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Quantum Stochastic Calculus and Applications – a Review†

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It has been a little more than a decade since this subject, as it is understood today, came into being with the seminal paper of Hudson and Parthasarathy [1]. Since then the subject has seen rapid development and many of these can be found in the monographs of Parthasarathy [2] and Meyer [3]. Here I want to discuss some of the more recent developments, many of which took place in Indian Statistial Institute, Delhi.

The first section contains notations and a collection of some basic results, the proofs of which can be found in the two monographs mentioned above. The second section deals with quantum stochastic differential equations (q.s.d.e.) with unbounded operator coefficients and Feller condition. Section 3 is devoted to quantum martingales and their representations while in section 4 we discuss stoptimes and the strong Markov property of the Fock space w.r.t. finite stop times. Finally in section 5 we discuss some applications.

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1 Notations and Preliminaries

We shall work exclusively in the bosonic (symmetric) Fock space and shall give a few background results, referring the reader to [2] and [3] for the details.

Let h be a complex separable Hilbert space, $\Gamma(h)$ be the symmetric Fock space over h, spanned by the total set of 'exponential vectors'

$$e(f) = 1 \oplus f \oplus \cdots \oplus \frac{f^{(n)}}{\sqrt{n!}} \oplus \cdots, \tag{1}$$

where $f \in h$ and $f^{(n)}$ is the n-fold tensor product of f. In most of our discussions, h will be $L^2(\mathbb{R}_+, \mathcal{K})$ with $\mathcal{K}$ another separable Hilbert space, the dimension of which will signify the noise degree of freedom. Let $\mathcal{H}_0$ (a separable Hilbert space) be the initial space or the system space. The $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(h)$ is the Hilbert space in which we shall work.

Let $\{e_n\}_{n=1}^{N \equiv \dim \mathcal{K}}$ denote a complete orthonormal system of $\mathcal{K}$. Then the *basic quantum processes* in $\mathcal{H}$ are

$$\left.\begin{array}{rcl} \Lambda_j^k(t)e(f) & = & -\, i\, \frac{d}{d\varepsilon} e\left(e^{i\varepsilon\chi_{[0,t]}\otimes|e_j\rangle\langle e_k|}f\right)\Big|_{\varepsilon=0}, (1 \le j,k \le N) \\ A_j(t)e(f) & = & \int_0^t ds\; f_j(s)e(f) \equiv \int_0^t ds\langle e_j, f(s)\rangle e(f), \\ A_j^{+}(t)e(f) & = & \frac{d}{d\varepsilon} e(f + \varepsilon\chi_{[0,t]}e_j)|_{\varepsilon=0}, \end{array}\right\} \tag{2}$$

where $\chi_{[0,t]}$ is looked upon in the first expression as the multiplication operator (projection) by the indicator function of $[0,t]$ while in the last it is the indicator function itself. If we set

$$Q_j(t) = A_j(t) + A_j^{+}(t) \text{ and } P_j(t) = -i[A_j(t) - A_j^{+}(t)],$$

then these define two sets of independent countable families of Brownian motions, but $[P_j(t), Q_k(s)] = -2i(t \wedge s)$. Thus there are two sets of non-commuting Brownian motions in Fock space. Similarly, the selfadjoint operators: $\Lambda_j^j(t) + \sqrt{\ell}\, Q_j(t) + \ell\, t$ define a family of quantum Poisson processes.

Let us write $h_{[a,b]} = L^2([a,b],\mathcal{K})$, $h_{t]} = h_{[0,t]}$ and $h_{[t} = h_{[t,\infty)}$ and note that the natural continuous tensor product structure of $\Gamma(h)$, viz. $\Gamma(h) = \Gamma(h_{t]}) \otimes \Gamma(h_{[t})$ allows one to define an operator family $\{L(t)\}$ to be *adapted* if $L(t)$ is of the form $L_0(t) \otimes I_{[t}$ for every $t \geq 0$, where $L_0(t)$ is a linear operator in $\mathcal{H}_{t]} \equiv \mathcal{H}_0 \otimes \Gamma(h_{t]})$ and $I_{[t}$ is the idenitty on $\Gamma(h_{[t})$. The *quantum Ito formulae* are given as :

$$\left.\begin{array}{rcl} dA_j dA_k^+ & = & \delta_{jk}dt,\ dA_j dA_k = dA_j^+ dA_k^+ = 0, \\ d\Lambda_j^k dA_\ell^+ & = & \delta_{\ell k} dA_j^+, dA_\ell d\Lambda_k^j = \delta_{\ell k} dA_j, \\ d\Lambda_j^k d\Lambda_m^\ell & = & \delta_{km} d\Lambda_j^\ell, d\Lambda_j^k dA_\ell = dA_\ell^+ d\Lambda_k^j = 0. \end{array}\right\} \tag{3}$$

An adapted process $\{L(t)\}$ is integrable w.r.t. the basic quantum processes if it is square-integrable $(\mathcal{D}, \mathcal{E}(M))$ i.e. if $\int_0^t \|L(s)ue(f)\|^2 ds < \infty$ for every $t \geq 0$, $u \in \mathcal{D}$, a suitable dense subset of $\mathcal{H}_0$ and $f \in M$, a dense subset of h such that $\mathcal{D} \otimes \mathcal{E}(M) \subseteq \mathrm{Dom}(L_0(t))$. Here by $\mathcal{E}(M)$ we mean the linear span of $e(f)$, $f \in M$. One has the estimates,

$$\|\int_0^t L(s) dA_j(s) ue(f)\| \leq \int_0^t |f_j(s)|\, \|L(s)ue(f)\| ds$$

$$\left.\begin{array}{l} \|\int_0^t L(s) dA_j^+(s) ue(f)\| \\ \text{and} \\ \|\int_0^t L(s) d\Lambda_j^k(s) ue(f)\| \end{array}\right\} \leq C(f,t)[\int_0^t \|L(s)ue(f)\|^2 ds]^{1/2}. \tag{4}$$

The above estimates, in fact show that the integral w.r.t. $A_j(t)$-process is defined whenever the R.H.S. of the estimate in (4) is finite and that can happen in circumstances more general than the square integrability requirement.

The simplest q.s.d.e. that one can solve is the following

$$X(t) = X_0 + \int_0^t X(s)[F_k^j d\Lambda_j^k(s) + E_j dA_j^+(s) + G_j dA_j(s) + N ds], \tag{5}$$

where $X_0, F^j_k, E_j, G_j (1 \leq j, k \leq \dim \mathcal{K})$ and N are constant bounded operators in $\mathcal{H}_0$, and we have used the summation convention. In the case when $\dim \mathcal{K} = \infty$, one needs a further condition, viz. $\sum_j \|F^j_k u\|^2, \ \sum_j \|E_j u\|^2 \leq C_k \|u\|^2 \ \ \forall \ u \in \mathcal{H}_0$ with some family of positive constants C_k. These issues were studied in [4]. In usual quantum mechanics one is interested in the unitarity of the evolution group and similarly here a natural question would be: under what conditions on the coefficient operators is the solution X of (5) unitary and what are its properties. The answer is that the solution U_t of (5) with the initial value $X_0 = I$ is unitary iff its coefficients satisfy:

$$\left.\begin{array}{rcl} F^j_k & = & W^j_k - \delta_{jk} \text{ where } W \equiv ((W^j_k)) \text{ is unitary on } \mathcal{H}_0 \otimes \mathcal{K}, \\ G_j & = & -E^*_k W^k_j, \\ N & = & -\frac{1}{2} E^*_k E_k + iH \text{ where } H \text{is bounded selfadjoint operator in } \mathcal{H}_0 \ . \end{array}\right\} \quad (6)$$

It is useful to replace (5) by a closely related one:

$$U(s,t) = I + \int_s^t U(s,\tau)[F^j_k d\Lambda^k_j(\tau) + \cdots + N d\tau]. \quad (5')$$

Then one finds that (i) $U(s,t)$ for $s \leq t$ is unitary iff the coefficients satisfy (6) as in the case of (5), (ii) $\{U(s,t)\}$ is an evolution, i.e. $U(r,s)U(s,t) = U(r,t), \ \ r \leq s \leq t; \ \ U(s,s) = I$. (iii) $U(0,t) = U_t$, (iv) $U(s,t)$ is time-homogeneous, i.e. depends only on $(t-s)$, iff all the coefficients except N are zero, and (v) the expectation operator $P(s,t)$ defined as $\langle u, P(s,t)v \rangle \equiv \langle ue(0), \ U(s,t)ve(0) \rangle$ is time-homogeneous, in fact $P(s,t) \equiv P_{t-s} = e^{N(t-s)}$. Thus $\{P_t\}_{t \geq 0}$ forms a norm-continuous semigroup on $\mathcal{H}$ with bounded generator N. Obviously the situation becomes more complicated if the coefficients still formally satisfy (6) but are *unbounded.* We shall discuss this in the next section.

A quantum stochastic process in Fock space $\mathcal{H}$ is a family of maps $\{j_t\}_{t \geq 0} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H}_0)$, satisfying :

(i) $\{j_t(x)\}_{t\geq 0}$ is an adapted family $\forall\, x \in \mathcal{A}$,

(ii) j_t is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, i.e. $j_t(x^*) = j_t(x)^*$, $j_t(xy) = j_t(x)j_t(y)\ \forall\, x, y \in \mathcal{A},\ t \geq 0$.

It is said to be *conservative* if $j_t(I) = I\ \forall\, t$ and a *quantum stochastic flow* (q.s.f.) if it satisfies furthermore the q.s.d.e.

$$j_t(x) = x + \int_0^t [j_s(\theta^j_k(x))d\Lambda^k_j(s) + j_s(\theta^0_j(x))dA^\dagger_j(s) + j_s(\theta^j_0(x))dA_j + j_s(\theta^0_0(x))ds] \tag{7}$$

where the *structure maps* $\theta^\alpha_\beta (0 \leq \alpha, \beta \leq \dim\ \mathcal{K})$ are linear operators on $\mathcal{A}$ satisfying for $x, y \in \mathcal{A}$

$$\theta^\alpha_\beta(x^*) = \theta^\beta_\alpha(x)^*, \ \ \theta^\alpha_\beta(xy) = x\theta^\alpha_\beta(y) + \theta^\alpha_\beta(x)y + \theta^\alpha_k(x)\theta^k_\beta(y). \tag{8}$$

It is shown in [2] and [5] that for $\dim\ \mathcal{K} < \infty$ if θ^α_β's are norm-bounded and satisfy (8), then equation (7) has a unique solution $j_t(x)$ which defines a q.s.f. such that it is contractive : $\|j_t(x)\| \leq \|x\|$ and $\mathbb{R}_+ \times \mathcal{A} \ni (s, x) \to j_s(x)$ is strongly jointly continuous on $\mathcal{H}$ w.r.t. the strong operator topology on $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$. Furthermore, it is conservative iff $\theta^\alpha_\beta(I) = 0$ for all α, β. Most of these results were extended in [4] to the case $\dim\ \mathcal{K} = \infty$ subject to an additional summability condition on θ^α_β 's . As in the preceding paragraph one can pose the question: what happens if θ^α_β 's are *not* norm-bounded on $\mathcal{A}$. Not much is known in this area, but some results can be found in [6, 14].

I shall end this section by giving some applications, viz. the description of classical Markov chains as q.s.f. in Fock space with the degree of freedom equaling the cardinality of the state space. In this context, the following lemma is instructive ([2]).

Lemma 1 (for simplicity $\dim\ \mathcal{K} < \infty$) : Let j_t be a q.s.f. satisfying (7) with bounded structure maps θ^α_β obeying (8). Assume furthermore that $\mathcal{A}$

is abelian. Then $\{j_s(x)|0 \leq s \leq \infty, x \in \mathcal{A}_h$, the self-adjoint part of $A\}$ is a classical process; in fact $[j_s(x), j_t(y)] = 0 \; \forall \; s, t \geq 0$ and $x, y \in \mathcal{A}$.

This lemma allows us to embed classical stochastic processes in the quantum receptacle and this we briefly describe for Markov chain [26]. Let $\mathcal{X}$ be the state space of a countably infinite continuous time Markov porocess and let $p_t(x,y)(x,y \in \mathcal{X})$ be the (stationary) transition probabilities such that $\ell(x,y) \equiv \frac{d}{dt}p_t(x,y)|_{t=0}$ satisfy the Markov conditions:

$$\ell(x,y) \geq 0 \text{ for } \; x \neq y \text{ and } \sum_{y\in\mathcal{X}} \ell(x,y) = 0. \tag{9}$$

It is convenient (but not necessary) to put a group structure G on $\mathcal{X}, G$ acting on $\mathcal{X}$ by left translation and let μ be the counting measure on $\mathcal{X}$.

Set $m_x(y) = \sqrt{\ell(y,xy)}$ if $x \neq id$ of G and $= 0$ otherwise, and write for $\phi \in L_\infty(\mathcal{X},\mu) \equiv \mathcal{A}$:

$$\begin{aligned}
\theta^x_0(\phi)(y) &= \text{Multiplication operator by } m_x(y)[\phi(xy) - \phi(y)], \\
\theta^0_x(\phi)(y) &= \text{Multiplication operator by } \overline{m_x(y)}[\phi(xy) - \phi(y)], \\
\theta^x_{x'}(\phi)(y) &= \text{Multiplication operator by } [\phi(xy) - \phi(y)]\delta_{xx'}, \\
\theta^0_0(\phi)(y) &= \text{Multiplication operator by } \sum_{x\in\mathcal{X}} |m_x(y)|^2[\phi(xy) - \phi(y)].
\end{aligned}$$

Then the q.s.d.e. (7) with the above structure maps on the (abelian) *-algerba $\mathcal{A}$ has a q.s.f. $j_t(\phi)$ as its unique solution if $\sup_{x\in\mathcal{X}}|\ell(x,x)| < \infty$. This is because under this condition, the abovementioned structure maps are norm bounded and we can apply the theory discussed above ([5], [2]). Furthermore the expectation semigroup $T_t(\phi) \equiv \mathbb{E}j_t(\phi) \equiv \langle \cdot \otimes e(0), \; j_t(\phi) \cdot \otimes e(0)\rangle$ has the bounded generator θ^0_0 given by $\theta^0_0(\phi)(y) = \sum_{\mathcal{X}\ni x\neq id} \ell(y,xy)[\phi(xy) - \phi(y)] = \sum_{x\neq y} \ell(y,x)\phi(x) - \{\sum_{z\neq y}\ell(y,z)\}\phi(y) = \sum_{x\in\mathcal{X}}\ell(y,x)\phi(x)$, the action is exactly the same as that for the Markov chain.

2 Q.S.D.E. with unbounded operator coefficients

Most of what I shall describe here is part of the thesis of A. Mohari in the Indian Statistical Institute, Delhi, and of publications arising from it [7,8]. For simplicity, we take dim $\mathcal{K} = 1$ and drop the $d\Lambda$ term from q.s.d.e.'s :

$$V(t) = I + \int_0^t V(s)[EdA^+(s) - E^*dA(s) - \frac{1}{2}E^*Eds]. \tag{10}$$

As can be easily seen, the operator coefficients in (10) satisfy formally the unitarity condition (6). The major problem, however, is that Dom $(E) \cap$ Dom (E^*) may be too small, even trivial in which case the equation (10) has hardly any content (see [9] for counter-example). To proceed further, we make

Assumption A : Let E be closed and assume furthermore that there exists a dense subset $\mathcal{D} \subset$ Dom $(E) \cap$ Dom (E^*) and a sequence of bounded operators $\{E_n\}$ in $\mathcal{H}_0$ such that E_n, E_n^* and $E_n^*E_n$ converge strongly on $\mathcal{D}$ to E, E^* and E^*E respectively.

Assumption B : $\mathcal{D}$ is stable under the action of T_t, (the expectation semi-group of $V(t)$ with generator $-\frac{1}{2}E^*E$).

The basic result is contained in the next theorem.

Theorem 2 : (i) Assume (A) and (B). Then (10) admits a unique adapted contractive solution V.

(ii) If furthermore, E satisfies the Feller Condition (F) : For some $\lambda > 0$ (and hence for all $\lambda > 0$) the Feller set

$$\beta_\lambda \equiv \{x \in \mathcal{B}(\mathcal{H}_0), 0 \le x \le 1 | \langle v, \theta_0^0(x)u\rangle \equiv \langle Ev, xEu\rangle$$

$$-\frac{1}{2}\langle E^*Ev, xu\rangle - \frac{1}{2}\langle x^*v, E^*Eu\rangle = \lambda\langle v, xu\rangle \;\forall\, u, v \in \mathcal{D}\} = \{0\}$$

, then V is unitary.

Sketch of proof : Consider the equation

$$V_n(t) = I + \int_0^t V_n(s)[E_n dA^+ - E_n^* dA - \frac{1}{2}E_n^* E_n ds], \tag{11}$$

which by the results in section 1 admits a unique adapted unitary solution V_n in $\mathcal{H}$. Let $u \in \mathcal{D}, 0 \leq t_1 < t_2 < T < \infty$, and $f \in C_0(0,\infty) \subseteq h$. Then

$$\|[V_n(t_2) - V_n(t_1)]ue(f)\|^2 \leq C \int_{t_1}^{t_2} \{\|E_n u\|^2 + |f(s)|^2 \|E_n^* u\|^2 + \frac{1}{4}\|E_n^* E_n u\|^2\} ds$$

and hence by assumption A we have that for every $\psi \in \mathcal{H}$ $\{\langle \psi, V_n(t)ue(f)\rangle\}$ is a bounded equicontinuous family. Thus we can extract a subsequence converging uniformly on $[0,T]$. Using the separability of $\mathcal{H}$, a diagonal trick, the uniform boundedness of $V_n(t)$ and the totality of vectors of the form $ue(f)$, one can show that $V_n(t)$ converges weakly (by relabelling the subsequence) to an adapted contraction $V(t)$ uniformly on $[0,T]$. From the properties (2) of the basic processes and (11), we have for $u, v \in \mathcal{D}$ and f, g as before,

$$\langle ve(g), V_n(t)ue(f)\rangle = \langle ve(g), ue(f)\rangle$$
$$+ \int_0^t \langle ve(g), V_n(s)[E_n u\bar{g}(s) - E_n^* u f(s) - \frac{1}{2}E_n^* E_n u]e(f)\rangle ds. \tag{12}$$

Choosing an appropriate subsequence, we see that the LHS of (12) converges to $\langle ve(g), V(t)ue(t)\rangle$ whereas by assumption (A), weak convergence of $V_n(s)$ to $V(s)$ uniformly in $[0,T]$ implies that the RHS of (12) converges to $\int_0^t \langle ve(g), V(s)[Eu\bar{g}(s) - E^* u f(s) - \frac{1}{2}E^* Eu]e(f)\rangle ds$. This proves that $V(t)$ is a solution of (10).

Let $V'(t)$ be another solution of (10) and set $X(t) = V(t) - V'(t)$ so that $\|X(t)\| \leq 2,\ X(0) = 0$ and

$$X(t) = \int_0^t X(s)[EdA^+(s) - E^* dA(s) - \frac{1}{2}E^* E ds]. \tag{13}$$

Fix $f, g \in C_0(0,\infty)$ and define a bounded operator $M(t)$ on $\mathcal{H}_0$ by $\langle u, M(t)v\rangle = \langle ue(f), X(t)ve(g)\rangle$ for $u, v \in \mathcal{D}$. Then by (13) one has

$$\langle u, M(t)v\rangle = \int_0^t \langle u, M(s)[E\bar{f}(s) - E^*g(s) - \frac{1}{2}E^*E]v\rangle ds. \tag{14}$$

Replacing f and g by αf and βg $(\alpha, \beta \in \mathbb{C})$ respectively, differentiating m times w.r.t. α and n times w.r.t. β and equating coefficients of both sides, we get

$$\begin{aligned}\langle u, M(t;m,n)v\rangle &\equiv \langle uf^{\otimes m},\ X(t)vg^{\otimes n}\rangle \\ &= \int_0^t \{\langle u, M(s;m-1,n)Ev\rangle \bar{f}(s) - \langle u, M(s;m,n-1)E^*v\rangle g(s) \\ &\qquad - \frac{1}{2}\langle u, M(s;m,n)E^*Ev\rangle\}ds. \end{aligned}\tag{15}$$

For $m = n = 0$, this leads to $\frac{dM(t;0,0)v}{dt} = -\frac{1}{2}M(t;0,0)E^*Ev\ \forall\ v \in \mathcal{D}$. Thus using (B), $\frac{d}{ds}M(s;0,0)T_{t-s}v = 0$ for all $0 \le s \le T$. Since $M(0;0,0) = 0$ this implies that $M(t;0,0) = 0$ for $0 \le t \le T$. Now by induction let $M(t;k,\ell) = 0$ for $k + \ell \le n$ and consider k, ℓ such that $k + \ell = n + 1$. Then by (15), $\frac{M}{dt}(t;k,\ell)v = -\frac{1}{2}M(t;k,\ell)E^*Ev$ with $M(0;k,\ell) = 0$ and just as above one concludes that $M(t;k,\ell) = 0\ \forall\ t$ and all k, ℓ which implies $X(t) = 0$ or the uniqueness of the solution of (10).

Let $Y(t) = I - V(t)^*V(t)$ and $Y_\lambda = \int_0^\infty e^{-\lambda t}Y(t)dt$, $\lambda > 0$. Then by the strong continuity of $V(t)$ in $\mathcal{B}(\mathcal{H})$ and since $\|Y(t)\| \le 2$, it is clear that Y_λ is well-defined as a strong Riemann integral and that $\|Y_\lambda\| \le 2/\lambda$ and since $Y(t) \ge 0$, one also has $Y_\lambda \ge 0$. Since $Y(0) = 0$, by the quantum Ito formula (3) one arrives at

$$\begin{aligned}\langle ve(g), Y(t)ue(f)\rangle = \int_0^t &\langle ve(g), \{Y(s)[E\bar{g}(s) - E^*f(s) + G] \\ &+ [E^*f(s) - E\bar{g}(s) + G]Y(s) + E^*Y(s)E\}ue(f)\rangle ds, \end{aligned}\tag{16}$$

where we have written G for $-\frac{1}{2}E^*E$. As before, going down to the finite particle vectors and considering the diagonal terms only, we have

$$\langle vg^{\otimes m}, Y(s)uf^{\otimes m}\rangle = \int_0^t \langle vg^{\otimes m}, \{Y(s)G + GY(s) + E^*Y(s)E\}uf^{\otimes m}\rangle ds. \quad (17)$$

Since $Y(t) \geq 0$, there exists at least one m such that $Y(t)uf^{\otimes m} \neq 0$ for some $f \in M$ and $u \in \mathcal{D}$. Then for such m, f, u one has by integrating by parts and using (17),

$$\langle v, B_\lambda v\rangle \equiv \int_0^\infty e^{-\lambda t}\langle ug^{\otimes m}, Y(t)uf^{\otimes m}\rangle dt = -\frac{1}{\lambda}e^{-\lambda t}\langle vg^{\otimes m}, Y(t)uf^{\otimes m}\rangle|_{t=0}^{\infty}+$$

$$\lambda^{-1}\int_0^\infty e^{-\lambda t}dt\{\langle vg^{\otimes m}, Y(t)Guf^{\otimes m}\rangle+\langle Gvg^{\otimes m}, Y(t)uf^{\otimes m}\rangle+\langle Evg^{\otimes m}, Y(t)Euf^{\otimes m}\rangle\}$$

$$= \lambda^{-1}\{\langle v, B_\lambda Gu\rangle + \langle Gv, B_\lambda v\rangle + \langle Ev, B_\lambda Eu\rangle\}. \quad (18)$$

Now, if condition (F) is satisfied, i.e. if $\beta_\lambda = \{0\}$ then it follows from (18) that $B_\lambda = 0$ which by the uniqueness of the Laplace transform implies in particular that $\langle uf^{\otimes m}, Y(t)uf^{\otimes m}\rangle = 0$. Since $Y(t) \geq 0$ this means that $Y(t)uf^{\otimes m} = 0$ which is a contradiction. Therefore V is an isometry.

For proving the coisometry of $V(t)$ we use the reflection map [7,9]. On h define a selfadjoint unitary map ρ_T (reflection about $T \geq 0$) by $(\rho_T f)(t) = f(T-t)$ if $t \leq T$ and $= f(t)$ if $t > T$, and let R_T be its second quantization to $\Gamma(h)$. Set $\widetilde{V}_n(s,t) \equiv R_t V_n(t-s)^* R_t$ and then one can compute the q.s.d.e. satisfied by $\widetilde{V}_n$ w.r.t. t as

$$\widetilde{V}_n(s,t) = I + \int_s^t \widetilde{V}_n(s,\tau)[E_n^* dA(\tau) - E_n dA^+(\tau) - \frac{1}{2}E_n^* E_n d\tau]. \quad (19)$$

We can now proceed as in the first part of this proof and see that $\widetilde{V}_n(s,t)$ (or possibly a subsequence of this) converges weakly to an adapted contraction, say $\widetilde{V}(s,t)$. As before one can obtain the q.s.d.e. satisfied by $\widetilde{V}(t) \equiv \widetilde{V}(0,t)$

as :

$$\widetilde{V}(t) = I + \int_0^t \widetilde{V}(s)[E^* dA(s) - E dA^\dagger(s) - \frac{1}{2} E^* E ds], \tag{20}$$

which is very similar to the equation (10) with E replaced by $-E$. This means that the Feller set $\tilde{\beta}_\lambda$ for the reflected problem is the same as β_λ, the original one. This, by the last paragraph, implies the isometry of $\widetilde{V}(t)$. Finally the definition of $\widetilde{V}$ and the unitarity of R_t shows that this means the coisometry of $V(t)$. ■

Remark 3 : (i) Classical birth and death processes can be described in the framework of theorem 2, see e.g. [10]. More general results can be derived when the noise is classical [11].

(ii) As can be seen, the Feller set β_λ plays an important role in the analysis. Many examples are known in which β_λ is not trivial and therefore the solutions of (10) even when they exist are not unitary [12]. It is clear that if $\beta_\lambda \neq \{0\}$, then $P_t(I) = EV(t)^* V(t) \neq I$. Then an important question arises: Can one extend the semigroup P_t to a conservative one? Some answers to this can be found in [13].

(iii) The difficulty in satisfying hypothesis (A) in general should be clear. However if E is normal (though unbounded) then (A) can be easily satisfied by taking $E_n = E(E^*E+n)^{-1}$ so that $E_n^* = \overline{(E^*E+n)^{-1}E^*} = E^*(EE^*+n)^{-1}$. However, in this case we can explicitly solve (10) : $V(t) = \int_{\mathbf{C}} P(dz) \otimes W(z\chi_{[0,t]})$, where $E = \int_{\mathbf{C}} zP(dz)$ is the spectral resolution of E and W is the Weyl operator in the Fock space $\Gamma(h)$ ([2]).

3 Martingales and their representation

An adapted process X defined on $\mathcal{D} \otimes \mathcal{E}(m)$ is a *martingale* if for $s \leq t$

$$\langle ve(g_{s]}), X(t)ue(f_{s]})\rangle = \langle ve(g_{s]}), X(s)ue(f_{s]})\rangle \tag{21}$$

for $u, v, \in \mathcal{D}$ and $F, G \in M \subseteq h$. If furthermore $X(s)$ is known to be bounded then (21) extends to whole of $\mathcal{H}_{s]}$ and one has

$$\langle \tilde{v}_{s]},\ X(t)\tilde{u}_{s]}\rangle = \langle \tilde{v}_{s]},\ X(s)\tilde{u}_{s]}\rangle, \tilde{u}_{s]}, \tilde{v}_{s]} \in \mathcal{H}_{s]}.$$

It is clear that the basic quantum processes $A(t),\ A^{+}(t),\ \Lambda(t)$ are martingales though *not* bounded (for simplicity we take dim $\mathcal{K} = 1$ in this section).

A bounded martingale is said to be *regular* if there exists a Radon measure μ on $[0, \infty)$ such that for all $t \geq a \geq 0,\ \tilde{u} \in \mathcal{H}_{a]}$

$$\|[X(t) - X(a)]\tilde{u}\|^2 + \|[X(t)^* - X(a)^*]\tilde{u}\|^2 \leq \|\tilde{u}\|^2 \mu[a, t]. \tag{22}$$

If X admits the representation :

$$X(t) = X(0) + \int_0^t \{F(s)d\Lambda(s) + E(s)dA^{+}(s) + G(s)dA(s) + N(s)ds\}, \tag{23}$$

it is a martingale iff $N(s) = 0$. On the other hand, one has a counterexample ([15]) in which $\mathcal{H}_0 = \mathbf{C},\ X(t)e(f) = e(\chi_{[0,t]} H \chi_{[0,t]} f)$ with H the Hilbert transform. Then X is a bounded martingale, but it does not have an integral representation as above. This is due to the fact that this X is not a regular martingale, though it can be shown that $X(t)$ is the strong limit of a sequence of regular martingales. Next we state without proof two results including the representation theorem for bounded regular martingales ([16]).

Theorem 4 : Let X be a bounded martingale on $\mathcal{H}$ having the representation (23) with $N = 0, \{F, E, G\}$ as well as $\{F^*, E^*, G^*\}$ bounded adapted square-integrable processes such that both $\|E(t)\|$ and $\|G(t)\|$ are locally square integrable. Then X is a regular martingale with $\mu[a, b] = \int_a^b \{\|E(s)\|^2 + \|G(s)\|^2\} ds$.

Thoerem 5 : Let X be a bounded regular martingale with Radon measure μ and let $\mathcal{E}_b$ be the dense subset of $\Gamma(h)$ generated by $\{e(f)|f \in L^2(I\!R_+), f$ locally bounded $\}$. Then X admits the representation (23) with $\dot{N} = 0$ and with (F, E, G) - three bounded adapted processes square integrable w.r.t. $(h_0, \mathcal{E}_b)$ such that max $\{\|E(s)\|^2, \|G(s)\|^2\} \leq \mu'_{ac}(s)$.

Some simple yet interesting examples of bounded regular martingales are unitary Hilbert-Schmidt and Fermion martingales. A Fermion martingale with $\mathcal{H}_0 = \mathbb{C} = \mathcal{K}$ is a family $\{F_f(t)\}$ satisfying

$$\text{(i) CAR : for all } f, q, \in h \begin{cases} F(f)F(g) + F(g)F(f) &= 0 \\ F(f)F(g)^* + F(g)^*F(f) &= \langle g, f \rangle, \end{cases}$$

$$\text{(ii)} \qquad F_f(t) = F(f\chi_{[0,t]}).$$

Then it is clear that it is a bounded (in fact $\|F_f(t)\|^2 = \int_0^t |f(s)|^2 ds$), regular $(\|[F_f(t) - F_f(a)]u\|^2 + \|[F_f(t)^* - F_f(a)^*]u\|^2 = \|u\|^2 \int_a^t |f(s)|^2 ds)$ martingale. Thus by theorem 5, F_f admits a representation. One can show [16] that for such a martingale, the coefficients of $d\Lambda$ and dA^+ (and of course of ds) all vanish. If furthermore one assumes that the $*$-algebra generated by $\{F_f(t)|t \geq 0, \ f \in h\}$ acts irreducibly on $\mathcal{H}$, then one has the known representation viz. $F_f(t) = \int_0^t \theta(s)\overline{f(s)}J(s)dA(s)$, where $|\theta(s)| = 1$ and $J(s)e(f) = e(-f_{s]} + f_{[s})$.

The more recent developments in this theme can be found in [17, 18]. Before we describe these results, we need to go back to the definition of quantum stochastic integrals in section 1 and note that by quantum Ito's formula one has that :

$$\int_0^t F(s)d\Lambda(s) \text{ is well-defined on } ue(f) \text{ if } \int_0^t |f(s)|^2 \|F(s)ue(f)\|^2 ds < \infty,$$

$$\int_0^t E(s)dA^+(s) \text{ is well-defined on } ue(f) \text{ if } \int_0^t \|E(s)ue(f)\|^2 ds < \infty,$$

$$\int_0^t G(s)dA(s) \text{ is well-defined on } ue(f) \text{ if } \int_0^t |f(s)|\ \|G(s)ue(f)\| ds < \infty,$$

and

$$\int_0^t N(s)ds \text{ is well-defined on } ue(f) \text{ if } \int_0^t \|N(s)ue(f)\| ds < \infty, \qquad (24)$$

instead of (2). It is clear that the integrability conditions (2) imply (24).

Also note that the integral representation of regular martingales as given in theorem 5 is valid on $\mathcal{H}_0 \otimes \mathcal{E}_b$ and this makes multiplication of such martingales impossible in general. To circumvent this, one uses the representation of a vector process in $\Gamma(h) : \Phi_t = I\!\!E(\Phi) + \int_0^t \hat{\Phi}_s dw(s)$, where the expectation $I\!\!E(\Phi) = \langle e(0), \Phi\rangle$ and ω is the standard Brownian motion. If $\Phi \in \mathcal{E}_b$ and if X has representation (23) on $\mathcal{H}_0 \otimes \mathcal{E}_b$, then one has

$$X(t)\Phi_t = \int_0^t X(s)\hat{\Phi}_s d\omega(s) + \int_0^t F(s)\hat{\Phi}_s d\omega(s)$$

$$+ \int_0^t E_s\hat{\Phi}_s ds + \int_0^t G_s\Phi_s d\omega(s) + \int_0^t N(s)\Phi_s ds, \qquad (25)$$

as one would expect if Ito formula has to be valid on this domain. This allows an extension of the definition of quantum stochastic integrals as follows.

Thoerem 6 : Let X be an adapted process such that both $\{X(t)\}$ and $\{X(t)^*\}$ admit integral representations of the type (23) on $\mathcal{H}_0 \otimes \mathcal{E}_b$. Then both representations can be extended to $\mathcal{H}_0 \otimes \mathcal{D}$ with $\mathcal{D} \supseteq \mathcal{E}_b$ if the corresponding expressions (25) makes sense for $\Phi \in \mathcal{D}$. In particular, if X is bounded and if $t \to \|E(t)\|, \|G(t)\|$ are locally square integrable as well as $t \to \|F(t)\|, \|N(t)\|$ are locally bounded then the integral representation (23) can be extended to whole of $\mathcal{H}$.

The conditions appearing in the second part of theorem 6 seems to be natural in view of (24). The next theorem deals with the $*$-algebra property of semi-martingales.

Thoerem 7 : Let $\mathcal{S}$ be the set of all bounded semi-martingales X on $\mathcal{H}$ admitting integral representation (23) with coefficients satisfying the properties in theorem 6. Then $\mathcal{S}$ is a $*$-algebra.

Sketch of the proof of theorem 7 : Since $\|\int_0^t N(s)ds\| \leq \int_0^t \|N(s)\|ds < \infty$, $Y(t) \equiv X(t) - \int_0^t N(s)ds$ is clearly a bounded martingale. By hypothesis and theorm 6, $Y(t)$ has an integral representation (23) on whole $\mathcal{H}$ and hence, $\|Y(s)\| = \sup\limits_{\Phi,\Psi} \dfrac{|\langle\psi_{s]}, Y(s)\Phi_{s]}\rangle|}{\|\psi_{s]}\| \, \|\Phi_{s]}\|} = \sup\limits_{\psi,\Phi} \dfrac{|\langle\psi_{s]}, Y(t)\Phi_{s]}\rangle|}{\|\psi_{s]}\| \, \|\Phi_{s]}\|} \leq \|Y(t)\| \ \forall \ s \leq t$. Therefore $\|Y(t)\|$ is locally bounded. The rest of the proof follows from quantum Ito formula (2).

As in [18], an adapted process X of bounded operators in $\mathcal{H}$ is said to be a regular *semi-martingale* if there is a dense subset M of the set of bounded $L^2(I\!R_+)$ functions stable under $u \to u_{t]}$ for all $t \geq 0$ and if there is a Radon measure μ and an absolutely continuous measure ν on $I\!R_+$ such that $\forall \ \Phi \in \mathcal{E}(M)$, and $a < s < t$.

(i) $\|[X(t) - X(s)]\Phi_{a]}\|^2 + \|[X(t) - X(s)^*]\Phi_{a]}\|^2 \leq \|\Phi_{a]}\|^2 \mu[s,t]$,

(ii) $\|[I\!E_s X(t) - X(s)]\Phi_{a]}\| \leq \|\Phi_{a]}\| \nu[s,t]$,

where $I\!E_s$ is the conditional expectation map defined by $\langle\Phi_{s]}, I\!E_s X(t)\psi_{s]}\rangle = \langle\Phi_{s]} \otimes e(0_{[s}), \ X(t) \ \ \psi_{s]} \otimes e(0_{[s})\rangle, \Phi, \psi \in \mathcal{H}$. Then one has the following theorem ([18]) which completely characterizes regular semi-martingales.

Theorem 8 : Let X be a bounded adapted process in $\mathcal{H}$. Then $X \in S$ iff it is a regular semi-martingale.

The representation theory of martingales has been successfully applied in [19] to derive q.s.d.e. for a class of unitary cocycles with bounded generator of the expectation semigroup and in [20] to obtain a normal q.s.f. again with a norm-bounded generator of its corresponding expectation semigroup.

4 Stop-time and Stopped processes

Here we take $\mathcal{H}_0 = \mathcal{K} = \mathbb{C}$ for simplicity. A *stop-time* S on $\mathcal{H} = \Gamma(h)$ is a spectral measure on $[0,\infty)$ such that $\{S[0,t]\}_{t\geq 0}$ is an adapted process. It is *finite* if $S\{\infty\} = 0$ and is *bounded* if S has bounded support. Suppose $\{X(t)\}$ is an adapted process of commuting self-adjoint operators in $\mathcal{H}$ so that $\varphi(X(t)) \in \mathcal{B}(\mathcal{H}_{t]})\ \forall\ t \geq 0$ and every bounded Borel function φ on $\mathbb{R}$, then by the spectral theorem we can construct a Borel space $(\mathcal{X}, \mathcal{F})$, a spectral measure P on it and a family of real Borel functions $x(t,\cdot)$ on $\mathcal{X}$ such that $X(t) = \int_{\mathcal{X}} x(t,w)P(dw)\ \forall\ t$. Let $\overline{\mathcal{F}}_t$ and $\overline{\mathcal{F}}$ be the completions of $\mathcal{F}_t$ and $\mathcal{F}$ w.r.t. P and let τ be a classical stop-time w.r.t. the filtration $(\mathcal{X}, \overline{\mathcal{F}}_t, \overline{\mathcal{F}})$ i.e. $\tau : \mathcal{X} \to [0,\infty)$ Borel such that $\{w|\tau(w) \leq t\} \in \overline{\mathcal{F}}_t\ \forall\, t \geq 0$. Then $Po\tau^{-1}$ is a spectral measure in $\mathcal{H}$ on $[0,\infty]$, $Po\tau^{-1}[0,t] \in \mathcal{B}(\mathcal{H}_{t]})$ and hence is a stop-time. Now Brownian motion and poisson porcesses can be realized as commuting self-adjoint operator processes in $\mathcal{H}$ and therefore all the classical stop times relative to these processes can be interpreted as stop-times in Fock space. Just as in classical probability, one can define stopped processes, in particular, stopped Weyl processes. However, one can have right, left- or both left- and right- stopped processes. The next theorem sums up some of these results from [21].

Thoerem 9 : (i) Let W be a Weyl process, i.e. $W(s) = W(f_{s]}, e^{i\phi_{s]}})$ $(f \in h, \phi$ bounded on $R_+)$, S a stop-time, $g, h \in h; \xi, \eta : [0,\infty] \to \mathcal{H}$ Borel, adapted to the future (i.e., $\xi(t) \in \mathcal{H}_{[t}\ \forall\, t \geq 0)$ satisfying

$$\int_{[0,\infty]} e^{-\|g_{[s}\|^2} \|\xi(s)\|^2 \langle e(g), S(ds)e(g)\rangle + \int_{[0,\infty]} e^{-\|g_{[s}\|^2]} \|\eta(s)\|^2 \langle e(h), S(ds)e(h)\rangle < \infty.$$

Set $x(s) = e(g_{s]})\xi(s), y(s) = e(h_{s]})\eta(s)$. Then $\int W(s)S(ds)x(s)$ is well-defined as the strong limit of Riemann sums in $\mathcal{H}$ and

$$\langle \int W(s)S(ds)x(s), \int W(\sigma)S(d\sigma)y(\sigma)\rangle$$
$$= \int_{[0,\infty]} e^{-\langle g_{[s},h\rangle}\langle \xi(s), \eta(s)\rangle\langle e(g), S(ds)e(h)\rangle.$$

(ii) Define the shift θ_s on h as $\theta_s f(t) = f(t-s)$ if $t \geq s$ and $= 0$ otherwise and note that θ_s is an isometry. Let $\Gamma(\theta_s)$ be its second quantization. Then $U^S \equiv \int S(ds)\Gamma(\theta_s)$ exists as a contraction in $\mathcal{H}$ and is an isometry if $S\{\infty\} = 0$, i.e. if S is finite.

One can interpret the range $\mathcal{H}_{[S}$ of the isometry U^S as the Fock space beyond the "random" time S while the Fock space upto time $S, \mathcal{H}_{S]}$, is expected to be spanned by vectors of the type $\int S(ds)e(f_{s]})\phi(s)(f \in h, \phi$ bounded on $\mathbb{R}_+)$. We end this section by stating without proof [21] a result which can be called the *Strong Markov property* for the Fock space.

Theorem 10 : Let S be a finite stoptime in $\mathcal{H}; \mathcal{H}_{S]}$ and $\mathcal{H}_{[S}$ be as above. Then there exists a unique unitary isomorphism $\mathcal{I}_S : \mathcal{H}_{S]} \otimes \mathcal{H}_{[S} \rightarrow \mathcal{H}$ by $\mathcal{I}_S(\int S(ds)e(f_{s]})\phi(s) \otimes U^S\psi) = \int S(ds)e(f_{s]})\phi(s)\Gamma(\theta_s)\psi$.

5 Applications and Discussion

As is well known, the classical damped harmonic oscillator is described by the equation of motion :

$$\ddot{q} + 2\alpha\dot{q} + \omega^2 q = 0, \quad (0 < \alpha < \omega) \tag{26}$$

and such a (non-conservative) system cannot be described in terms of a Hamiltonian i.e. the above second order equation cannot be recast as a pair of canonical equations of motion.

Nevertheless, we can introduce a pair of "conjugate variables" q and p satisfying a pair of first order differential equations which is equivalent to

(26). Set $\delta = \sqrt{\omega^2 - \alpha^2}$ and write

$$\dot{q} = p - \alpha q, \ \ \dot{p} = -\delta^2 q - \alpha p; \tag{27}$$

and a simple calculation verifies that indeed (27) leads to (26). In fact we would like to introduce 'annihilation' and 'creation' variables.

$$a = (2\delta)^{-1/2}(p + i\delta q), \ a^+ = (2\delta)^{-1/2}(p - i\delta q) \tag{28}$$

and observe that (27) can be further rewritten in a convenient form

$$\dot{a} = (-\alpha + i\delta)a. \tag{29}$$

The equation for a^+ is just the complex conjugate of (29) and does not add any new information. The solution of (29) is simply given as $a(t) = a(0)e^{(-\alpha+i\delta)t}, a^+(t) = a^+(0)^{(-\alpha-i\delta)t}$ or equivalently $p(t) = e^{-\alpha t}(p_0 \cos \ \delta t - \delta q_0 \ \sin \ \delta t)$ and $q(t) = e^{-\alpha t}(q_0 \cos \ \delta t + \frac{p_0}{\delta} \sin \ \delta t)$. From this it follows that even if (q_0, p_0) were a true canonically conjugate pair at $t = 0$, they cannot remain so for any $t > 0$. This is well known and expected. Of course quantization does not bring any change to the above observation and leads to the conclusion that there is *no* unitary time evolution to give rise to the equation of motion (29).

Now we want to change the picture and imagine that the damping in the motion of the quantum harmonic oscillator is due to the presence of some environmental friction which we shall model by quantum Brownian motion as described above. In other words, we consider a q.s.d.e.

$$U(t) = I + \int_0^t U(s)[\sqrt{2}\alpha(a^* dA(s) - a\, dA^+(s)) + (-\alpha + i\delta)a^* a\, ds]. \tag{30}$$

Here a and a^* have the same expression as for a and a^+ in (29), but we assume canonical conjugacy between p and q for all times i.e. $[p, q] = -iI$ or equivalently $[a, a^*] = I$ in $\mathcal{H}_0 = L^2(I\!R)$, the quantum state space for the harmonic oscillator. We also note that if $\alpha = 0$ i.e. if there is no damping that U satisfies $\frac{dU}{dt} = i\delta U a^* a$ whose solution is $U(t) = e^{i\delta a^* a t}$ with $\delta = \omega$ and this is the well known standard quantum harmonic oscillator evolution group.

That equation (30) has a unique unitary solution was proven in [22] using a method different from the one we have described in section 2. Nevertheless we give a sketch of a proof using theorem 2. As has been shown in [12] the map $Q_\lambda(\lambda > 0)$ defined on $\mathcal{B}(\mathcal{H}_0)$ as :

$$\langle u, Q_\lambda(x) v\rangle = \int_0^\infty e^{-\lambda t} \langle E e^{-Gt} U, x E e^{-Gt} v\rangle dt \tag{31}$$

is a well-defined completely positive contraction and the Feller set δ_λ of the problem is trivial if $Q_\lambda^n(I)$ converges strongly to 0. In the problem at hand, $E = -(2\alpha)^{1/2} a$, $G = (-\alpha + i\delta) a^* a = -\frac{1}{2} E^* E + iH$ (with $H = \delta a^* a$) $= (-\alpha + i\delta) N$ where $N = a^* a$ is the number opeator in $\mathcal{H}_0 = L^2(I\!R)$. It is easy to see that the 'finite particle vectors' form a total set in $\mathcal{H}_0$ and if we choose the (dense) linear span of these to be $\mathcal{D}$, and $E_n = -(2\alpha)^{1/2} n a (N+n)^{-1}$ then E_n, E_n^* and $G_n = -\frac{1}{2} E_n^* E_n + i\delta N n (N+n)^{-1}$ converge strongly and to E, E^* and G respectively. It is also equally easy to prove that $T_t = e^{Gt} = e^{(-\alpha + i\delta)Nt}$ leaves $\mathcal{D}$ invariant, thus verifying assumptions (A) and (B) preceeding theorem 2. In order to prove the unitarity of $U(t)$ it sufficies to show that the Feller set $\delta_\lambda = \{0\}$ for some $\lambda > 0$ and for this we use the observations made in the earlier part of this paragraph.

We can write Q_λ for this problem as:

$$Q_\lambda(x) = 2\alpha \int_0^\infty e^{-\lambda t} e^{-\bar{\gamma} t N} a^* x a e^{-\gamma t N} dt$$

with $\gamma = \alpha - i\delta$ and $\lambda > 0$. Then

$$Q_\lambda(I) \;=\; 2\alpha N \int_0^\infty e^{-(\lambda + 2\alpha N)t} dt$$

$$= \quad 2\alpha N(2\alpha N+\lambda)^{-1} = \frac{N}{N+\lambda'}, \tag{32}$$

where we have set $\lambda' = \lambda(2\alpha)^{-1}$. A simple calculation shows that

$$Q^n_\lambda(I) = \frac{N}{N+\lambda'}\frac{N-1}{N-1+\lambda'}\cdots\frac{N-n+1}{N-n+1+\lambda'}. \tag{33}$$

Thus for any $u \in \mathcal{H}_0$,

$$\begin{aligned} \|Q^n_\lambda(I)u\|^2 &= \sum_{m=0}^{\infty}\left(\frac{m}{m+\lambda'},\ldots,\frac{m-n+1}{m-n+t+\lambda'}\right)^2\|P_m u\|^2 \\ &\equiv \sum_{m=0}^{\infty} g_n(m)\|P_m u\|^2, \end{aligned}$$

where P_m is the projection onto the m-particle subspace. It is clear that $g_n(m) = 0$ if $n \geq m+1$ and $|g_n(m)| \leq 1$ for all n, m. Therefore, by the dominated convergence theorem, it follows that $Q^n_\lambda(I)$ converges strongly to zero and as discussed above we have the unitarity of $U(t)$.

If we set $a(t) = U(t)^* a U(t)$ and $a(t)^* = U(t)^* a U(t)$ so that $[a(t), a(t)^*] = I_{\mathcal{H}}$ for all $t \geq 0$, in contrast to what we have discussed earlier. It is clear that the evolved q and p in the presence of noise retains this kind of extended conjugacy. To get back to the description of the harmonic oscillator, we have to take expectation or average out the noise degrees of freedom. This leads to the expectation semigroup T_t on the algebra generated by a and a^* :

$T_t(x) \equiv \mathbb{E}U(t)^* x U(t)$ with its infinitesmal generator, the Lindbladian, given formally as

$$\mathcal{L}(x) = \frac{\alpha}{2}[2a^* x a - \alpha a^* a x - x a^* a] - i\delta[a^* a, x].$$

Thus $\mathcal{L}(a) = (-\alpha + i\delta)a$ or equivalently $T_t(a) = e^{(-\alpha+i\delta)t}a$ just as we had observed earlier. By introducing the noise degree of freedom we have gained back the unitarity of the evolution $U(t)$ though not the group propoerty as in the case with $\alpha = 0$ (viz. $U(t) = \exp(i\delta t\, a^* a)$).

There have been other attempts at various applications, e.g. scattering theory between a class of Markov cocycles ([23]), measurement theory of observables with continuous spectra in quantum mechanics ([24]) and input - output channels in quantum systems ([25]). We also mention that a class of Hamiltonian theories have been studied in the quantum field theoretic set-up to show that in the limit when the coupling constant tends to zero, one can derive an equation similar to (5) if one chooses the scaled macroscopic (or collective) variables appropriately ([27]).

Bibliography

1. R.L. Hudson, K.R. Parthasarathy, Quantum Ito's formula and Stochastic evolutions, Comm. Math. Phys. **93**, 301-323 (1984).

2. K.R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Birkhauser 1992.

3. P.A. Meyer, Quantum Probability for Probabilists, LNM series No. 1538, Springer 1993.

4. A. Mohari, K.B. Sinha, Quantum stochastic flows with infinite degrees of freedom and countable state Markov chains, Sankhya A **52**, 43-57 (1990).

5. M.P. Evans, Existence of quantum diffusions, Prob. Th. Rel. Fields **81**, 473-483 (1989).

6. F. Fagnola, K.B. Sinha, Quantum flows with unbounded structure maps and finite degrees of freedom, J. London Math. Soc. (2) **48**, 537-551 (1993).

7. A. Mohari, K.R. Parthasarathy, A quantum probabilistic analogue of Feller's condition for the existence of unitary Markovian cocycles in

Fock space, 475-497, Statistics and Probability : A Raghu Raj Bahadur Festschrift, Ed: J.K. Ghosh et al., Wiley Eastern 1993.

8. A. Mohari, K.B. Sinha, Stochastic dilation of minimal quantum dynamical semigroup, Proc. Ind. Acad. Sc. (Math. Sc.) **102 (3)**, 159-173 (1992).

9. J.L. Journé, structure des cocycles Markovian sur lépace de Fock, Prob. Th. Rel. Fields **75**, 291-316 (1987).

10. A. Mohari, Quantum stochastic differential equations with unbounded coefficients and dilation of Feller's minimal solution, Sankhya **A 53**, 255-287 (1991).

11. B.V.R. Bhat, K.B. Sinha, A stochastic differential equation with time-dependent unbounded operator coefficients, J. Func. Anal. **114**, 12-31 (1993).

12. B.V.R. Bhat, K.B. Sinha, Examples of unbounded generators leading to non conservative minimal semigroups, 89-104, Quantum Prob. Rel. Topics, Vol. 9, World Scientific 1994.

13. K.B. Sinha, Quantum Dynamical Semigroups, Operator theory — Advances and Applications Vol. **70**, 161-169, Birkhauser 1994.

14. J.M. Lindsay, K.B. Sinha, Feyman-kac representation of some non-commutative elliptic operators, preprint 1994.

15. J.L. Journé,, P.A. Meyer, Une martingale d'opérateurs bornés, non-représentable en intégrale stochastique, Sem. Prob. XX, LNM 1204, 313-316 (1986).

16. K.R. Parthasarathy, K.B. Sinha, Stochastic integral representation of bounded quantum martingales in Fock space, J. Func. Anal. **67**, 126-151 (1986).

17. S. Attal, P.A. Meyer, Interprétation probabiliste et extension des intégrales stochastique non-commutative, Sém. Prob. XXVII, LNM 1557, 312-327, Springer 1994.

18. S. Attal, An algebra of non-commutative semi-martingales : square and angle quantum brackets, J. Func. Anal. **124**, 292-332 (1994).

19. R.L. Hudson, J.M. Lindsay, On characterising quantum stochastic evolutions, Math. Proc. Camb. Phil. Soc. **102**, 363-369 (1987).

20. W.S. Bradshaw, Quantum diffusions and stochastic cocycles, Ph.D. thesis, university of Nottingham, U.K. 1990.

21. K.R. Parthasarathy, K.B. Sinha, Stop times in Fock space stochastic calculus, Prob. Th. Rel. Fields **75**, 431-458 (1987).

22. F. Fagnola, On quantum stochastic differential equatioons with unbounded coefficients, Prob. th. Rel. Fields **86**, 501-517 (1990).

23. F. Fagnola, K.B. Sinha, scattering theory for unitary cocycles, stochastic processes : A festschrift in honour of G. Kallianpur, ed: S. Cambanis et al, 81-88, Springer 1993.

24. K.B. Sinha, On the collapse postulate of Quantum mechanics, Mathematical Physics towards the 21st century, ed: R.N. Sen et al, 344-350, Ben Gurion Univ., Israel 1994.

25. A. Barchielli, Input and output channels in quantum systems and quantum stochastic differential equations, LNM 1303, 37-51, Springer 1988.

26. K.R. parthasarathy, K.B. Sinha, Markov chains as Evans-Hudson diffusions in Fock space, Sem. Prob. XXIV, LNM 1426, 363-369., Springer 1988/89.

27. L. Accardi, A. Frigerio, Y.G.Lu, The weak coupling limit as a quantum functional central limit, Comm. Math. Phys. **131**, 537-570 (1990).