$W(E_7)$-invariant polynomial of degree 10 and 28 bitangents of plan equartic curves

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$W(E_7)$–invariant polynomial of degree 10 and 28 bitangents of plan equartic curves

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序文

ルート系に対する複比多様体という概念を筆者は定義した (cf. [Se4]) が、$E_7$型ルート系の場合にそれを見直し調べる。平面の非特異 4 次曲線には 28 本の複接線が存在するが、この古典的問題と関係がある。

本文の結果を説明する。

・$E_7$型ルート系に対する複比多様体の $E_6$ 型部分ルート系に対して定義される部分多様体。
・射影平面の 7 点の配置空間の特別な配置（変えられた 7 点に対して、この中のある点で cusp になるようなこれらの 7 点を通る 3 次曲線が存在するような配置）。
・平面の非特異 4 次曲線には 28 本の複接線が存在するが、それらの複接線の接点は普通 2 点あるがそれらが一致するような複接線が存在する。

以上 3 つの条件がワイル群 $W(E_7)$ のある 10 次の不変式を使って記述できる。この主張を示すことが本文の目的だが、証明には数式処理システム risa/asir を利用する。

§1. The root system of type $E_7$.

We first recall the definition of the root system $\Delta(E_7)$ of type $E_7$. We always denote it by $\Delta$ for simplicity in this paper. Let $\tilde{E}$ be an inner product space of dimension 8 with an orthonormal basis $\{\epsilon_j; 1 \leq j \leq 8\}$ with respect to an inner product $\langle \cdot, \cdot \rangle$ and let $E$ be its linear subspace
orthogonal to $\varepsilon_7 + \varepsilon_8$. As in [Se4], §4, we define the following 63 vectors of $E$:

$$
\gamma_1 = \varepsilon_8 - \varepsilon_7, \quad \gamma_j = \varepsilon_{j-1} - \gamma_0 + \gamma_1, \quad \gamma_{1j} = -\varepsilon_{j-1} + \gamma_0, \quad (1 < j < 8)
$$

$$
\gamma_{jk} = \varepsilon_{j-1} - \varepsilon_{k-1}, \quad \gamma_{1jk} = -\varepsilon_{j-1} - \varepsilon_{k-1}, \quad (1 < j < k < 8)
$$

where $\gamma_0 = \frac{1}{2} \sum_{j=1}^{8} \varepsilon_j - \varepsilon_7$. The totality $\Delta$ of $\pm \gamma_j, \pm \gamma_{jk}, \pm \gamma_{1jk}$ is a root system of type $E_7$ (cf. [B]).

As a fundamental set of roots of $\Delta$, we may take

$$
\alpha_1 = \gamma_{12}, \quad \alpha_2 = \gamma_{123}, \quad \alpha_3 = \gamma_{23}, \quad \alpha_4 = \gamma_{34}, \quad \alpha_5 = \gamma_{45}, \quad \alpha_6 = \gamma_{56}, \quad \alpha_7 = \gamma_{67}.
$$

Then the corresponding Dynkin diagram is:

```
  α₁ ——— α₃ ——— α₄ ——— α₅ ——— α₆ ——— α₇
     |           |           |
         α₂
```

We denote by $\Delta^+$ the set of positive roots in $\Delta$. It is easy to see that $\Delta^+$ consists of $\gamma_1, \gamma_{1j}, \gamma_{1jk}$.

If $g_j$ is the reflection on $E$ with respect to the root $\alpha_j$, the group generated by $g_1, \cdots, g_7$ is the Weyl group $W(E_7)$ of type $E_7$. In the sequel, we frequently identify $W(A_8) \simeq \Sigma_7$ (resp. the Weyl group $W(E_6)$ of type $E_6$) with the subgroup of $W(E_7)$ generated by $g_1, g_j$ ($j = 3, 4, 5, 6, 7$) (resp. $g_j$ ($j = 1, 2, 3, 4, 5, 6$)).

Using the 63 positive roots defined above, we define linear forms on $E$ by

$$
h_j = \gamma_j(t), \quad h_{jk} = \gamma_{jk}(t), \quad h_{1jk} = \gamma_{1jk}(t), \quad (t \in E).
$$

§2. The configuration space of 7 points in $P^2$.

We briefly review the definition of the configuration space of 7 points of $P^2$ which we denote by $P(2, 7)$. We first define the vector space $M_{3,7}$ of $3 \times 7$ matrices. Then $M_{3,7}$ admits $GL(3) \times GL(7)$-action in a natural manner. Let $D(7)$ be the maximal torus of $GL(7)$ consisting of diagonal matrices. Let $D_{ijk}(X)$ be the determinant of the $3 \times 3$ matrix consisting of the $i, j, k$-th column vectors of $X \in M_{3,7}$. If $M^*_{3,7}$ is the subset of $M_{3,7}$ consisting of $X$ with $D_{ijk}(X) \neq 0 \quad \forall (i, j, k) (i < j < k)$, we denote by $P(2, 7)$ the quotient of $M^*_3_{7}$ by the action $GL(3) \times D(7)$. It is possible to choose as a representative of any element of $P(2, 7)$ a matrix of the form

$$
X = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\
0 & 0 & 1 & 1 & y_1 & y_2 & y_3
\end{pmatrix}
$$
In this way, $\mathbb{P}(2, 7)$ is regarded as a quasi-affine subset of $\mathbb{C}^6$ by the correspondence

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & x_1 & x_2 \\
0 & 0 & 1 & 1 & y_1 & y_2 \\
\end{pmatrix}
\rightarrow (x_1, x_2, x_3, y_1, y_2, y_3).
$$

In fact, $\mathbb{P}(2, 7)$ is identified with $\mathbb{C}^6 - S_0(A_6)$, where $S_0(A_6)$ is the union of the 28 hypersurfaces below:

- $x_i = 0$,  $x_i - 1 = 0$,  $y_i = 0$,  $y_i - 1 = 0$,  $x_i - x_j = 0$,  $y_i - y_j = 0$,  $x_i - y_i = 0$,
- $x_i y_j - x_j y_i = 0$,  $(1 - x_i)(1 - y_j) - (1 - x_j)(1 - y_i) = 0$,
- $\varphi_1(x_1, x_2, x_3, y_1, y_2, y_3) = \det \begin{pmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
\end{pmatrix} = 0$.

We introduce the following seven birational transformations $s_1, \cdots, s_6, s_R$:

- $s_1 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (1/x_1, 1/x_2, 1/x_3, y_1/x_1, y_2/x_2, y_3/x_3)$
- $s_2 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (y_1, y_2, y_3, x_1, x_2, x_3)$
- $s_3 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (x'_1, x'_2, x'_3, y'_1, y'_2, y'_3)$
- $s_4 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (1/x_1, x_2/x_1, x_3/x_1, 1/y_1, y_2/y_1, y_3/y_1)$
- $s_5 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (x_2, x_1, x_3, y_2, y_1, y_3)$
- $s_6 : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (x_1, x_3, x_2, y_1, y_3, y_2)$
- $s_R : (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (1/x_1, 1/x_2, 1/x_3, 1/y_1, 1/y_2, 1/y_3)$

where

$$x'_j = \frac{x_j - y_j}{1 - y_j}, \quad y'_j = \frac{y_j}{y_j - 1}, \quad j = 1, 2, 3.$$

The correspondence

$$g_1 \rightarrow s_1, \quad g_2 \rightarrow s_R, \quad g_j \rightarrow s_{j-1} \quad (j = 3, \cdots, 7)$$

induces a group isomorphism of $W(E_7)$ to the group generated by $s_1, \cdots, s_6, s_R$.

We introduce 7 polynomials of $(x_1, x_2, x_3, y_1, y_2, y_3)$ defined by

$$\sigma_j(x_1, x_2, x_3, y_1, y_2, y_3) = \varphi_{6-j}(x_1, x_2, x_3, y_1, y_2, y_3), \quad (j = 1, 2, 3, 4),$$
\[\sigma_5(x_1, x_2, x_3, y_1, y_2, y_3) = x_2 y_3 (1 - x_3) (1 - y_2) - x_3 y_2 (1 - x_2) (1 - y_3),\]
\[\sigma_6(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 y_3 (1 - x_3) (1 - y_1) - x_3 y_1 (1 - x_1) (1 - y_3),\]
\[\sigma_7(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 y_2 (1 - x_2) (1 - y_1) - x_2 y_1 (1 - x_1) (1 - y_2),\]

where \(\varphi_j\) (\(j = 2, 3, 4, 5\)) are polynomials introduced in [Se4], §4. In particular,
\[\varphi_2(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 x_2 x_3 y_1 y_2 y_3 \varphi_1(1/x_1, 1/x_2, 1/x_3, 1/y_1, 1/y_2, 1/y_3)\]

Let \(Q_j\) be the hypersurface in \(\mathbb{C}^6\) defined by \(\sigma_j = 0\) (\(j = 1, \ldots, 7\)). Then it is easy to see that \(\Sigma_7\) acts on the set \(\{Q_1, \ldots, Q_7\}\) as a permutation group. If \(\tilde{\sigma}_7 = \sigma_7\) and \(\tilde{\sigma}_j = \tilde{\sigma}_{j+1} \circ s_j\) (\(j = 1, 2, \ldots, 6\)), and \(Q'_j\) is the hypersurface in \(\mathbb{C}^6 - S_0(A_6)\) defined by \(\tilde{\sigma}_j = 0\), then \(Q_j\) is the Zariski closure of \(Q'_j\) in \(\mathbb{C}^6\). A geometric meaning of \(Q_j\) will be given in §6. In the sequel, we denote by \(P_0(2, 7)\) the complement of the union \(S(E_7)\) of \(S_0(A_6)\) and \(Q_1, \ldots, Q_7\). Clearly all the elements of \(W(E_7)\) induce biregular transformations on \(P_0(2, 7)\).

§3. The cross ratio variety \(C(\Delta(E_7), D_4)\).

For any subroot system \(\Delta_1\) of type \(D_4\) in \(\Delta\), we defined a \(D_4\)-cross ratio map of the Zariski open subset \(Z(\Delta)\) of the projective space \(P^6 = P(E\mathbb{C})\) associated to the complexification \(E\mathbb{C}\) of \(E\) to \(CR(P) \cong P^1\). There are totally 315 subroot systems of type \(D_4\) in \(\Delta\). The corresponding \(D_4\)-cross ratio maps are denoted by
\[cr^1_{[i_3, i_1, i_7]} = (h_{i_1i_4} h_{i_2i_3i_4} h_{i_1i_6} h_{i_1i_9i_5} : -h_{i_1i_4} h_{i_1i_3i_4} h_{i_2i_5} h_{i_2i_3i_5} : h_{i_1i_2} h_{i_1i_7} h_{i_4i_6} h_{i_3i_4i_5})\]
\[cr^2_{[i_1i_2, i_3i_4, i_6i_7]} = (h_{i_1i_3i_4} h_{i_2i_4i_5} h_{i_2i_3i_4} h_{i_1i_4i_6} : -h_{i_2i_3i_4} h_{i_1i_4i_5} h_{i_2i_4i_6} h_{i_1i_4i_7} : h_{i_1i_2} h_{i_3i_4} h_{i_4i_6} h_{i_3i_4i_5})\]
\[cr^3_{[i_1, i_2, i_3i_4]} = (h_{i_1i_2i_7} h_{i_3i_4i_5} h_{i_3i_4i_7} h_{i_1i_7} : -h_{i_1i_2i_7} h_{i_3i_4i_5} h_{i_3i_4i_7} h_{i_1i_7} : h_{i_1i_2i_7} h_{i_3i_4i_5} h_{i_3i_4i_7} h_{i_1i_7})\]
(cf. [Se4], §4). By taking the product of all the 315 maps above, we obtain a map \(cr_{D_4, \Delta}\) of \(Z(\Delta)\) to \(CR(P)^{315}\). Let \(C'(\Delta, D_4)\) be the image \(cr_{D_4, \Delta}(Z(\Delta))\) and let \(C(\Delta, D_4)\) be its closure in \(CR(P)^{315}\).

For any subroot system \(\Delta_1\) of \(\Delta\), we defined a subvariety \(Y_{\Delta_1, D_4}(\Delta_1)\) in [Se4], §4. There are four kinds of hypersurfaces of \(C(\Delta, D_4)\) defined as the form \(Y_{\Delta, D_4}(\Delta_1)\) for suitable subroot systems.

§4. Hypersurfaces corresponding to subroot systems of type \(E_6\).

We introduce hypersurfaces of \(C(\Delta, D_4)\) which are fixed by \(W(E_6)\)-actions (cf. [Se4], §4,(4.15.10)). If \(\Delta_1\) is a subroot system of type \(E_6\) in \(\Delta\), it is easy to show that \(Y_{\Delta_1, D_4}(\Delta_1)\) is a hypersurface of
\(C(\Delta, D_4)\). Such a hypersurface is called that of the \(5^{th}\) kind. As a basic property of hypersurfaces of the \(5^{th}\) kind, we have the lemma below.

**Lemma 4.1.** (cf. [Se4a]) \(Y_{\Delta, D_4}(\Delta(E_6)) \simeq C(\Delta(E_6), \{A_3, D_4\})\).

Lemma 4.1 establishes an embedding of the cross ratio variety \(C(\Delta(E_6), \{A_3, D_4\})\) into \(C(\Delta, D_4)\). To show an identification of \(C(\Delta(E_6), \{A_3, D_4\})\) with the variety defined \([L]\), we need some preparation on cubic curves in \(\mathbb{P}^2\) passing through 7 points. For simplicity, we take 7 points \(P_1, \ldots, P_7\) of \(\mathbb{P}^2\) as follows:

\[
\begin{align*}
P_1 &= (1 : 0 : 0), & P_2 &= (0 : 1 : 0), & P_3 &= (0 : 0 : 1), & P_4 &= (1 : 1 : 1), \\
P_5 &= (1 : x_1 : y_1), & P_6 &= (1 : x_2 : y_2), & P_7 &= (1 : x_3 : y_3).
\end{align*}
\]

We assume that the 7 points above are in a general position which means the corresponding matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\
0 & 0 & 1 & 1 & y_1 & y_2 & y_3
\end{pmatrix}
\]

is a representative of the configuration space \(\mathbb{P}(2, 7)\).

Let \(C(P_1, \ldots, P_6; P_7)\) be the cubic curve in \(\mathbb{P}^2\) passing through \(P_1, \ldots, P_7\) such that \(P_7\) is a double point (cf. [M], [L]). We now consider the case where \(C(P_1, \ldots, P_6; P_7)\) has a cusp at \(P_7\) (cf. [L]). This condition implies a relation \(\Psi(x, y) = 0\) among \((x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)\).

The explicit form of the polynomial \(\Psi(x, y)\) is too lengthy to write down here. It is provable that \(\deg_x \Psi = \deg_y \Psi = 8\).

Noting that \(C(\Delta, D_4)\) is a compactification of \(\mathbb{P}_0(2, 7)\), we obtain a hypersurface \(Y_{\text{cusp}}\) of \(C(\Delta, D_4)\) as the Zariski closure of the hypersurface of \(\mathbb{P}_0(2, 7)\) defined by \(\Psi(x, y) = 0\).

**Theorem 4.2.** \(Y_{\text{cusp}} = Y_{\Delta, D_4}(\Delta_1) \cap \mathbb{P}_0(2, 7)\).

The basic idea of the proof employed here is the comparison between the defining equations of \(Y_{\Delta, D_4}(\Delta_1)\) and \(Y_{\text{cusp}}\). Before entering the details of its outline, we state a result on the polynomial \(\Psi(x, y)\).

**Lemma 4.3.** We put

\[
\Phi(x, y) = \Phi_1(x, y)^2 - 4\Phi_2(x, y),
\]

where
\[ \Phi_1(x, y) = x_1x_2y_1 - x_1x_3y_1 + x_1y_1y_3 + x_1y_2 - x_1y_3 + x_2y_1y_3 - x_2y_1 - x_3y_1y_2 + x_3y_1, \]
\[ \Phi_2(x, y) = (x_1 - y_1)(x_2y_3 - x_3y_2)(y_1 - 1)(y_2 - y_3)x_1. \]

Then there is \( s \in W(E_7) \) such that \( \Phi \circ s = \Psi \).

We are going to explain the outline of the proof of Theorem 4.2.

We first compute the condition that the cubic curve \( C(P_1, \cdots, P_6; P_7) \) has a cusp at \( P_7 \). For this purpose, we assume that \( F(\xi_1, \xi_2, \xi_3) = 0 \) is the defining equation of \( C(P_1, \cdots, P_6; P_7) \), where
\[ F = c_1\xi_1^3 + c_2\xi_2^3 + c_3\xi_3^3 + c_4\xi_1\xi_2^2 + c_5\xi_1^2\xi_3 + c_6\xi_2\xi_3^2 + c_7\xi_1\xi_2\xi_3 + c_8\xi_1\xi_2^2 + c_9\xi_2\xi_3 + c_{10}\xi_1\xi_3^2. \]

In the discussion above, we have taken \( \xi = (\xi_1 : \xi_2 : \xi_3) \) as a homogeneous coordinate of \( \mathbb{P}^2 \).

The condition that \( C(P_1, \cdots, P_6; P_7) \) passes through \( P_1, \cdots, P_7 \) is equivalent to \( (C.1) \)
\[ F(P_j) = 0, \quad j = 1, \cdots, 7. \]

The condition that \( P_7 \) is a double point of \( C(P_1, \cdots, P_6; P_7) \) is equivalent to \( (C.2) \)
\[ F_{\xi_i}(P_7) = 0, \quad i = 1, 2, 3. \]

The condition that \( P_7 \) is moreover a cusp point of \( C(P_1, \cdots, P_6; P_7) \) is equivalent to \( (C.3) \)
\[ F_{\xi_1\xi_1}(P_7)F_{\xi_2\xi_2}(P_7) - F_{\xi_1\xi_2}(P_7)^2 = 0. \]

From \( (C.1), (C.2) \), we conclude that the ratio of \( c_1, \cdots, c_{10} \) is uniquely determined. Substituting such \( c_1, \cdots, c_{10} \) to the equation \( (C.3) \), we obtain an algebraic relation
\[ \Psi(x, y) = 0 \]
if \( (x, y) \in C^6 - S(A_6) \). We need a long computation to obtain \( (4) \) and it is hard to reproduce here.

Our next purpose is to compute the defining equation of the hypersurface \( Y_{\Delta, D_4}(\Delta_1) \) in \( \mathbb{P}(2,7) \).

For this purpose, we first recall the definition of the rational map of \( \mathbb{P}^6 \) to \( \mathbb{P}(2,7) \) in [Se4].

Lemma 4.2. We put
\[ x_1(t) = \frac{h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135}}{h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235}}, \quad x_2(t) = \frac{h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136}}{h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}}, \quad x_3(t) = \frac{h_{24} \cdot h_{234} \cdot h_{17} \cdot h_{137}}{h_{14} \cdot h_{134} \cdot h_{27} \cdot h_{237}}, \]
\[ y_1(t) = \frac{h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125}}{h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235}}, \quad y_2(t) = \frac{h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126}}{h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236}}, \quad y_3(t) = \frac{h_{34} \cdot h_{234} \cdot h_{17} \cdot h_{127}}{h_{14} \cdot h_{124} \cdot h_{37} \cdot h_{237}}. \]

and define the map \( F_{E_7} \) of \( Z(\Delta) \) to the \((x, y)\)-space by
\[ F_{E_7}(t) = (x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t)), \]
where \( \mathbb{P}(2,7) \) is identified with a Zariski open subset of the \((x, y)\)-space (cf.[Se4], §4) and \( h_1, h_{ij}, h_{ijk} \).
are linear forms on $E$ associated with the roots of $\Delta$. Now we put

$$\zeta_1 = h_{12}, \quad \zeta_2 = h_{123}, \quad \zeta_3 = h_{23}, \quad \zeta_4 = h_{45}, \quad \zeta_5 = h_{56}, \quad \zeta_7 = h_{67}.$$ 

It is clear from the definition that linear forms in question corresponding to the roots of $\Delta(E_6)$ are expressed as linear combinations of $\zeta_j$ ($j = 1, \cdots, 6$). We may take $\zeta = (\zeta_1 : \cdots : \zeta_6)$ as the homogeneous coordinate of $P^6$. Now we write $\zeta_j = \zeta_j' \tau$ ($j = 1, \cdots, 6$). Then we put

$$\tilde{\tau} = ((\zeta_1', \cdots, \zeta_6'), \tau)$$

as also a local coordinate of an affine open subset defined by $\tau \neq 0$ in $P^6$.

Regarding (5) $f_1 = \cdots = f_6 = 0,$

as a system of equations for $\zeta_1, \cdots, \zeta_6$ with coefficients in the function field $C(x, y)$, we are going to solve the system (5). If $x_j, y_j$ ($j = 1, 2, 3$) satisfy an algebraic equation $\Psi'(x, y) = 0$, the system (5) has a non-trivial solution. From the construction, the hypersurface $\Psi'(x, y) = 0$ in $P(2, 7) = C'(D_{4}, D_4)$ is nothing but the subvariety $Y_{\Delta, D_4}(\Delta(E_6))$ (cf.[Se4], §§1,4). By a little
lengthy computation, we conclude that \( \Psi'(x, y) \) coincides with \( \Psi(x, y) \) up to a constant factor, where \( \Psi(x, y) \) is the polynomial introduced before.

In this way, we can prove Theorem 2.

§5. Comparison between \( Y_{\Delta(E_6), D_4}(\Delta(D_4)) \) and \( Y_{\Delta(E_7), D_4}(\Delta(E_6)) \).

It is worthwhile to compare similarities between hypersurfaces of the 5th kind and the subvariety \( Y_{\Delta(E_6), D_4}(\Delta(D_4)) \) of \( C(\Delta(E_6), D_4) \) introduced in [Se4], §3 (cf.[N], [L], [Sh2]).

<table>
<thead>
<tr>
<th>( Y_{\Delta(E_6), D_4}(\Delta(D_4)) )</th>
<th>( Y_{\Delta(E_7), D_4}(\Delta(E_6)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a cubic surface with an Eckardt point</td>
<td>a plane quartic curve with a special flex</td>
</tr>
<tr>
<td>( \lambda - 1 = 0 )</td>
<td>( \Psi(x, y) = 0 )</td>
</tr>
<tr>
<td>a cross ratio variety for ( \Delta(D_4) ) (?)</td>
<td>( C(\Delta(E_6), {A_3, D_4}) )</td>
</tr>
<tr>
<td>associated quintic</td>
<td>a ( W(E_7) )-invariant of 10th degree</td>
</tr>
<tr>
<td>configuration of 6 points</td>
<td>configuration of 7 points</td>
</tr>
</tbody>
</table>

We give here an explanation on TABLE I.

(6.1) Let \( S \) be a non-singular cubic surface in \( \mathbb{P}^3 \). An Eckardt point on \( S \) is the intersection of three lines on \( S \) (cf. [N]). Every cubic surface does not have an Eckardt point. On the other hand, a flex of a non-singular plane quartic \( C \) is a point \( p \in C \) such that there is a line \( l \) triply tangent to \( C \) at \( p \) (cf. [Sh2]). A flex is ordinary if \( l \cap C \) consists of two points and a flex is special if \( l \cap C = \{p\} \). Every plane quartic does not have a special flex.

(6.2) In [N], the parameter \( \lambda \) was introduced. It was shown in [Se3] (cf.[H]) that \( \lambda \) is regarded as a rational function on \( \mathbb{P}(2, 6) \). In fact, using the notation in [Se3], we have

\[
\lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)}. \]

(6.3) Is it possible to regard \( Y_{\Delta(E_6), D_4}(\Delta(D_4)) \) as a cross ratio variety for the root system \( \Delta(D_4) \) of type \( D_4 \) ?

(6.4) If \( \delta_5(t) \) is a \( W(E_6) \)-invariant polynomial of degree 5 (which is unique up to a constant factor), it is shown in [Se3] that the polynomial \( P_5(t) = \delta_5(t_1, t_2, t_3, t_4, t_5, -3t_5) \) is \( W(F_4) \)-semi-invariant under the notation there. Hence, by \( W(F_4) \)-action, we obtain totally 45 quintic polynomials on the standard representation space of \( W(E_6) \). For the sake of convenience, we call these polynomials associated quintics. There is a 1-1 correspondence between the set of associated quintics and that of the 45 triple tangent planes.

Similarly, there is a \( W(E_7) \)-invariant polynomial \( \delta_{E_7}(t) \) of degree 10 which plays a role analogues...
to $\delta$. (The construction of $\delta_{E_7}(t)$ will be given later.)

(6.5) (cf. [L]) Let $P_1, \cdots, P_6$ be 6 points of $P^2$. We consider a conic $C$ passing through five points $P_1, \cdots, P_5$ and a line $L$ passing through $P_5, P_6$. The condition corresponding to $\lambda - 1 = 0$ is that the line $L$ is also a tangent of $C$ at $P_6$. Let $P_1, \cdots, P_7$ be 7 points of $P^2$. The condition corresponding to $\Psi(x, y) = 0$ is the main subject in the previous section. Namely, let $P_1, \cdots, P_7$ be 7 points of $P^2$. We consider a cubic curve $C$ passing through seven points $P_1, \cdots, P_7$ such that $P_7$ is a double point. The condition corresponding to $\Psi(x, y) = 0$ is that $C$ has a cusp at $P_7$.

We are going to explain the construction on $\delta_{E_7}(t)$.

Let $\omega_j$ be a fundamental weight of $\Delta(E_7)$ such that $\omega_j$ belongs to the set of weights of the 56 dimensional irreducible representation of the simple Lie algebra of type $E_7$. By definition, $\omega_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7$. The totality $\Pi$ of $\alpha \in \Delta$ such that $\omega_7, \alpha >= 0$ form a root system of type $E_6$. Let $\Omega_{27}$ be the set of weights of 27 irreducible representation of the simple Lie algebra of type $E_6$ corresponding to the root system $\Pi$. Then $z_p = \sum_{\omega \in \Omega_{27}} \omega^p$ ($p = 1, 2, \cdots$) are $W(\Pi)$-invariant polynomials. (From the definition, $W(\Pi) \cong W(E_6)$.)

**Lemma 5.1.** Under the notation above,

$$P(t) = 43545600\omega_7^{10} - 3628800z_2\omega_7^8 + 100800z_2^2\omega_7^6 + 725760z_5\omega_7^5$$
$$+ (4200z_2^3 - 604800z_5)\omega_7^4 - 20160z_2z_5\omega_7^3 + (175z_2^4 - 40320z_2z_5 + 181440z_5)\omega_7^2$$
$$+ (3360z_2^2z_5 - 57600z_5)\omega_7 + 1008z_5^2$$

is a $W(E_7)$-invariant polynomial of degree 10.

For simplicity, we put $P = P(\omega_7, z_2, z_5, z_6, z_8, z_9)$. From the definition, $P$ is defined on the standard representation space $E$ of $W(E_7)$. Therefore $P = 0$ is a hypersurface on $P^6 = P(E_C)$. Similarly, $P(-2\omega_7, z_2, z_5, z_6, z_8, z_9) = 0$ defines a hypersurface $H_{\omega_7}$ in $P^6$.

**Theorem 5.2.** The closure of $cr_{D_4, \Delta}(H_{\omega_7})$ is isomorphic to $Y_{\Delta(E_7), D_4}(\Delta(E_6))$ by $W(E_7)$-action.

From the theorem above, $\delta_{E_7}(t) = P(\omega_7, z_2, z_5, z_6, z_8, z_9)$ is a required $W(E_7)$-invariant polynomial. Therefore $P(-2\omega_7, z_2, z_5, z_6, z_8, z_9)$ is an associated $W(E_7)$-invariant polynomial of degree 10. There are totally 28 associated hypersurfaces in $C(\Delta, D_4)$.

It is interesting to characterize the polynomial $\delta_{E_7}(t)$ among $W(E_7)$-invariant polynomials of homogeneous degree 10.
Appendix. Theorem 4.2, Theorem 5.2 について — 数式処理システムの利用 —

(1) §4 の式 $\Psi(x, y)$ の計算はかなり大変。数式処理システム REDUCE3.4 を Toshiba J3100 で利用して計算して得られた 2 通りの式が等しいことを示した。10,000 行程度の式を因数分解して結果的に 1,800 行程度の式になった。最近、富士通の開発した risa/asir を PC9801 NS/R で利用して計算し直してみたが、これならかなり楽に結論を導けた。

(2) $Y_{\Delta, D_{4}}(\Delta(E_{6}))$ の形の $C(\Delta, D_{4})$ の部分多様体は 56 次元実現の weight と対応する。56 次元表現の weight は

$$\pm(e_{1} + e_{2} + e_{3} + e_{4} + e_{5} + e_{6}), \pm(e_{1} + e_{2} + e_{3} + e_{4} - e_{5} - e_{6}), \pm(\gamma_{1} \pm 2e_{j})$$

である。$\omega_{7} = (\gamma_{1} - 2e_{6})$ および $-\omega_{7}$ に対応するのが $\Psi(x, y) = 0$ で定義される超平面。$\pm(e_{1} + e_{2} + e_{3} + e_{4} - e_{5} - e_{6})$ に対応するのが Lemma 4.3 で導入した $\Phi(x, y)$ で定義される超平面。この計算は risa/asir に頼った。simple reflection で weight を次々に移すのと平行して 1,800 行程度の式を有理变换で座標変換していけばよい。結果を確認することは容易。

(3) $x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}$ を $t$ の斉次有理式で表して，$\Phi(x, y)$ に代入すると $t$ の有理式が得られるが，その分子の自明でない因子は 22 次斉次式である。この式をさらに因数分解すると 1 次の因子が 12 個あり、残りの因子が $F(t) = \Phi(-2\omega_{7}, z_{2}, z_{5}, z_{6}, z_{9})$ になる。一次の因子はすべて鏡面に対応する。$F(t)$ は $t$ の式とみると 5,000 行程度である。適当な変数変換によって $F(t)$ から $W(E_{7})$ 不変式を導き出すのは自明ではないようにかんじる。$E_{6}$ の場合にそうなっていたのが唯一の根拠。それにもとづいて，$P(\omega_{7}, z_{2}, z_{5}, z_{6}, z_{9})$ が $W(E_{7})$ 不変になるような定数 $p$ を求めよ，という問題を考えた。もし正しいとするならば，$t$ として特殊値を代入しても成り立つはずであると考え，適当な値を代入して調べてみた。正しいか正しくないか computer に計算させたところ，$p = 1$ ならば少なくともある特値では正しいことがわかった。それで，いくつか特殊値を代入して確かめたところいずれも正しいことが示せた。それで $t$ を変数として計算して主張を示した。奇妙なことだが $P(-2\omega_{7}, z_{2}, z_{5}, z_{6}, z_{8}, z_{9})$ は $t$ 式とみて 5,000 行程度とかなり長い式だが $P(\omega_{7}, z_{2}, z_{5}, z_{6}, z_{8}, z_{9})$ は 270 行程度のもとと比べればかなり短い式になった。

§6. Relations between $C(\Delta, D_{4})$ and cubic surfaces.

At a meeting organized by H. Yamada held in RIMS, Kyoto University (December, 1993), I. Naruki and J. Matsuzawa gave talks on a root system construction of universal cubic surfaces. They constructed a fibre space $\tilde{C}$ of cubic surfaces over Naruki's cross ratio variety $C(= C(\Delta(E_{6}), D_{4}))$ so that the natural projection $\varpi : \tilde{C} \rightarrow C$ is $W(E_{6})$-equivariant.

In this section, we discuss a relation between $C(\Delta, D_{4})$ and $\tilde{C}$.

For this purpose, we introduce the set $\tilde{P}(2, k)$ of $k$ points $(P_{1}, \cdots, P_{k})$ in $\mathbb{P}^{2}$ such that $P_{i} \neq P_{j}$ if $i \neq j$. By definition, $\tilde{P}(2, k)$ admits a $PGL(3)$-action. Let $P(2, k)$ be the quotient of $\tilde{P}(2, k)$
by \( \text{PGL}(3) \). It is clear that \( \text{P}(2,7) \) is nothing but the one introduced in \( \S 2 \). There is a natural projection \( p \) of \( \text{P}(2,k+1) \) to \( \text{P}(2,k) \) defined by \( p((P_1,\cdots,P_k,P_{k+1}))=(P_1,\cdots,P_k) \).

From now on, we focus our attention to the cases \( k=6,7 \). It is known (cf. [Se4]) that there is a birational \( W(E_k) \)-action on \( \text{P}(2,k) \) \( (k=6,7) \). This easily implies that the projection \( p: \text{P}(2,7) \to \text{P}(2,6) \) is \( W(E_6) \)-equivariant. Denoting by \( \hat{p} \) the extension of \( p \) to \( C(\Delta, D_4) \), we obtain a birational \( W(E_6) \)-equivariant map \( \hat{p}: C(\Delta, D_4) \to C(\Delta(E_6), D_4) \).

We consider the \( W(E_6) \)-orbits of the set of hypersurfaces of the \( 1^{st} \) kind in \( C(\Delta, D_4) \). There are two orbits. The first one denoted by \( \Omega_1 \) consists of those corresponding to roots contained in \( \Delta(E_6) \):

\[
Y_i, \quad Y_{ij} \ (1 \leq i < j < 7), \quad Y_{ijk} \ (1 \leq i < j < k < 7).
\]

The second one denoted by \( \Omega_2 \) consists of the remaining 27 hypersurfaces:

\[
Y_i \ (1 \leq i < 7), \quad Y_{i7} \ (1 \leq i < 7), \quad Y_{ij} \ (1 \leq i < j < 7).
\]

For any \( (P_1,\cdots,P_k) \in \text{P}(2,6) \), the closure \( S(P_1,\cdots,P_k) = \overline{p^{-1}((P_1,\cdots,P_k))} \) of its fibre in \( C(\Delta, D_4) \) is of dimension 2. The surface \( S(P_1,\cdots,P_k) \) intersects with all the hypersurfaces of \( \Omega_2 \). By the intersection relations among hypersurfaces of the \( 1^{st} \) kind, we easily find that the intersection relations among the 27 curves \( S(P_1,\cdots,P_k) \cap Y (\forall Y \in \Omega_2) \) on \( S(P_1,\cdots,P_k) \) are same as those of the 27 lines on a non-singular cubic surface.

If the interpretation of the work of Naruki and Matsuzawa is correct, \( \tilde{C} \) coincides with \( C(\Delta, D_4) \) and \( \hat{p}: C(\Delta, D_4) \to C(\Delta(E_6), D_4) \) defined above is the natural projection \( \varpi \). As an easy consequence (\(?\)), \( S(P_1,\cdots,P_k) \) is a cubic surface. Therefore it is hopeful that \( S(P_1,\cdots,P_k) \cap Y (\forall Y \in \Omega_2) \) are the 27 lines on it. If this is true, hypersurfaces of \( \Omega_2 \) are global sections of 27 lines of cubic surfaces in the total space \( \tilde{C} \).

From the definition, \( \text{P}(2,7) \) is identified with the open subset of \( (x_1, x_2, x_3, y_1, y_2, y_3) \)-space outside the union \( S_0(A_6) \) of 28 hyperplanes (cf. \( \S 2 \)). Moreover, we introduced 7 hypersurfaces \( Q_1,\cdots,Q_7 \) of the \( (x_1, x_2, x_3, y_1, y_2, y_3) \)-space in order to define \( \text{P}_0(2,7) \). It is clear that the closure of the hypersurface \( D_{ijk}=0 \in C(\Delta, D_4) \) is nothing but \( Y_{ijk} \) and that of \( Q_j \) is \( Y_j \).

We now take seven points \( P_1,\cdots,P_6, P_7 \) as in \( \S 4 \) and fix \( P_j \ (j=1,\cdots,6) \) for the moment. Then \( P_7 \) is regarded as a point on \( \text{P}^2 - \{P_1,\cdots,P_6\} \). Therefore \( (x_3,y_3) \) are interpreted as an inhomogeneous coordinate of \( \text{P}^2 \). Under this identification, the defining equation \( D_{ij7}=0 \) corresponds to the line on \( \text{P}^2 \) passing through \( P_i \) and \( P_j \) for \( i,j \ (1 \leq i < j < 7) \). On the other hand, the defining equation of \( Q_i \) corresponds to the conic on \( \text{P}^2 \) passing through the five points \( \{P_j; j=1,\cdots,6, j \neq i\} \). This is a geometric interpretation of hypersurfaces \( Q_1,\cdots,Q_7 \) (cf. [M], Theorem 26.2).
In [Se4] §3, we have studied the structure of subvarieties of the form $Y(M)$ in $C(\Delta, D_4)$, where $M$ is a subset of $\Delta$ consisting of mutually orthogonal positive roots. In particular, we now treat the intersection $Y_\alpha \cap Y_\beta$ for $\alpha, \beta \in \Delta$ such that $Y_\alpha, Y_\beta \in \Omega_2$. The intersection $Y_\alpha \cap Y_\beta$ may be regarded as a global section of the intersection of two lines of cubic surfaces $S(P_1, \cdots, P_6)$, being assumed that $S(P_1, \cdots, P_6)$ is a cubic surface for any $(P_1, \cdots, P_6) \in \mathbb{P}(2, 6)$. It is interesting to make clear a relation between cubic surfaces and the global section of the intersections of two lines on them.

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