3.2, \( f \) is bad. From theorem 1.3 in [Lav78], there is a barrier \( C \subseteq B(2) \) and an \( g \) defined on \( C \) s.t. for \( c = b_1 \cup b_2 \) where \( b_1 < b_2 \) and \( b_1, b_2 \in B \) \( g(c) \in g(b_1) \) and \( g \) is bad. Since \( g(c) \in S_{n+1}(Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<}\tau_b, l_b) \) and \( g \) is bad, from theorem 3.4 there is a barrier \( D \) with \( C \subseteq D \) and \( h \) defined on \( D \) s.t. \( h(d) \in Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<}\tau_b, l_b) \) for \( (b <) d \in D \) and \( h \) is bad. Since \( Q^+ \) and \( \mathcal{F}_{n-1}(Q) \) are BQO, from theorem 2.1 there is a barrier \( E \subseteq D \) and \( j \) defined on \( E \) s.t. \( j(e) <' (T_b, l_b) \) for \( (b <) e \in E \) and \( j \) is bad. Thus \( g \subseteq j \) and this is contradiction.

**Theorem 3.6** \( \mathcal{M}_n(Q) = \mathcal{F}_n(Q) \).

We will prove theorem 3.6 by induction on \( n \). For \( n = 0, \leq \) and \( \leq_G \) are equivalent and this is shown by lemma 4 in theorem 2.4 in [Lav78]. Note that if \( (T, l) \in \mathcal{M}_n(Q) \) does not contain \( 0, \) by induction hypothesis \( \text{sub}(T, l) \in \mathcal{M}_{n-1}(Q) \) and \( (T, l) \in \mathcal{F}_n(Q) \).

**Definition 3.6** Let \( (T, l) \in \mathcal{M}_n(Q) \) and \( T = \cup T_i \) with \( (T_i, l) \in \mathcal{S}_n(Q) \) s.t. if \( x \in T_i \) is maximal wrt \( <_T \) then either \( br(x) \) does not contain \( 0 \) or \( l(x) = 0 \). Let \( Q^+ = Q \cup \{\omega\} \) with \( q(\omega) = 0 \).

Define \( B_{(T, l)}(i) = (T_i, l) \in S_{n+1}(Q^+ \cup \mathcal{F}_n(Q)) \) where

1. If \( x \in T_i \) is not maximal wrt \( <_T \), then \( \ell(x) = l(x) \).
2. If \( x \in T_i \) is maximal wrt \( <_T \) and \( (br(x), l) \) does not contain \( 0 \), then add a new vertex \( x^+ \) below \( x \) and set \( \ell(x^+) = n + 1, \ell(x^+) = (br(x), l) \).
3. If \( x \in T_i \) is maximal wrt \( <_T, l(x) = 0 \) and \( br(x) \in \mathcal{F}_n(Q) \), then \( \ell(x) = (br(x), l) \).
4. If \( x \in T_i \) is maximal wrt \( <_T, l(x) = 0 \) and \( (br(x), l) \in \mathcal{M}_n(Q) \) \( \to \mathcal{F}_n(Q) \), then \( \ell(x) = \Omega \).

Define \( B((T, l)) = \{B_{(T, l)}(i) | i < \omega\} \in \mathcal{P}(S_{n+1}(Q^+ \cup \mathcal{F}_n(Q))) \) For \( (T, l), (U, m) \in \mathcal{M}_n(Q) \setminus \mathcal{F}_n(Q) \), define \( B((T, l)) \leq B((U, m)) \) if for each \( B_{(T, l)}(i) \in B((T, l)) \) there exists \( B_{(U, m)}(j) \in B((U, m)) \) s.t. \( B_{(T, l)}(i) \leq_G B_{(U, m)}(j) \).

**Lemma 3.3** Let \( (T, l), (U, m) \in \mathcal{M}_n(Q) \setminus \mathcal{F}_n(Q) \) s.t. \( l(\text{root}(T)) = m(\text{root}(U)) = 0 \). If \( B((T, l)) \leq B((br(u), m)) \) for each \( u \in U \) s.t. \( m(u) = 0 \) and \( (br(u), m) \notin \mathcal{F}_n(Q) \), then \( (T, l) \leq_G (U, m) \).

**Proof** We will construct an embedding \( I: (T, l) \to (U, m) \) (keeping gap-condition) in \( \omega \) steps. The induction hypothesis is:
If \( x \in T_i \) is maximal \( \text{wrt} \ <_{T_i} \), there is a 1-1 function \( J_i \) s.t.

1. if \( (br(y), l) \) does not contain 0 then \( (br(J_i(y)), m) \) does not contain 0.
2. if \( l(y) = 0 \) and \( (br(y), l) \in \mathcal{F}_n(Q) \) then \( m(J_i(y)) = 0 \) and \( (br(J_i(y)), m) \in \mathcal{F}_n(Q) \).
3. if \( l(y) = 0 \) and \( (br(y), l) \not\in \mathcal{F}_n(Q) \) then \( m(J_i(y)) = 0 \) and \( (br(J_i(y)), m) \not\in \mathcal{F}_n(Q) \).

Since \( B((T, l)) \leq B((U, m)) \), there exists \( B(U, m)(j) \in B((U, m)) \) s.t. \( B(T, l)(0) = (T_0, l) \leq_{\bar{G}} B(U, m)(j) = (U_j, m) \). Then set \( T_0 \) by the embedding \( T_0 \to U_j \).

Suppose that \( I_i \) has been defined, \( y \in T_i \) is maximal. If either (1) \( br(y) \) does not contain 0 or (2) \( l(y) = 0 \) and \( (br(y), l) \in \mathcal{F}_n(Q) \) then \( (br(y), l) \leq_{\bar{G}} (br(J_i(y)), l) \). Thus extend \( I_i \) with an embedding of \( \text{br}(y) \) into \( \text{br}(J_i(y)) \).

Suppose that (3) \( l(y) = 0 \) and \( (br(y), l) \not\in \mathcal{F}_n(Q) \), then from induction hypothesis \( m(J_i(y)) = 0 \) and \( (br(J_i(y)), m) \not\in \mathcal{F}_n(Q) \). Thus from the assumption, \( B((T, l)) \leq B((br(J_i(y)), m)) \) and there exists \( j \) s.t. \( B(T, l)(i + 1) \leq_{\bar{G}} B(br(J_i(y)), m)(j) \) via an embedding \( K \). Then \( I_i \) can be extended on \( (br(y) \cap T_{i+1}) \) with \( K \), and the induction hypothesis is preserved. \[ \]

**Proof of induction step for theorem 3.6** Let \( (T, l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q) \) and \( S = \{ x \in T \mid l(x) = 0 \} \) and \( (br(x), l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q) \). For each \( s, t \in S \) s.t. \( s <_T t \), \( B((br(s), l)) \geq B((br(t), l)) \) by an identity embedding.

If \( (br(x), l) \) does not contain 0 then \( (br(x), l) \in \mathcal{F}_n(Q) \). Thus \( S \) (wrt \( <_T \)) is an infinite tree of the height \( \omega \).

Since \( B((T, l)) \in \mathcal{P}(S_{n+1}(Q^+ \cup \mathcal{F}_n(Q))) \), \( \{ B((U, m)) \mid (U, m) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q) \} \) is a BQO, thus well-founded. Then there exists \( s \in S \) s.t. for each \( t \in S \) s.t. \( s \leq_T t \), \( B((br(s), l)) \not\simeq B((br(t), l)) \) (thus \( B((br(s), l)) \equiv B((br(t), l)) \)). From lemma 3.3, \( (br(s), l) \leq_{\bar{G}} (br(t), l) \). But since \( (br(s), l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q) \), from definition there must be an infinite sequence \( s = s_0 \leq_T s_1 \leq_T \cdots \) s.t. \( (br(s_i), l) \geq_{\bar{G}} (br(s_{i+1}), l) \) for each \( i \). This is contradiction.

**Remark 3.1** The natural conjecture would be the extension of Kruskal-Friedman theorem for arbitrary large infinite trees. However, this has a counter example. Suppose \( \omega_0 = (\omega) \) be the least countable ordinal, \( \omega_1 \) be the least ordinal with cardinality \( 2^{\omega_0} \), etc. Then, an infinite sequence \( a_0, a_1, a_2, \cdots \) where \( a_i = 0^{\omega_1} \cdot i \) is bad \( \text{wrt} \leq_{\bar{G}} \). The extension of Kruskal-Friedman theorem for countable trees remains open.

## 4 Simple gap termination for term graph rewriting systems

**Definition 4.1** \[ \] JKdV94] A term graph \( s \) is a finite directed graph satisfying:

1. \( s \) has one root.
2. each non-terminal vertex of \( s \) has a label of a function symbol which has a fixed arity.
3. each terminal vertex of \( s \) has a label of either a constant symbol (i.e., function symbol with arity 0) or a variable symbol.

An \( \omega \)-term obtained by unfolding \( s \) is denoted \text{unravel}(s).

A term graph rewrite rule \( r \) is a graph with two (not necessary distinct) roots, called the left and right roots, satisfying:

1. each terminal vertex with a label of variable is accessible from the left root.
2. the subgraphs consisting of those vertices accessible from the left and the right roots, which are denoted as \text{left}(r) \text{ and } \text{right}(r), \text{ are term graphs.}
3. \textit{left}(r) is a finite tree.

A redex \(g\) of a term graph rewrite rule is a graph homomorphism from \(\mathit{left}(r)\). A \textit{term graph rewriting system} (TGRS, for short) \(R\) is a finite set of term graph rewrite rules.

Roughly speaking, reduction relation \(\rightarrow\) is defined similar to those which of a term rewriting system, except that a TGRS regards a variable as an address.\(^6\) We say an \textit{acyclic TGRS} for a TGRS on acyclic term graphs, and a \textit{cyclic TGRS} for a TGRS on possibly cyclic term graphs.

A rewrite system \(\rightarrow\) is \textit{terminating} if there is no infinite sequence s.t. \(s_1 \rightarrow s_2 \rightarrow \cdots\). Since a redex of a term graph rewrite rule \(r\) is defined as a graph homomorphism of \(\mathit{left}(r)\), a reduction includes an unfolding mechanism. This mimics the termination of a cyclic TGRS. For instance, a term graph rewrite rule \(r = (\mathit{LeftRoot} : a(\mathit{RightRoot}))\) corresponding to \(a(x) \rightarrow x\) is nonterminating for \(x = a^\omega\) (i.e., precisely a cyclic term graph \(x : a(x)\)).

\begin{definition}
Let \(\rightarrow_R\) be a reduction system on possibly cyclic term graphs defined by a TGRS \(R\). A reduction system \(\rightarrow_{\mathit{unravel}(R)}\) on \(\omega\)-terms is defined to be \(\mathit{unravel}(s) \rightarrow_{\mathit{unravel}(R)} \mathit{unravel}(t)\) iff \(s \rightarrow_R t\).
\end{definition}

From the definition of the redex of \(\rightarrow_R\), the next lemma holds. This implies the termination of \(\rightarrow_R\) is equivalent to the termination of \(\rightarrow_{\mathit{unravel}(R)}\).

\begin{lemma}
A term graph \(s\) is a normal form wrt \(\rightarrow_R\) iff an \(\omega\)-term \(\mathit{unravel}(s)\) is a normal form wrt \(\rightarrow_{\mathit{unravel}(R)}\).
\end{lemma}

Simple termination [Der82] is the frequently used criteria for a term rewriting system. However, the naive extension of simple termination based on Kruskal-type theorem on infinite trees [NW65, Lav78] does not work well for a cyclic TGRS. Let \(R = \{a(a(b(x))) \rightarrow a(b(x))\}\). Then \(R\) is terminating. \(R\) rewrites a term graph \(y : a(a(b(y)))\) to \(y : a(b(y))\), but both \(\mathit{unravel}(y : a(a(b(y)))) \geq \mathit{unravel}(y : a(b(y)))\) and \(\mathit{unravel}(y : a(a(b(y)))) \leq \mathit{unravel}(y : a(b(a(b(y))))\) = \(\mathit{unravel}(y : a(b(y)))\), because only fairness of occurrences of \(a, b\) on each path relates to \(\leq\).

Our termination criteria, named \textit{simple gap termination}, excludes \(\mathit{unravel}(y : a(a(b(y)))) \leq \mathit{unravel}(y : a(b(a(b(y))))\) as \(\mathit{unravel}(y : a(b(y))) \leq_G \mathit{unravel}(y : a(b(y)))\) with the gap condition for \(a > b\).

\begin{maintheorem}
Let \(R = \{l \rightarrow r\}\) be a TGRS. Assume that a set of function symbols is totally ordered. If there is a \(QO \leq \omega\) terms s.t.
\end{maintheorem}

\begin{enumerate}
\item For term graphs \(s, t\), \(\mathit{unravel}(s) \geq \mathit{unravel}(t)\) implies \(C[\mathit{unravel}(s)] \geq C[\mathit{unravel}(t)]\) for each context \(C[\ ]\).
\item \(C[\mathit{unravel}(s)] \geq \mathit{unravel}(s)\) where each function symbol \(f\) on a path from the root of \(C[\square]\) to \(\square\) satisfies \(f \geq \mathit{root}(s)\).
\item For each ground term graphs \(s, t, s \xrightarrow{l \rightarrow r} t\) (i.e., reduction at the root by the rule \(l \rightarrow r\)) implies \(\mathit{unravel}(s) > \mathit{unravel}(t)\).
\item \(\geq\) is infinitely transitive (i.e., if \(a_0 \leq a_1 \leq \cdots \leq a_\omega\) then \(a_0 \leq a_\omega\)).
\end{enumerate}

Then \(R\) is terminating.

\begin{proof}
From (1), (2), (4), \(\leq_G \leq_G\) on \(\omega\) terms. Suppose there exists an infinite reduction sequence \(s_1 \rightarrow s_2 \rightarrow \cdots\). Without loss of generality, we can assume that each \(s_i\) is a ground term graph. Thus

\footnote{For precise definition, please refer [JKdV94].}
from (1), (3), \( \text{unravel}(s_1) > \text{unravel}(s_2) > \cdots \). However, from main theorem 1, there exists \( i, j \) s.t. 
\( i < j \) and \( \text{unravel}(s_i) \leq_G \text{unravel}(s_j) \). Thus \( \text{unravel}(s_i) \leq \text{unravel}(s_j) \). This is contradiction.

**Example 4.1** Let \( R = \{ \text{a}(\text{a}(\text{b}(x))) \rightarrow \text{a}(\text{b}(x)) \} \). Then \( R \) is terminating, such as \( y : \text{a}(\text{a}(\text{b}(y))) \rightarrow_R \text{a}(\text{b}(y)) \) with \( a > b \).

**Example 4.2** Let \( R = \{ \text{a}(\text{b}(\text{a}(\text{b}(x)))) \rightarrow \text{a}(\text{b}(\text{b}(x))) \} \). Then \( R \) is terminating as a cyclic TGRS. (Furthermore \( R \) is simply terminating as an acyclic TGRS or TRS.) But, simple gap termination cannot show its termination. For instance, \( y : \text{a}(\text{b}(y)) \rightarrow_R y : \text{a}(\text{b}(\text{b}(y))) \) satisfies \( y : \text{a}(\text{b}(y)) \leq_G y : \text{a}(\text{b}(\text{b}(y))) \) with either \( a > b \) or \( a < b \). \( y : \text{a}(\text{b}(y)) \geq_G y : \text{a}(\text{b}(\text{b}(y))) \) is satisfied only with \( a > b \).

**Example 4.3** Let \( R = \{ \text{a}(\text{b}(\text{a}(\text{b}(x)))) \rightarrow \text{a}(\text{b}(\text{b}(x))) \} \). Then there is an instance \( y : \text{a}(\text{b}(y)) \rightarrow_R y : \text{a}(\text{b}(\text{b}(y))) \) satisfies \( y : \text{a}(\text{b}(y)) \leq_G y : \text{a}(\text{b}(\text{b}(y))) \) with either \( a > b \) or \( a < b \). Thus the termination of \( R \) cannot be shown by simple gap termination. Actually, \( R \) is not terminating such as

\[
\text{a}(\text{b}(y : \text{a}(\text{b}(y)))) \normalarrow_R \text{a}(\text{b}(y : \text{a}(\text{b}(y)))) \normalarrow_R \text{a}(\text{a}(\text{b}(y : \text{a}(\text{b}(y))))) \rightarrow_R \cdots
\]

**References**


