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TRACE FORMULAS IN BOSON FOCK SPACES AND APPLICATIONS

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ABSTRACT. Trace formulas for the heat semi-groups of second quantization operators and their perturbations in the abstract Boson Fock space are given in terms of path (functional) integral representations. As applications, an inequality of Golden-Thompson type and a classical limit are derived for the trace of the heat semi-group of a perturbed second quantization operator. The abstract results are applied to a model of $P(\phi)$-type in quantum field theory.

I. Introduction

In the previous work [1,2], we introduced (infinite dimensional) Dirac-type operators acting in the abstract Boson-Fermion Fock space, which, in concrete realizations, describe supercharges of some models of supersymmetric quantum field theory, and studied their properties (see also [3]–[9] for further developments and related aspects). In particular, we derived a formula for their index in terms of a path (functional) integral. From a technical view-point, however, the conditions assumed in [1,2] to derive the index formula are not optimal. It is desirable to formulate more optimal conditions in this respect. The present paper concerns this problem.

In deriving the index formula, trace formulas for the heat semi-groups of second quantization operators and their perturbations in both the Boson and Fermion Fock spaces play important roles. Hence, one of the basic tasks to refine the previous result on the index formula should be to make more elaborate analysis on the trace formulas just mentioned. A first step of work in this direction has been taken forward in [11], where some technically improved (possibly most general) results on trace formulas in the abstract Boson Fock space were obtained. As applications of the trace formulas, one can derive an inequality of Golden-Thompson type and a classical limit for the trace of the heat semi-group of a perturbed second quantization operator [11]. We also apply the abstract results to a model of $P(\phi)$-type in QFT. This kind of results is well known in the case of Schrödinger operators in finite dimensions (e.g., [24, §9, §10] and references therein), but, the corresponding results in the case of Schrödinger-type operators in *infinite dimensions*, including the case of perturbed second quantization operators in the abstract Boson Fock space, seems to be lacking in the literature (see, however, [14], a pioneering work in the direction of mathematical theories of models in quantum field theory (QFT) *at finite positive temperatures*).

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In this paper, we summarize some results obtained in [11].

II. Preliminaries

In this section we review some fundamental facts in the abstract Boson Fock space.

2.1. *Some definitions*

Let $\mathcal{H}$ be a real separable Hilbert space with norm $||\cdot||_{\mathcal{H}}$ and $\{\phi(f)|f \in \mathcal{H}\}$ be the Gaussian random process indexed by $\mathcal{H}$. We denote by $(E, \mathcal{B}, \mu)$ the underlying probability space of the process, so that the Borel field $\mathcal{B}$ is generated by $\{\phi(f)|f \in \mathcal{H}\}$) and

$$\int_E e^{i\phi(f)} d\mu = e^{-||f||^2_{\mathcal{H}}/2}, \quad f \in \mathcal{H}.$$

The complex Hilbert space $L^2(E, d\mu)$ is called the Q-space representation of the Boson Fock space over $\mathcal{H}$ [21, §I.3].

Let A be a *strictly positive* self-adjoint operator acting in $\mathcal{H}$ with domain $D(A)$ (i.e., there exists a constant $c > 0$ such that $||Af||^2_{\mathcal{H}} \geq c||f||^2_{\mathcal{H}}, f \in D(A)$). Then, for each $s \in \mathbf{R}$, we can define an inner product $(\cdot\,,\cdot)_s$ on $D(A^{s/2})$ by

$$(f, g)_s = (A^{s/2}f, A^{s/2}g)_{\mathcal{H}}, \quad f, g \in D(A^{s/2}),$$

where $(\cdot\,,\cdot)_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$ (note that, for $s < 0$, $D(A^{s/2}) = \mathcal{H}$). For $s \geq 0$, $D(A^{s/2})$ with the inner product $(\cdot\,,\cdot)_s$ becomes a Hilbert space. We denote this Hilbert space by $\mathcal{H}_s$. For $s < 0$, we denote by $\mathcal{H}_s$ the completion of $\mathcal{H}$ in the norm $||\cdot||_s$.

For all $s \in \mathbf{R}$, the dual space of $\mathcal{H}_s$ can be identified with $\mathcal{H}_{-s}$ through the bilinear form ${}_{-s}<\cdot,\cdot>_s$ on $\mathcal{H}_{-s} \times \mathcal{H}_s$ such that for all $f \in \mathcal{H}_{-s} \cap \mathcal{H}, g \in \mathcal{H}_s \cap \mathcal{H}$, ${}_{-s}< f, g >_s = (f, g)_{\mathcal{H}}$.

We denote by $\mathcal{I}_1(\mathcal{H})$ the ideal of the trace class operators on $\mathcal{H}$. Throughout the present paper, we assume the following:

Assumption I. *For some* $\gamma_0 > 0$, $A^{-\gamma_0}$ *is in* $\mathcal{I}_1(\mathcal{H})$.

Let $\gamma > \gamma_0$ be fixed. Then the embedding mapping of $\mathcal{H}$ into $\mathcal{H}_{-\gamma}$ is Hilbert-Schmidt. Hence, by a theorem of Minlos-Sazonov-Gross, we can take

$$E = \mathcal{H}_{-\gamma}$$

and

$$\phi(f) = {}_{-\gamma}< \phi, f >_{\gamma}, \quad \phi \in E, f \in \mathcal{H}_{\gamma}.$$

For a probability measure ν on $(E, \mathcal{B})$, we denote by $:\phi(f_1)\cdots\phi(f_n):_{\nu}$ the Wick product of the random variables $\phi(f_1), \cdots, \phi(f_n)$ $(f_j \in \mathcal{H}, j = 1, \cdots, n)$ with respect to ν [21, §I.1]. For each $n \geq 1$, let $\Gamma_n(\mathcal{H})$ be the closed subspace (in $L^2(E, d\mu)$) generated by $:\phi(f_1)\cdots\phi(f_n):_{\mu}$, and set $\Gamma_0(\mathcal{H}) = \mathbf{C}$ (the space of constant functions on E). Then one has the orthogonal decomposition

$$L^2(E, d\mu) = \bigoplus_{n=0}^{\infty} \Gamma_n(\mathcal{H}).$$

(see [21, §I.1, Theorem I.6].)

As usual, we denote by $d\Gamma(A)$ the *second quantization* of A [21, §I.4] and set

$$H_0 = d\Gamma(A).$$

In the context of QFT, H_0 describes the Hamiltonian of the free Boson field with one-particle Hamiltonian A.

2.2. *Imaginary time Green functions*

Under Assumption I, $e^{-\beta A}$ is in $\mathcal{I}_1(\mathcal{H})$ for all $\beta > 0$, since $e^{-\beta A} \le \beta^{-s} C A^{-s}$ for all $s > 0$ with $C = \sup_{x>0}(x^s e^{-x}) < \infty$. Using the spectral property of $d\Gamma(A)$ (see [17, §VIII.10]), one can easily prove that, for all $\beta > 0$, $e^{-\beta H_0}$ is in $\mathcal{I}_1(L^2(E, d\mu))$ with

$$Z_A(\beta) := \mathrm{Tr}\ e^{-\beta H_0} = \frac{1}{\det(1 - e^{-\beta A})},$$

where Tr denotes trace and $\det(1+T)$ with T being a trace class operator is the determinant of $1+T$ ([19, §XIII.17], [23, Chapt.3]). In the context of QFT, $Z_A(\beta)$ is called the "partition function" of the free Hamiltonian H_0 with the "inverse temperature" β.

The following estimates are well known (cf. [2], [18, §X.7]): For all $n = 1, 2, \cdots$,

$$||\phi(f)^n \Psi||_{L^2(E,d\mu)} \le C_{f,n} ||(H_0 + 1)^{n/2} \Psi||_{L^2(E,d\mu)}, \quad f \in \mathcal{H}, \Psi \in D(H_0^{n/2}), \tag{2.1}$$

where $C_{f,n}$ is a constant depending on f and n. Hence, for all $t > 0, n = 1, 2, \cdots$, and $f \in \mathcal{H}$, $\phi(f)^n e^{-tH_0}$ is a bounded linear operator on $L^2(E, d\mu)$, which implies that its adjoint $(\phi(f)^n e^{-tH_0})^*$ is also bounded. It follows that $e^{-tH_0}\phi(f)^n$ is bounded and its closure is equal to $(\phi(f)^n e^{-tH_0})^*$. For notational convenience, we denote the closure of $e^{-tH_0}\phi(f)^n$ by the same symbol. Thus, for $z_1, \cdots, z_n \in \mathbf{C}$ with $0 \le \mathrm{Re}\ z_1 \le \mathrm{Re}\ z_2 \le \cdots \le \mathrm{Re}\ z_n \le \beta$, and $f_j \in \mathcal{H}, j = 1, \cdots, n$, we can define the "complex time Green function" $G_n(z_1, f_1; z_2, f_2; \cdots; z_n, f_n)$ at the inverse temperature β by

$$\begin{aligned} &G_n(z_1, f_1; \cdots; z_n, f_n) \\ &= \frac{\mathrm{Tr}\ \left(e^{-z_1 H_0}\phi(f_1) e^{-(z_2 - z_1)H_0}\phi(f_2)\cdots e^{-(z_n - z_{n-1})H_0}\phi(f_n) e^{-(\beta - z_n)H_0}\right)}{Z_A(\beta)}. \end{aligned}$$

For any set $\{z_1, \cdots, z_n\}$ with $\mathrm{Re}\ z_j \in [0, \beta], j = 1, \cdots, n$, we define $G_n(z_1, f_1; \cdots; z_n, f_n)$ by

$$G_n(z_1, f_1; \cdots; z_n, f_n) := G_n(z_{\sigma(1)}, f_{\sigma(1)}; \cdots; z_{\sigma(n)}, f_{\sigma(n)}),$$

if $\mathrm{Re}\ z_{\sigma(1)} \le \mathrm{Re}\ z_{\sigma(2)} \le \cdots \le \mathrm{Re}\ z_{\sigma(n)}$, where σ denotes a permutation of $(1, 2, \cdots, n)$.

The *two-point function* $G_2(z, f; w, g)$ can be explicitly computed [14,2]:

$$\begin{aligned} G_2(z, f; w, g) &= \left(f, (1 - e^{-\beta A})^{-1}\left(e^{-\beta A + \varepsilon(\mathrm{Re}\,(w-z))(w-z)A} + e^{-\varepsilon(\mathrm{Re}\,(z-w))(z-w)A}\right) g\right)_{\mathcal{H}}, \\ &\qquad f, g \in \mathcal{H},\ \mathrm{Re}\ z, \mathrm{Re}\ w \in [0, \beta], \end{aligned} \tag{2.2}$$

where $\varepsilon(t) = 1$ for $t \ge 0$ and $\varepsilon(t) = -1$ for $t < 0$. Moreover, for all $n \ge 1$,

$$\begin{aligned} &G_{2n-1}(z_1, f_1; \cdots; z_{2n-1}, f_{2n-1}) = 0, \\ &G_{2n}(z_1, f_1; \cdots; z_{2n}, f_{2n}) = \sum_{\text{pairs}} G_2(z_{i_1}, f_{i_1}; z_{j_1}, f_{j_1}) \cdots G_2(z_{i_n}, f_{i_n}; z_{j_n}, f_{j_n}), \end{aligned} \tag{2.3}$$

where $\sum_{\text{pairs}}$ means the sum over all $(2n)!/2^n n!$ ways of choosing n distinct pairs $\{i_1, j_1\}, \cdots, \{i_n, j_n\}$ from $\{1, \cdots, 2n\}$ with $i_1 < i_2 < \cdots < i_n; i_1 < j_1, \cdots, i_n < j_n$.

Remark : Obviously the right hand side (R.H.S.) of (2.2) is defined for all $z, w \in \mathbf{C}$ with symmetricity in $\{z, f\}$ and $\{w, g\}$, provided that $f, g \in \cap_{\alpha>0} D(e^{\alpha A})$. Hence, if $f_j \in \cap_{\alpha>0} D(e^{\alpha A}), j = 1, \cdots, n$, then $G_n(z_1, f_1; \cdots; z_n, f_n)$ can be extended to a function on $\mathbf{C}^n$ in time varibles $z_j, j = 1, \cdots, n$.

In what follows, we are concerned with functions $G_n(t_1, f_1; \cdots; t_n, f_n)$ with $t_j \in [0, \beta]$, called the *imaginary time Green fucntions* (ITGF's), and their generalizations. By (2.2) we have

$$G_2(t, f; s, g) = \left(f, (1 - e^{-\beta A})^{-1}(e^{(-(\beta - |t-s|)A} + e^{-|t-s|A})g\right)_{\mathcal{H}},$$
$$f, g \in \mathcal{H}, t, s \in [0, \beta]. \tag{2.4}$$

2.3. *Path Integral Representations of the ITGF's*

It was proven in [1,2] that the ITGF's introduced in the last subsection can be represented in terms of path (functional) integrals. In this subsection we review this aspect.

We first recall a fundamental result. Let $\beta > 0$ and

$$E_\beta = C([0, \beta]; E)$$

be the space of E-valued continuous functions on $[0, \beta]$. For each $\Phi \in E_\beta$, we denote the value of Φ at $t \in [0, \beta]$ by $\Phi_t \in E$. Let $\mathcal{F}$ be the Borel field on E_β generated by $\Phi_t(f), f \in \mathcal{H}_\gamma, t \in [0, \beta]$. The following theorem is a key to deriving the path integral representations of the ITGF's.

Theorem 2.1 [1,2]. *There exists a probability measure ν_β on $(E_\beta, \mathcal{F})$ such that $\{\Phi_t(f) | f \in \mathcal{H}_\gamma, t \in [0, \beta]\}$ is a family of jointly Gaussian random variables on $(E_\beta, \mathcal{F}, \nu_\beta)$ with covariance*

$$\int_{E_\beta} \Phi_t(f)\Phi_s(g) d\nu_\beta(\Phi) = \left(f, (1 - e^{-\beta A})^{-1}(e^{-(\beta - |t-s|)A} + e^{-|t-s|A})g\right)_{\mathcal{H}},$$
$$s, t \in [0, \beta], f, g \in \mathcal{H}_\gamma. \tag{2.5}$$

Remark: (1) The measure ν_β is an abstract form of a measure introduced in [14] to describe finite positive temperature states of Boson field models. The measure ν_β with $\beta = +\infty$ ("zero-temperature state") is discussed in [12].

(2) It follows from (2.4) that, for all $f \in \mathcal{H}_\gamma$,

$$\int_{E_\beta} |\Phi_0(f) - \Phi_\beta(f)|^2 d\nu_\beta(\Phi) = 0,$$

which, together with the separability of $\mathcal{H}_\gamma$, implies that $\Phi_0 = \Phi_\beta$, a.e. Φ. Hence, if we denote by $L([0, \beta], E)$ the space of continuous loops of E with parameter space $[0, \beta]$, then we have $\operatorname{supp} \nu_\beta \subset L([0, \beta], E)$. Thus ν_β can be regarded as a probability measure on the loop space $L([0, \beta], E)$.

(3) The random variable $\Phi \to \Phi_t(f)$ $(t \in [0, \beta], f \in \mathcal{H}_\gamma)$ can be extended to all $f \in \mathcal{H}$ as an element in $\cap_{p<\infty} L^p(E_\beta, \nu_\beta)$. We denote it by the same symbol.

By Theorem 2.1 and (2.3)–(2.5), we obtain the following:

Theorem 2.2. *Let $f_j \in \mathcal{H}, t_j \in [0,\beta], j = 1, \cdots, n$. Then*

$$G_n(t_1, f_1; \cdots ; t_n, f_n) = \int_{E_\beta} \Phi_{t_1}(f_1) \cdots \Phi_{t_n}(f_n) d\nu_\beta(\Phi).$$

We can also derive more general trace formulas. For this purpose, we introduce a class of measurable functions on $(E, \mathcal{B})$.

Definition 2.3. Let H be a self-adjoint operator in $L^2(E, d\mu)$ such that, for all $t > 0$, $\exp(-tH)$ is in $\mathcal{I}_1(L^2(E, d\mu))$. We say that a measurable function F on $(E, \mathcal{B})$ is in the set $\mathcal{I}_H$ if $e^{-tH}|F|e^{-tH}$ is in $\mathcal{I}_1(L^2(E, d\mu))$ for all $t > 0$.

Theorem 2.4 [1]. *Let $F_1, \cdots, F_n \in \mathcal{I}_{H_0}$ and $0 < t_1 < t_2 \cdots < t_n < \beta$. Then*

$$\frac{\mathrm{Tr}\left(e^{-t_1 H_0} F_1 e^{-(t_2-t_1)H_0} F_2 \cdots e^{-(t_n-t_{n-1})H_0} F_n e^{-(\beta-t_n)H_0}\right)}{Z_A(\beta)}$$
$$= \int_{E_\beta} F_1(\Phi_{t_1}) \cdots F_n(\Phi_{t_n}) d\nu_\beta(\Phi).$$

2.4. *A circle action*

Let $\beta > 0$. Since $\coth x > 0$ for all $x > 0$, we can define, via the functional calculus, a self-adjoint operator

$$B(\beta) := \left(\coth \frac{\beta A}{2}\right)^{1/2},$$

on $\mathcal{H}$, which is strictly positive and bounded with

$$1 \leq B(\beta) \leq \sqrt{\coth \frac{\beta \lambda_1}{2}},$$

where $\lambda_1 > 0$ is the lowest eigenvalue of A.

Lemma 2.5. *The operator $B(\beta) - 1$ is in $\mathcal{I}_1(\mathcal{H})$.*

By Lemma 2.5, $B(\beta) - 1$ is Hilbert-Schmidt. It follows from Shale's theorem ([20], [21, p.41, Theorem I.23]) that there exists a probability measure $\mu_{B(\beta)}$ on $(E, \mathcal{B})$ mutually absolutely continuous to μ such that

$$\int_E e^{i\phi(f)} d\mu_{B(\beta)}(\phi) = \int_E e^{i\phi(B(\beta)f)} d\mu(\phi) = e^{-\|B(\beta)f\|^2_{\mathcal{H}}/2}, \quad f \in \mathcal{H}_\gamma,$$

and $d\mu_{B(\beta)} = G_\beta d\mu$ with $G_\beta \in L^p(E, d\mu)$ for some $p > 1$ and $G_\beta^{-1} \in L^q(E, d\mu_{B(\beta)})$ for some $q > 1$.

By Remark (2) after Theorem 2.1, for a.e.$\Phi \in E_\beta$, we can extend Φ_t as a function of t to a periodic function on $\mathbf{R}$ with period β. It follows that, for each $t \in \mathbf{R}$, there exists a unique linear isometry J_t from $L^2(E, d\mu_{B(\beta)})$ into $L^2(E_\beta, d\nu_\beta)$ such that

$$J_t 1 = 1,$$
$$J_t : \phi(f_1) \cdots \phi(f_n) :_{\mu_{B(\beta)}} = : \Phi_t(f_1) \cdots \Phi_t(f_n) :_{\nu_\beta},$$
$$n \geq 1, f_j \in \mathcal{H}_\gamma, j = 1, \cdots, n.$$

It is easy to show that $t \to J_t$ is strongly continuous and

$$J_{t+\beta} = J_t, \quad t \in \mathbf{R}.$$

Moreover, in the same way as in a standard case (e.g., [21, p.34, Theorem I.17]), we can show that J_t is positivity preserving and extends uniquely to a contraction from $L^p(E, d\mu_{B(\beta)})$ to $L^p(E_\beta, d\nu_\beta)$ for each $p \in [1, \infty]$. By a standard limiting argument, we can show that, for all $F \in L^2(E, d\mu_{B(\beta)})$ and $t \in \mathbf{R}$,

$$(J_t F)(\Phi) = F(\Phi_t).$$

In particular, the mapping $t \to F(\Phi_t)$ is continuous in $L^2(E_\beta, d\nu_\beta)$.

III. Perturbation of H_0

We now consider a perturbation of H_0 by the multiplication operator defined by a real-valued measurable function V on $(E, \mathcal{B})$. For generality, we take the perturbation in the sense of quadratic forms. We first consider the case where V is bounded from below and then the case where V is not necessarily bounded from below.

3.1. *The case where V is bounded from below*

In this subsection, we assume the following:

(V.1) V is bounded from below and $D(H_0^{1/2}) \cap D(|V|^{1/2})$ is dense in $L^2(E, d\mu)$.

Under this condition, we have the self-adjoint operator

$$H_V := H_0 \dot{+} V$$

determined by the quadratic form sum of H_0 and V (we denote by $B \dot{+} C$ the self-adjoint operator determined by the quadratic form sum $q_{A,B}$ of self-adjoint operators A and B if $q_{A,B}$ is bounded from below and closed, see, e.g., [17, §VIII.6]).

Theorem 3.1. *Suppose that V satisfies (V.1). Then, for all $\beta > 0$, $\exp(-\beta H_V)$ is in $\mathcal{I}_1(L^2(E, d\mu))$ and, for all $t_j \in [0, \beta], 0 \le t_1 < t_2 < \cdots < t_n \le \beta$, and $F_j \in \mathcal{I}_{H_V} (j = 2, \cdots, n-1), F_1, F_n \in L^\infty(E, d\mu)$,*

$$\frac{\operatorname{Tr}\left(e^{-t_1 H_V} F_1 e^{-(t_2-t_1)H_V} F_2 \cdots e^{-(t_n-t_{n-1})H_V} F_n e^{-(\beta-t_n)H_0}\right)}{Z_A(\beta)}$$
$$= \int_{E_\beta} F_1(\Phi_{t_1}) \cdots F_n(\Phi_{t_n}) e^{-\int_0^{t_n} V(\Phi_s) ds} d\nu_\beta(\Phi).$$

Remark. (1) This theorem is a refinement of [1, Appendix D, Proposition D.3] (see Theorem 3.5 below).

(2) Under condition (V.1), we have

$$||H_0^{1/2}\Psi|| \le ||(H_V + c)^{1/2}\Psi||, \quad \Psi \in D(H_0^{1/2}) \cap D(|V|^{1/2}),$$

where c is a constant such that $V + c \ge 0$. It follows from this estimate and (2.1) that $\phi(f) \in \mathcal{I}_{H_V}$ for all $f \in \mathcal{H}$.

Theorem 3.1 can be proven by applying the Trotter product formula [16] and limit theorems on trace class operators [13,23] and quadratic forms [15,22].

3.2. *The case where V is not necessarily bounded from below*

In this case, we introduce a class $\mathfrak{S}_V$ of self-adjoint operators in $L^2(E, d\mu)$:

Definition 3.2. We say that a self-adjoint operator H is in $\mathfrak{S}_V$ if the following conditions (i) and (ii) are satisfied: (i) H is bounded from below ; (ii) there exists a sequence $\{V_n\}$ of real-valued measurable fucntions on $(E, \mathcal{B})$ satisfying (V.1) such that $V_n \geq V$ for all n, $V_n \to V$ a.e. as $n \to \infty$ and, for all $t > 0$,

$$e^{-tH_{V_n}} \to e^{-tH}$$

weakly as $n \to \infty$.

Remark. (1) It is easy to see that the weak convergence condition on $\exp(-tH_{V_n})$ in Definition 3.2 implies in fact that, for all $t > 0$, $\exp(-tH_{V_n}) \xrightarrow{s} \exp(-tH)$ as $n \to \infty$.

(2) Let ν_∞ be the measure ν_β with $\beta = +\infty$. Assume that V satisfies (V.1). Then, in the same way as in the proof of Theorem 3.1, we can show that, for all $t > 0$,

$$(\Psi_1, e^{-tH_V}\Psi_2) = \int_{E_\infty} \Psi_1(\Phi_0)^* \Psi_2(\Phi_t) e^{-\int_0^t V(\Phi_s)ds} d\nu_\infty(\Phi), \quad \Psi_j \in L^2(E, d\mu), j = 1, 2,$$

which is a standard Feynman-Kac-Nelson (FKN) formula [12,21]. Using this formula, we can show that, in this case, $\mathfrak{S}_V = \{H_V\}$.

The following proposition shows that, for wide classes of V, $\mathfrak{S}_V$ is not empty.

Proposition 3.3. *(i) Suppose that*

$$V,\ e^{-V} \in \cap_{0<p<\infty} L^p(E, d\mu).$$

Then $H = H_0 + V$ is essentially self-adjoint on $C^\infty(H_0) \cap D(V)$ and bounded from below (we denote by $\bar{H}$ the closure of H). Moreover, $\mathfrak{S}_V = \{\bar{H}\}$.

(ii) Suppose that, for a constant $\alpha > 1$, $D(H_0^{1/2}) \cap D(|V|^{\alpha/2})$ is dense in $L^2(E, d\mu)$ and

$$||H_0^{1/2}\Psi||^2 + \int_E V(\phi)|\Psi(\phi)|^2 d\mu(\phi) \geq -c||\Psi||^2, \quad \Psi \in D(H_0^{1/2}) \cap D(|V|^{\alpha/2}),$$

with a constant $c > 0$. Then $\mathfrak{S}_V \neq \emptyset$.

Remark. In Proposition 3.3, we do not need Assumption I for A ; as for part (i) (resp. (ii)), it is sufficient to assume that A is strictly positive (then $e^{-tH_0} (t > 0)$ becomes a hypercontractive semi-group [18, Theorem X.61]) (resp. nonnegative).

In what follows, we assume that $\mathfrak{S}_V \neq \emptyset$. We state the main result of this subsection.

Theorem 3.4. *Let $H \in \mathfrak{S}_V$. Then the following (i) and (ii) hold:*

(i) *If*

$$\int_{E_t} e^{-\int_0^t V(\Phi_s)ds} d\nu_t(\Phi) < \infty, \tag{3.1}$$

for $t > 0$, then $\exp(-tH)$ *is in* $\mathcal{I}_1(L^2(E, d\mu))$ *and*

$$\frac{\text{Tr } e^{-tH}}{Z_A(t)} \leq \int_{E_t} e^{-\int_0^t V(\Phi_s)ds} d\nu_t(\Phi).$$

(ii) *Let $\beta > 0$ be fixed and $0 < \delta < \beta$. Suppose that, for all $t \in [\delta, \beta]$, (3.1) is satisfied. Then, for all $t \in [\delta, \beta]$,* $\exp(-tH)$ *is in* $\mathcal{I}_1(L^2(E, d\mu))$ *and*

$$\frac{\text{Tr } e^{-tH} e^{-(\beta-t)H_0}}{Z_A(\beta)} = \int_{E_\beta} e^{-\int_0^t V(\Phi_s)ds} d\nu_\beta(\Phi).$$

Theorem 3.4 can be generalized as follows.

Theorem 3.5. *Let $H \in \mathfrak{S}_V$ and (3.1) holds for all $t \in (0, \beta]$. Then, for all $t \in (0, \beta]$,* $\exp(-tH)$ *is in* $\mathcal{I}_1(L^2(E, d\mu))$ *and, for all $t_j \in [0, \beta], 0 \leq t_1 < t_2 < \cdots < t_n \leq \beta$, and $F_j \in \mathcal{I}_H (j = 2, \cdots, n-1), F_1, F_n \in L^\infty(E, d\mu)$,*

$$\frac{\text{Tr } \left(e^{-t_1 H} F_1 e^{-(t_2-t_1)H} F_2 \cdots e^{-(t_n-t_{n-1})H} F_n e^{-(\beta-t_n)H_0}\right)}{Z_A(\beta)}$$
$$= \int_{E_\beta} F_1(\Phi_{t_1}) \cdots F_n(\Phi_{t_n}) e^{-\int_0^{t_n} V(\Phi_s)ds} d\nu_\beta(\Phi).$$

Remark. Theorem 3.5 may be the most general form for trace formulas w.r.t. the heat semi-group generated by a second quantization operator perturbed by a multiplication operator.

IV. A Golden-Thompson Inequality

By applying Theorem 3.4, we can establish an inequality of Golden-Thompson type for the partition function Tr $e^{-\beta H}$ with $H \in \mathfrak{S}_V$.

Theorem 4.1. *Let $\beta > 0$ and $H \in \mathfrak{S}_V$. Assume that V satisfies*

$$\int_E e^{-\beta V} d\mu_{B(\beta)} < \infty. \tag{4.1}$$

Then $e^{-\beta H}$ is in $\mathcal{I}_1(L^2(E, d\mu))$ and

$$\frac{\text{Tr } e^{-\beta H}}{Z_A(\beta)} \leq \int_E e^{-\beta V} d\mu_{B(\beta)}.$$

Moreover, if

$$\int_E |V| d\mu_{B(\beta)} < \infty$$

in addition to (4.1), then

$$e^{-\beta \int_E V d\mu_{B(\beta)}} \leq \frac{\text{Tr } e^{-\beta H}}{Z_A(\beta)}.$$

As a Corollary of Theorem 4.1, we can obtain some information about properties of a limiting operator given as a limit of operators $H^{(N)} \in \mathfrak{S}_{V_N}$.

Corollary 4.2. *Let $V_N(N = 1, 2, \cdots)$ be a real-valued measurable function on $(E, \mathcal{B})$ with*

$$C_\beta := \sup_{N \geq 1} \int_E e^{-\beta V_N} d\mu_{B(\beta)} < \infty.$$

Let $H^{(N)} \in \mathfrak{S}_{V_N}$. Suppose that there exists a self-adjoint operator H bounded from below such that $\exp(-\beta H^{(N)}) \to \exp(-\beta H)$ weakly as $N \to \infty$. Then, $\exp(-\beta H)$ is in $\mathcal{I}_1(L^2(E, d\mu))$ and

$$\frac{\mathrm{Tr}\ e^{-\beta H}}{Z_A(\beta)} \leq C_\beta.$$

V. Classical Limit

For $\lambda > 0$, we define V_λ by

$$V_\lambda(\phi) = V(\sqrt{\lambda}\phi), \quad \phi \in E.$$

Let $\hbar > 0$ be a parameter, which physically means the Planck constant devided by 2π, and let $H_\hbar \in \mathfrak{S}_{V_\hbar/\hbar}$. We are interested in the limiting behavior of the scaled partition function $\mathrm{Tr}\ e^{-\beta\hbar H_\hbar}$ as $\hbar \to 0$, which, in concrete realizations, corresponds to the classical limit of the quantum system whose Hamiltonian is given by $\hbar H_\hbar$.

5.1. *A simpler case*

We first consider the case where V obeys the following condition:

(V.2) V is bounded from below and there exists a polynomial $P(x, y)$ of two real variables with positive coefficients such that

$$|V(\phi) - V(\phi')| \leq ||\phi - \phi'||_E P(||\phi||_E, ||\phi'||_E), \quad \phi, \phi' \in E.$$

Note that (V.2) implies that V is continuous on E and V is polynomially bounded. Using the fact that

$$||\phi||_E^2 = \sum_{n=0}^{\infty} \frac{|\phi(e_n)|^2}{\lambda_n^\gamma}, \quad \phi \in E,$$

one can show that

$$\int_E ||\phi||_E^p d\mu(\phi) < \infty$$

for all $0 < p < \infty$. Hence, under condition (V.2), it follows that, for all $\lambda > 0$, $D(H_0) \cap D(V_\lambda)$ is dense in $L^2(E, d\mu)$; In particular, (V.1) is satisfied with V replaced by V_λ/λ for all $\lambda > 0$. Hence we have

$$H_\hbar = H_{V_\hbar/\hbar} = H_0 \dot{+} \frac{1}{\hbar} V_\hbar.$$

For any constant $c > 0$, the operator $cA^{-1/2}$ is a continuous bijection from $\mathcal{H}_\gamma$ to itself. Hence it extends to a continuous bijection from E to itself. We set

$$V^{(A)}(\phi) = V(\sqrt{2}A^{-1/2}\phi), \quad \phi \in E.$$

Theorem 5.1. *Suppose that V satisfies (V.2) and $H_\hbar$ be as above. Then*

$$\lim_{\hbar\to 0}\frac{\mathrm{Tr}\ e^{-\beta\hbar H_\hbar}}{\mathrm{Tr}\ e^{-\beta\hbar H_0}} = \int_E e^{-\beta V(\beta A)}d\mu. \tag{5.1}$$

The method of proof of this theorem is similar to that of [10].

5.2. *A more general case*

We next consider the case where V obeys a more general condition than (V.2). To describe it, we introduce two bounded operators:

$$C(\beta;\varepsilon) := \sqrt{\frac{\varepsilon}{\beta}}B(\varepsilon), \quad \varepsilon > 0,$$
$$C(\beta) = \sqrt{\frac{2}{\beta}}A^{-1/2}.$$

By the functional calculus, we can show that

$$C(\beta;\varepsilon) \xrightarrow{s} C(\beta)$$

as $\varepsilon \to +0$. Since $||C(\beta;\varepsilon)f||_{\mathcal{H}}$ defines a norm equivalent to $||f||_{\mathcal{H}}$, there exists a probability measure $\mu_{C(\beta;\varepsilon)}$ on $(E,\mathcal{B})$ such that

$$\int_E e^{i\phi(f)}d\mu_{C(\beta;\varepsilon)}(\phi) = e^{-||C(\beta;\varepsilon)f||^2_{\mathcal{H}}/2} = \int_E e^{i\sqrt{\varepsilon/\beta}\phi(f)}d\mu_{B(\varepsilon)}, \quad f \in \mathcal{H}_\gamma.$$

Similarly there exists a probability measure $\mu_{C(\beta)}$ on $(E,\mathcal{B})$ such that

$$\int_E e^{i\phi(f)}d\mu_{C(\beta)}(\phi) = e^{-||C(\beta)f||^2_{\mathcal{H}}/2} = \int_E e^{i\phi(C(\beta)f)}d\mu, \quad f \in \mathcal{H}_\gamma.$$

It follows that, for all $F \in L^1(E, d\mu_{C(\beta;\varepsilon)})$,

$$\int_E F(\sqrt{\varepsilon/\beta}\phi)d\mu_{B(\varepsilon)}(\phi) = \int_E F(\phi)d\mu_{C(\beta;\varepsilon)}$$

and, for all $G \in L^1(E, d\mu_{C(\beta)})$,

$$\int_E G(C(\beta)\phi)d\mu(\phi) = \int_E G(\phi)d\mu_{C(\beta)}.$$

We now consider the case where V satisfies the following condition:

(V.3) There exists a sequence $\{V_N\}_N$ of functions on E obeying condition (V.2) with the following properties:

(i) For all $p > 0$,

$$F_1(p) := \sup_{0<\varepsilon\leq\varepsilon_0, N\geq 1} \int_E e^{-pV_N}\, d\mu_{C(\beta;\varepsilon)} < \infty,$$

$$F_2(p) := \sup_{N\geq 1} \int_E e^{-pV_N}\, d\mu_{C(\beta)} < \infty.$$

(ii) There exists some $q \in (1, \infty)$ such that

$$\lim_{N\to\infty} ||V_N - V||_{L^q(E, d\mu_{C(\beta;\varepsilon)})} = 0$$

uniformly in $\varepsilon \in (0, \varepsilon_0]$ and

$$\lim_{N\to\infty} ||V_N - V||_{L^q(E, d\mu_{C(\beta)})} = 0.$$

We can prove the following theorem.

Theorem 5.2. *Suppose that V satisfies (V.3). Then, for all $t > 0$ and $\hbar \in (0, \varepsilon_0/\beta]$, $\exp(-tH_\hbar)$ is in $\mathcal{I}_1(L^2(E, d\mu))$ and (5.1) holds.*

VI. Application to a Model in QFT

In this section we apply the results in the preceding sections to a QFT model of $P(\phi)$-type on a finite volume in the d-dimensional space $\mathbf{R}^d$ $(d \geq 1)$(e.g.,[21,14,10]). Let

$$\Lambda = [-\ell_1/2, \ell_1/2] \times \cdots \times [-\ell_d/2, \ell_d/2]$$

be a rectangle in $\mathbf{R}^d$ $(\ell_j > 0, j = 1, \cdots, d)$ and set

$$\Lambda^* = \left\{ p = (p_1, \cdots, p_d) = \left(\frac{2\pi}{\ell_1} n_1, \cdots, \frac{2\pi}{\ell_d} n_d \right) \middle| n_1, \cdots, n_d \in \mathbf{Z} \right\}.$$

We denote by $\mathcal{D}'_{\mathrm{real}}(\Lambda)$ the space of real distributions on Λ (regarded as a d-torus). For $\phi \in \mathcal{D}'_{\mathrm{real}}(\Lambda)$, we define its Fourier transform $\hat{\phi}$ by

$$\hat{\phi}(p) = \phi(f_p^*)$$

where

$$f_p(x) = \frac{1}{\sqrt{|\Lambda|}} e^{ipx} \qquad (|\Lambda| := \prod_{j=1}^d \ell_j).$$

Let a be a real-valued function on Λ^* such that

$$a(p) \geq C(p^2 + m_0^2)^{d/2}, \quad p \in \Lambda^*,$$

where $C > 0$ and $m_0 > 0$ are constants. Then the set

$$\mathcal{H}(\Lambda) = \left\{ \phi \in \mathcal{D}'_{\mathrm{real}}(\Lambda) \middle| \sum_{p\in\Lambda^*} \frac{|\hat{\phi}(p)|^2}{a(p)} < \infty \right\}$$

becomes a real Hilbert space with the inner product

$$(\phi, \psi)_{\mathcal{H}(\Lambda)} := \frac{1}{2} \sum_{p\in\Lambda^*} \frac{\hat{\phi}(p)^* \hat{\psi}(p)}{a(p)}.$$

In $\mathcal{H}(\Lambda)$, we define an operator $A(\Lambda)$ by

$$D(A(\Lambda)) := \left\{ \phi \in \mathcal{H}(\Lambda) \middle| \sum_{p\in\Lambda^*} \frac{|a(p)\hat{\phi}(p)|^2}{a(p)} < \infty \right\}$$

$$(\widehat{A(\Lambda)\phi})(p) := a(p)\hat{\phi}(p), \quad \phi \in D(A(\Lambda)).$$

The operator $A(\Lambda)$ is self-adjoint and satisfies

$$A(\Lambda) \geq C m_0^d.$$

It is easy to see that the spectrum of $A(\Lambda)$ is equal to $\{a(p) | p \in \Lambda^*\}$. If $\gamma > 1$, then

$$\sum_{p\in\Lambda^*} \frac{1}{a(p)^\gamma} \leq \frac{1}{C^\gamma} \sum_{p\in\Lambda^*} \frac{1}{(p^2 + m_0^2)^{d\gamma/2}} < \infty.$$

Hence, for all $\gamma > 1$, $A(\Lambda)^{-\gamma}$ is in $\mathcal{I}_1(\mathcal{H}(\Lambda))$.

In what follows, we consider the case where the Hilbert space $\mathcal{H}$ and the self-adjoint operator A in the abstract theory are realized as $\mathcal{H}(\Lambda)$ and $A(\Lambda)$, respectively. We remark that the case $a(p) = (p^2 + m_0^2)^{1/2}$ (independently of d) gives the standard framework for a neutral scalar QFT on the space-time $\Lambda \times \mathbf{R}$ (hence, for $d \geq 2$, the present model differs from the standard one).

We fix a constant $\gamma > 1$ and set $E = \mathcal{H}(\Lambda)_{-\gamma}$. For $N = 1, 2, \cdots$, we define

$$\phi_N(x) = \sum_p^N \phi(f_p^*) f_p(x), \quad x \in \Lambda, \ \phi \in E,$$

where $\sum_p^N = \sum_{|p_1| \leq 2\pi N/\ell_1, \cdots, |p_d| \leq 2\pi N/\ell_d}$. Note that

$$\int_E ||\phi_N - \phi||_{-\gamma}^2 d\mu \to 0$$

as $N \to \infty$.

Let $g \in L^q(\Lambda), g \geq 0$ $(1 < q \leq 2)$. Then we can show that, for all $p \geq 1$ and $j = 1, 2, \cdots$,

$$\lim_{N\to\infty} \int_\Lambda :\phi_N(x)^j:_\mu g(x)dx \stackrel{\text{def}}{=} \int_\Lambda :\phi(x)^j:_\mu g(x)dx \tag{6.1}$$

exists in $L^p(E, d\mu)$ [10, Appendix].

Let P be a polynomial of the form $P(X) = \sum_{j=1}^{2n} c_j X^j$, $X \in \mathbf{R}$, with $c_{2n} > 0, c_j \in \mathbf{R}, j = 1, \cdots, 2n-1$, and set

$$V_N(\phi) = \int_\Lambda :P(\phi_N(x)):_\mu g(x)dx, \tag{6.2}$$

$$V(\phi) = \int_\Lambda :P(\phi(x)):_\mu g(x)dx. \tag{6.3}$$

Then, by (6.1), we have for all $p \geq 1$

$$||V_N - V||_{L^p(E,d\mu)} \to 0$$

as $N \to \infty$. Moreover, in the same way as in the case of the standard $P(\phi)_2$ model [21], we can show that, for all $t > 0$ and $N \geq 1$,

$$e^{-tV_N},\ e^{-tV} \in L^1(E, d\mu).$$

(cf. also [10, §III].) Hence, applying a general theorem [18, p.261, Theorem X.58], we see that

$$H(V_N) := H_0 + V_N$$

and

$$H := H_0 + V$$

are essentially self-adjoint on $C^\infty(H_0) \cap D(V_N)$ and $C^\infty(H_0) \cap D(V)$, respectively, and bounded from below. Moreover, $\overline{H(V_N)}$ converges to $\bar{H}$ in norm-resolvent sense as $N \to \infty$.

The operator $\overline{H(V_N)}$ (resp. $\bar{H}$) desribes a Hamiltonian with (resp. without) momentum cutoff.

The potential V given by (6.3) satisfies the assumption of Proposition 3.3(i). Hence we have the following fact.

Lemma 6.1. *Let V be as in (6.3). Then $\mathfrak{S}_V = \{\bar{H}\}$.*

6.1. *Bounds for the partition function of $\bar{H}$*

We now apply Theorem 4.1 to obtain bounds for the partition function Tr $e^{-\beta\bar{H}}$ of $\bar{H}$.

Theorem 6.2. *For all $\beta > 0$, $e^{-\beta\bar{H}}$ is in $\mathcal{I}_1(L^2(E, d\mu))$ and*

$$e^{-\beta\int_E V d\mu_{B(\beta)}} \leq \frac{\mathrm{Tr}\ e^{-\beta\bar{H}}}{Z_{A(\Lambda)}(\beta)} \leq \int_E e^{-\beta V} d\mu_{B(\beta)}.$$

6.2. *Classical limit*

As for classical limit of the present model, we first consider the case of the cutoff Hamiltonian $\overline{H(V_N)}$.

Lemma 6.3. *Let V_N be given by (6.2). Then, for all $N \geq 1$, V_N satisfies (V.2).*

By Lemma 6.3, we can apply Theorem 5.1 to obtain the following result. Let $H_{N,\hbar}$ be $H(V_N)$ with V_N replaced by $(V_N)_\hbar/\hbar$.

Theorem 6.4. *For all $\beta > 0$ and $N \geq 1$,*

$$\lim_{\hbar\to 0} \frac{\mathrm{Tr}\ e^{-\beta\hbar H_{N,\hbar}}}{\mathrm{Tr}\ e^{-\beta\hbar H_0}} = \int_E e^{-\beta V_N^{\beta A(\Lambda)}} d\mu.$$

Finally we consider the classical limit for Tr $e^{-\beta\bar{H}}$.

Theorem 6.5. *Let $H_\hbar$ be $\bar{H}$ with V replaced by $V_\hbar/\hbar$. Suppose that*

$$\sum_{p\in\Lambda^*} \frac{1}{a(p)} < \infty.$$

Then, for all $\beta > 0$,

$$\lim_{\hbar\to 0} \frac{\mathrm{Tr}\ e^{-\beta\hbar H_\hbar}}{\mathrm{Tr}\ e^{-\beta\hbar H_0}} = \int_E e^{-\beta V^{\beta A(\Lambda)}} d\mu.$$

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