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Split $\mathbb{Z}$-forms of irreducible prehomogeneous vector spaces

AKIHIKO GYOJA

Introduction.

Let $G$ be a connected reductive group over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$ a rational representation, and $V := \mathbb{C}^n$. Such a triple $(G, \rho, V)$ is called a prehomogeneous vector space if $G$ has a Zariski dense orbit in $V$. If $(G, \rho, V)$ is an irreducible, $(G, \rho, V)$ is said to be irreducible. Now assume that $(G, \rho, V)$ is an irreducible prehomogeneous vector space such that there exist a non-trivial rational character $\phi \in \text{Hom}(G, \mathbb{C}^\times)$ and an irreducible polynomial function $f \in \mathbb{C}[V]$ on $V$ such that $f(gv) = \phi(g)f(v)$ for all $g \in G$ and $v \in V$. Put

$$\text{Aut}(V, f) := \{(g, \phi_g) \in GL(V) \times \mathbb{C}^\times \mid f(gv) = \phi_g f(v) \text{ for all } v \in V\},$$

and $\text{Aut}^0(V, f)$ be the identity component of $\text{Aut}(V, f)$. If the image of $\text{Aut}^0(V, f)$ by the first projection coincides with $\rho(G)$, then $(G, \rho, V)$ is said to be saturated.

The purpose of this note is to classify and to describe the split $\mathbb{Z}$-forms of the saturated, irreducible prehomogeneous vector spaces. (See [G] for "split $\mathbb{Z}$-form"). For this purpose, we need to describe a Chevalley system explicitly for each complex simple Lie algebra. Such a description is given in §1, which would be useful in a different context, and so we have included some information which is not used in the present note. (For example, all information concerning $E_8$ is not necessary here.)

Notation. For a ring $A (\ni 1)$, $M_n(A)$ denotes the totality of $n \times n$-matrices. The group of units in $A$ is denoted by $A^\times$. An element of $A^\times$ is identified with the
the \( n \times n \)-matrix whose \((i,j)\)-component is 1 and the other components are 0. We sometimes write \( E_i \) for \( E_{ii} \). We denote by \( \text{diag}(t_1, \cdots, t_n) \) the diagonal matrix whose diagonal components are \( t_1, \cdots, t_n \). For a set \( X \), its cardinality is denoted by \( \#X \).

§1. Chevalley system.

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \), \( \mathfrak{h} \) a Cartan subalgebra, \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{r \in \mathcal{R}} \mathfrak{g}(r) \) the root space decomposition, \( 0 \neq X(r) \in \mathfrak{g}(r) \), and \( H(r) (\in \mathfrak{h}) \) the coroot vector which corresponds to a root \( r \). A system \( (X(r))_{r \in \mathcal{R}} \) is called a Chevalley system, if

\[
[X(r), X(-r)] = H(r) \quad (r \in \mathcal{R})
\]

and, for \( r, s, r + s \in \mathcal{R} \),

\[
[X(r), X(s)] = \pm p X(r + s),
\]

where \( p \) is the smallest positive integer such that \( s + (p + 1)r \not\in \mathcal{R} \).

The purpose of this section is to describe explicitly a Chevalley system for each complex simple Lie algebra.

1.1. Type \( A_{n-1} \).

We may assume that

\[
\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid \text{tr}(x) = 0\}
\]

and

\[
\mathfrak{h} = \{\text{diag}(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}, \sum t_i = 0\}.
\]

Then

\[
\mathcal{R} = \{\epsilon_i - \epsilon_j \mid i \neq j\},
\]
where
\[ \epsilon_i(\text{diag}(t_1, \ldots, t_n)) = t_i. \]

The coroots are given by
\[ H(\epsilon_i - \epsilon_j) = E_i - E_j. \]

A Chevalley system is given by
\[ X(\epsilon_i - \epsilon_j) = E_{ij}. \]

We may take as a root basis
\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq n - 1). \]

Then the Dynkin diagram is given by
\[ \circ \alpha_1 \cdots \alpha_{n-1}. \]

1.2. Type \( B_n \).

Let us define an element \( J \) of \( M_{2n+1}(\mathbb{C}) \) by
\[ J = \sum_{i=1}^{n}(E_{i,n+1} + E_{n+1,i}) + 2E_{2n+1,2n+1}. \]

We may assume that
\[ \mathfrak{g} = \{ X \in M_{2n+1}(\mathbb{C}) \mid XJ + J^tX = 0 \} \]
and
\[ \mathfrak{h} = \{ \text{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n, 0) \}. \]
Then
\[ R = \{ \pm \epsilon_i \pm \epsilon_j \ (i \neq j), \pm \epsilon_i \}, \]

where
\[ \epsilon_i (\text{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n, 0)) = t_i. \]

The coroots are given by
\[
\begin{align*}
H(\epsilon_i - \epsilon_j) &= (E_i - E_j) - (E_{n+i} - E_{n+j}) \quad (i \neq j) \\
H(\epsilon_i + \epsilon_j) &= (E_i + E_j) - (E_{n+i} + E_{n+j}) \quad (i < j) \\
H(-\epsilon_i - \epsilon_j) &= (-E_i - E_j) - (-E_{n+i} - E_{n+j}) \quad (i < j) \\
H(\epsilon_i) &= 2(E_i - E_{n+i}) \\
H(-\epsilon_i) &= -2(E_i - E_{n+i}).
\end{align*}
\]

A Chevalley system is given by
\[
\begin{align*}
X(\epsilon_i - \epsilon_j) &= E_{ij} - E_{n+j,n+i} \quad (i \neq j) \\
X(\epsilon_i + \epsilon_j) &= E_{i,n+j} - E_{j,n+i} \quad (i < j) \\
X(-\epsilon_i - \epsilon_j) &= E_{n+j,i} - E_{n+i,j} \quad (i < j) \\
X(\epsilon_i) &= E_{i,2n+1} - 2E_{2n+1,n+i} \\
X(-\epsilon_i) &= 2E_{2n+1,i} - E_{n+i,2n+1}.
\end{align*}
\]

We may take as a root basis of \( R \)
\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i < n), \quad \alpha_n = \epsilon_n. \]

Then the Dynkin diagram is given by
\[
\begin{array}{cccccc}
& \circ & \circ & \cdots & \circ & \Rightarrow \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n
\end{array}
\]
1.3. Type $C_n$.

Let
\[ J = \sum_{i=1}^{n} (E_{i,n+i} - E_{n+i,i}). \]

We may assume that
\[ \mathfrak{g} = \{ X \in M_{2n}(\mathbb{C}) \mid XJ + J^t X = 0 \} \]
and
\[ \mathfrak{h} = \{ \text{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n) \}. \]

Then
\[ R = \{ \pm \epsilon_i \pm \epsilon_j \ (i \neq j), \ \pm 2\epsilon_i \}, \]

where
\[ \epsilon_i(\text{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n)) = t_i. \]

The coroots are given by
\[
H(\epsilon_i - \epsilon_j) = (E_i - E_j) - (E_{n+i} - E_{n+j}) \quad (i \neq j) \\
H(\epsilon_i + \epsilon_j) = (E_i + E_j) - (E_{n+i} + E_{n+j}) \quad (i < j) \\
H(-\epsilon_i - \epsilon_j) = -(E_i + E_j) + (E_{n+i} + E_{n+j}) \quad (i < j) \\
H(2\epsilon_i) = E_i - E_{n+i} \\
H(-2\epsilon_i) = -E_i + E_{n+i}. 
\]

A Chevalley system is given by
\[
X(\epsilon_i - \epsilon_j) = E_{i,j} - E_{n+j,n+i} \quad (i \neq j) \\
X(\epsilon_i + \epsilon_j) = E_{i,n+j} + E_{j,n+i} \quad (i < j) \\
X(-\epsilon_i - \epsilon_j) = E_{n+j,i} + E_{n+i,j} \quad (i < j) \\
X(2\epsilon_i) = E_{i,n+i} \\
X(-2\epsilon_i) = E_{n+i,i}. 
\]
We may take as a root basis

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i < n), \quad \alpha_n = 2\epsilon_n. \]

Then the Dynkin diagram is given by

\[ \alpha_1 \xrightarrow{\bullet} \alpha_2 \xrightarrow{\bullet} \ldots \xrightarrow{\bullet} \alpha_{n-1} \xleftarrow{\bullet} \alpha_n. \]

1.4. Type \( D_n \).

Let

\[ J = \sum_{i=1}^{n}(E_{i,n+i} + E_{n+i,i}). \]

We may assume that

\[ \mathfrak{g} = \{ X \in M_{2n}(\mathbb{C}) \mid XJ + J^tX = 0 \} \]

and

\[ \mathfrak{h} = \{ \text{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n) \}. \]

Then

\[ R = \{ \pm \epsilon_i \pm \epsilon_j \ (i \neq j) \}, \]

where

\[ \epsilon_i(\text{diag}(t_1, \ldots, t_n - t_n, \ldots, -t_n)) = t_i. \]

The coroots are given by

\[ H(\epsilon_i - \epsilon_j) = (E_i - E_j) - (E_{n+i} - E_{n+j}) \quad (i \neq j) \]
\[ H(\epsilon_i + \epsilon_j) = (E_i + E_j) - (E_{n+i} + E_{n+j}) \quad (i < j) \]
\[ H(-\epsilon_i - \epsilon_j) = -(E_i + E_j) + (E_{n+i} + E_{n+j}) \quad (i < j). \]
A Chevalley system is given by

\[
X(\epsilon_i - \epsilon_j) = E_{ij} - E_{n+j,n+i} \quad (i \neq j)
\]
\[
X(\epsilon_i + \epsilon_j) = E_{i,n+j} - E_{j,n+i} \quad (i < j)
\]
\[
X(-\epsilon_i - \epsilon_j) = E_{n+j,i} - E_{n+i,j} \quad (i < j).
\]

We may take as a root basis

\[
\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i < n), \quad \alpha_n = \epsilon_n - 1 + \epsilon_n.
\]

Then the Dynkin diagram is given by

\[
\alpha_n \circ \alpha_1 \circ \alpha_2 \cdots \alpha_{n-2} \circ \alpha_{n-1}.
\]

Up to now, we have worked with the vector representation of the simple Lie algebra of type \(D_n\), but we also need to work with the half-spin representation. In the remainder of this paragraph, we freely use the notations of [SK, pp.110-114], where a brief account of the theory of the spin representation is given.

The representation space \(\Lambda(E) = \Lambda(\mathbb{C}^n)\) of the spin representation is the Grassmann algebra of the vector space \(E = \bigoplus_{i=1}^{n} \mathbb{C}e_i\). We write \(e_{i_1}e_{i_2}\cdots e_{i_k}\) for \(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\). Let us consider two kinds of linear operators which are defined as follows:

\[
e_i(e_{i_1}e_{i_2}\cdots e_{i_k}) = e_ie_{i_1}e_{i_2}\cdots e_{i_k}.
\]
\[
f_i(e_{i_1}e_{i_2}\cdots e_{i_k}) = \begin{cases} (-1)^{p-1}e_{i_1}\cdots e_{i_p}\cdots e_{i_k}, & \text{if } i = i_p \text{ for some } p, \\ 0, & \text{otherwise.} \end{cases}
\]
Here \(e_{i_1}\ldots e_{i_p}\ldots e_{i_k}\) means \(e_{i_1}\ldots e_{i_{p-1}}e_{i_{p+1}}\ldots e_{i_k}\). Let \(\tilde{\mathfrak{g}}\) be the linear span of
\[
e_i f_j \quad (1 \leq i, j \leq n), \\
e_i e_j \quad (1 \leq i < j \leq n), \\
f_j f_i \quad (1 \leq i < j \leq n).
\]

Then \(\tilde{\mathfrak{g}}\) is a Lie algebra and an isomorphism between \(\mathfrak{g}\) and \(\tilde{\mathfrak{g}}\) is given as follows:
\[
\begin{align*}
\text{diag}(t_1, \ldots, t_n - t_1, \ldots, -t_n) & \leftrightarrow \frac{1}{2} \sum_{i=1}^{n} t_i (e_i f_i - f_i e_i). \\
X(\epsilon_i - \epsilon_j) = E_{ij} - E_{n+j,n+i} & \leftrightarrow e_i f_j \quad (i \neq j), \\
X(\epsilon_i + \epsilon_j) = E_{i,n+j} - E_{j,n+i} & \leftrightarrow e_i e_j \quad (i < j), \\
X(-\epsilon_i - \epsilon_j) = E_{n+j,i} - E_{n+i,j} & \leftrightarrow f_j f_i \quad (i < j).
\end{align*}
\]

Thus a Chevalley system of \(\tilde{\mathfrak{g}}\) is given by
\[
\begin{align*}
X(\epsilon_i - \epsilon_j) = e_i f_j & \quad (i \neq j), \\
X(\epsilon_i + \epsilon_j) = e_i e_j & \quad (i < j), \\
X(-\epsilon_i - \epsilon_j) = f_j f_i & \quad (i < j).
\end{align*}
\]

As is easily seen
\[
\Lambda^{odd} = \Lambda^{odd}(E) = \sum_{k=odd}^{\Lambda^k(E)}
\]

and
\[
\Lambda^{even} = \Lambda^{even}(E) = \sum_{k=even}^{\Lambda^k(E)}
\]

are \(\tilde{\mathfrak{g}}\)-stable subspaces of \(\Lambda(E)\). These \(\tilde{\mathfrak{g}}\)-modules \(\Lambda^{odd}\) and \(\Lambda^{even}\) are known to be irreducible and are called the \textit{odd half-spin representation} and the \textit{even half-spin representation}, respectively.
We define an involutory automorphism $\iota$ of the Clifford algebra $C(Q)$ (generated by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$) by $\iota(e_i) = f_i$ and $\iota(f_i) = e_i$ ($1 \leq i \leq n$). Then $\iota$ induces an automorphism of $Spin_{2n}$, which we shall denote by the same letter $\iota$. See [SK, pp.110–114] for the Clifford algebras and the spin groups.

1.5. Type $G_2$.

We may assume that $\mathfrak{g}$ is the totality of the matrixes

$$
\begin{pmatrix}
0 & 2d & 2e & 2f & 2a & 2b & 2c \\
a & x_{11} & x_{12} & x_{13} & 0 & f & -e \\
b & x_{21} & x_{22} & x_{23} & -f & 0 & d \\
c & x_{31} & x_{32} & x_{33} & e & -d & 0 \\
d & 0 & -c & b & -x_{11} & -x_{21} & -x_{31} \\
e & c & 0 & -a & -x_{12} & -x_{22} & -x_{32} \\
f & -b & a & 0 & -x_{13} & -x_{23} & -x_{33}
\end{pmatrix}
$$

with $x_{11} + x_{22} + x_{33} = 0$, and

$$\mathfrak{h} = \{\text{diag}(0,t_1,t_2,t_3,-t_1,-t_2,-t_3) \mid t_1 + t_2 + t_3 = 0\}.$$

Then

$$R = \{\epsilon_i - \epsilon_j \ (i \neq j), \ \pm \epsilon_i\},$$

where

$$\epsilon_i(\text{diag}(0,t_1,t_2,t_3,-t_1,-t_2,-t_3)) = t_i.$$ 

The coroots are given by

$$H(\epsilon_i - \epsilon_j) = (E_{1+i} - E_{1+j}) - (E_{4+i} - E_{4+j}) \ (i \neq j),$$

$$H(\epsilon_i) = (2E_{1+i} - E_{1+j} - E_{1+k}) - (2E_{4+i} - E_{4+j} - E_{4+k}),$$

$$H(-\epsilon_i) = -(2E_{1+i} - E_{1+j} - E_{1+k}) + (2E_{4+i} - E_{4+j} - E_{4+k}),$$
where \( \{i, j, k\} = \{1, 2, 3\} \). A Chevalley system is given by

\[
X(\epsilon_i - \epsilon_j) = E_{1+i,1+j} - E_{4+j,4+i},
X(\epsilon_i) = E_{1+i,1} + 2E_{1,4+i} + E_{4+k,1+j} - E_{4+i,1+k},
X(-\epsilon_i) = E_{4+i,1} + 2E_{1,1+i} + E_{1+j,4+k} - E_{1+k,4+j},
\]

where \((i, j, k)\) is an arbitrary even permutation of \((1, 2, 3)\). In fact,

\[
[X(\epsilon_i - \epsilon_j), X(\epsilon_j - \epsilon_k)] = X(\epsilon_i - \epsilon_k),
[X(\epsilon_i - \epsilon_j), X(\epsilon_j)] = X(\epsilon_i),
[X(\epsilon_i - \epsilon_j), X(-\epsilon_i)] = -X(-\epsilon_j),
[X(\epsilon_i), X(-\epsilon_j)] = 3X(\epsilon_i - \epsilon_j),
[X(\epsilon_i), X(\epsilon_j)] = 2X(-\epsilon_k),
[X(-\epsilon_i), X(-\epsilon_j)] = -2X(\epsilon_k).
\]

In the last two commutation relations, \( \{i, j, k\} = \{1, 2, 3\} \). Let \( \mathfrak{C} \) be the octonion algebra (=the algebra of Cayley numbers) over \( \mathbb{C} \) [F,1.1]. Define a basis of \( \mathfrak{C} \) by

\[
u_1 = e_0, \quad \nu_2 = e_7 \\
u_3 = e_1 + \sqrt{-1}e_6, \quad \nu_4 = e_2 + \sqrt{-1}e_5, \quad \nu_5 = e_4 + \sqrt{-1}e_3, \\
u_6 = -e_1 + \sqrt{-1}e_6, \quad \nu_7 = -e_2 + \sqrt{-1}e_5, \quad \nu_8 = -e_4 + \sqrt{-1}e_3.
\]

Here we use the notations of [F,1.5]. With respect to this basis, the Lie algebra of the infinitesimal automorphisms of \( \mathfrak{C} \) is identified with the Lie algebra \( \mathfrak{g} \) defined above. We may take as a root basis

\[
\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = -\epsilon_1.
\]
Then the Dynkin diagram is given by

$$\alpha_1 \Rightarrow \alpha_2.$$

### 1.6. Type $F_4$.

In this paragraph, we use the notations of [F]. Define a basis of $C$ by

\begin{align*}
  f_1 &= e_0 + \sqrt{-1}e_7, & f_5 &= -e_0 + \sqrt{-1}e_7, \\
  f_2 &= e_6 + \sqrt{-1}e_1, & f_6 &= -e_6 + \sqrt{-1}e_1, \\
  f_3 &= e_5 + \sqrt{-1}e_2, & f_7 &= -e_5 + \sqrt{-1}e_2, \\
  f_4 &= e_3 + \sqrt{-1}e_4, & f_8 &= -e_3 + \sqrt{-1}e_4.
\end{align*}

(1.6.1)

The multiplication table is given by

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
  & $f_1$ & $f_2$ & $f_3$ & $f_4$ & $f_5$ & $f_6$ & $f_7$ & $f_8$ \\
\hline
$f_1$ & $2f_1$ & $2f_2$ & $2f_3$ & $2f_4$ & 0 & 0 & 0 & 0 \\
$f_2$ & 0 & 0 & $-2f_8$ & $2f_7$ & $-2f_2$ & $2f_1$ & 0 & 0 \\
$f_3$ & 0 & $-2f_8$ & 0 & $-2f_6$ & $-2f_3$ & 0 & $2f_1$ & 0 \\
$f_4$ & 0 & $-2f_7$ & $2f_6$ & 0 & $-2f_4$ & 0 & 0 & $2f_1$ \\
$f_5$ & 0 & 0 & 0 & 0 & $-2f_5$ & $-2f_6$ & $-2f_7$ & $-2f_8$ \\
$f_6$ & $2f_6$ & $-2f_5$ & 0 & 0 & 0 & $2f_4$ & $-2f_3$ & \\
$f_7$ & $2f_7$ & 0 & $-2f_5$ & 0 & 0 & $-2f_4$ & 0 & $2f_2$ \\
$f_8$ & $2f_8$ & 0 & 0 & $-2f_5$ & 0 & $2f_3$ & $-2f_2$ & 0 \\
\hline
\end{tabular}
\end{table}

(1.6.2)

e.g., $f_1f_3 = 2f_3$, $f_3f_1 = 0$. Let us identify a linear endomorphism of $C$ with the corresponding matrix with respect to the basis $\{f_i\}$, e.g., $E_{ij}f_j = f_i$. Let us describe
the automorphisms $\lambda$ and $\lambda^2$ of $\mathfrak{D}_4 [\mathbb{F},2.2.4]$ in the matrix form. For

$$X = \begin{pmatrix}
0 & x_{12} & x_{13} & x_{14} & 0 & y_{21} & y_{31} & y_{41} \\
x_{21} & 0 & x_{23} & x_{24} & -y_{21} & 0 & y_{32} & y_{42} \\
x_{31} & x_{32} & 0 & x_{34} & -y_{31} & -y_{32} & 0 & y_{43} \\
x_{41} & x_{42} & x_{43} & 0 & -y_{41} & -y_{42} & -y_{43} & 0 \\
0 & -z_{12} & -z_{13} & -z_{14} & 0 & -x_{21} & -x_{31} & -x_{41} \\
z_{12} & 0 & -z_{23} & -z_{24} & -x_{12} & 0 & -x_{32} & -x_{42} \\
z_{13} & z_{23} & 0 & -z_{34} & -x_{13} & -x_{23} & 0 & -x_{43} \\
z_{14} & z_{24} & z_{34} & 0 & -x_{14} & -x_{24} & -x_{34} & 0
\end{pmatrix},$$

we have

$$(1.6.3) \quad \lambda(X) = \begin{pmatrix}
0 & -y_{43} & y_{42} & -y_{32} & 0 & -x_{21} & -x_{31} & -x_{41} \\
-y_{34} & 0 & x_{23} & x_{24} & x_{21} & 0 & z_{14} & -z_{13} \\
z_{24} & x_{32} & 0 & x_{34} & x_{31} & -z_{14} & 0 & z_{12} \\
-z_{23} & x_{42} & x_{43} & 0 & x_{41} & z_{13} & -z_{12} & 0 \\
0 & x_{12} & x_{13} & x_{14} & 0 & z_{34} & -z_{24} & z_{23} \\
-x_{12} & 0 & -y_{41} & y_{31} & y_{43} & 0 & -x_{32} & -x_{42} \\
-x_{13} & y_{41} & 0 & -y_{21} & -y_{42} & -x_{23} & 0 & -x_{43} \\
-x_{14} & -y_{31} & y_{21} & 0 & y_{32} & -x_{24} & -x_{34} & 0
\end{pmatrix}$$

and

$$(1.6.4) \quad \lambda^2(X) = \begin{pmatrix}
0 & -z_{12} & -z_{13} & -z_{14} & 0 & z_{34} & -z_{24} & z_{23} \\
-y_{21} & 0 & x_{23} & x_{24} & -z_{34} & 0 & -x_{14} & x_{13} \\
-y_{31} & x_{32} & 0 & x_{34} & z_{24} & x_{14} & 0 & -x_{12} \\
-y_{41} & x_{42} & x_{43} & 0 & -z_{23} & -x_{13} & x_{12} & 0 \\
0 & -y_{43} & y_{42} & -y_{32} & 0 & y_{21} & y_{31} & y_{41} \\
y_{43} & 0 & x_{41} & -x_{31} & z_{12} & 0 & -x_{32} & -x_{42} \\
-y_{42} & -x_{41} & 0 & x_{21} & z_{13} & -x_{23} & 0 & -x_{43} \\
y_{32} & x_{31} & -x_{21} & 0 & z_{14} & -x_{24} & -x_{34} & 0
\end{pmatrix}.$$
Let
\[
\begin{pmatrix}
(t_1^{(0)}) \\
(t_2^{(0)}) \\
(t_3^{(0)}) \\
(t_4^{(0)})
\end{pmatrix}
= 
\begin{pmatrix}
t_1 \\
t_2 \\
t_3 \\
t_4
\end{pmatrix},
\begin{pmatrix}
(t_1^{(j+1)}) \\
(t_2^{(j+1)}) \\
(t_3^{(j+1)}) \\
(t_4^{(j+1)})
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
(t_1^{(j)}) \\
(t_2^{(j)}) \\
(t_3^{(j)}) \\
(t_4^{(j)})
\end{pmatrix}.
\]

Then, for
\[
X = \text{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4),
\]
we have
\[
\lambda^j(X) = \text{diag}(t_1^{(j)}, t_2^{(j)}, t_3^{(j)}, t_4^{(j)}, -t_1^{(j)}, -t_2^{(j)}, -t_3^{(j)}, -t_4^{(j)}).
\]

As in [F,4.5.9], \( \mathfrak{J} \) denotes the exceptional simple Jordan algebra. We may assume that
\[
\mathfrak{g} = \{ \text{infinitesimal automorphisms of } \mathfrak{J} \}.
\]

Let us identify an element \( \delta \) of \( \mathfrak{D}_4 \) with the element \( \delta \) of \( \mathfrak{g} \) defined by
\[
\delta \begin{pmatrix}
\xi_1 & x_3 & \overline{x_2} \\
\overline{x_2} & \xi_2 & x_1 \\
x_2 & \overline{x_1} & \xi_3
\end{pmatrix}
= \begin{pmatrix}
0 & \delta_3x_3 & \overline{\delta_2x_2} \\
\overline{\delta_3x_3} & 0 & \delta_1x_1 \\
\delta_2x_2 & \overline{\delta_1x_1} & 0
\end{pmatrix},
\]
where \( \delta_i = \lambda^{i-1}(\delta) \). We may assume that
\[
\mathfrak{h} = \{ \text{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4) \},
\]
where we identify \( \mathfrak{h} (\subset \mathfrak{D}_4) \) with a subalgebra of \( \mathfrak{g} \) via the above defined identification.

Then
\[
R = \{ \pm \epsilon_i \pm \epsilon_j \ (i \neq j), \pm \epsilon_i, \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \},
\]
where
\[ \epsilon_i(\text{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4)) = t_i. \]

The coroots are given by
\[ H(s_i \epsilon_i + s_j \epsilon_j) = s_i(E_i - E_{4+i}) + s_j(E_j - E_{4+j}) \quad (i \neq j) \]
\[ H(s_i \epsilon_i) = s_i(2E_i - 2E_{4+i}) \]
(1.6.6)
\[ H\left(\frac{1}{2}(s_1 \epsilon_1 + s_2 \epsilon_2 + s_3 \epsilon_3 + s_4 \epsilon_4)\right) = \sum_{i=1}^{4} s_i(E_i - E_{4+i}), \]

where \( s_i = \pm 1 \). For \( a \in \mathbb{C} \), let
\[
(a)_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & a \\
0 & -\bar{a} & 0 \\
\end{pmatrix},
(a)_2 = \begin{pmatrix}
0 & 0 & -\bar{a} \\
0 & 0 & 0 \\
\bar{a} & 0 & 0 \\
\end{pmatrix},
(a)_3 = \begin{pmatrix}
0 & a & 0 \\
-\bar{a} & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]

For \( X \in \mathfrak{M}^{(3)} \), define a linear endomorphism \( \tilde{X} \) of \( \mathfrak{J} \) by
\[ \tilde{X}(Y) = \frac{1}{2}(XY + Y^{*}X^{*}), \]
where \( X^{*} \) is the transposed conjugate of \( X \) \([F,4.1]\). A Chevalley system is given by
\[ X(\epsilon_i - \epsilon_j) = E_{ij} - E_{4+j,4+i} \quad (i \neq j) \]
\[ X(\epsilon_i + \epsilon_j) = E_{i,4+j} - E_{j,4+i} \quad (i < j) \]
\[ X(-\epsilon_i - \epsilon_j) = E_{4+j,i} - E_{4+i,j} \quad (i < j) \]
(1.6.7)
\[ X(\epsilon_i) = (f_i)_1 \quad X(-\epsilon_i) = (f_{4+i})_1 \]
\[ X(\epsilon_i \circ \lambda) = (f_i)_2 \quad X(-\epsilon \circ \lambda) = (f_{4+i})_2 \]
\[ X(\epsilon_i \circ \lambda^2) = (f_i)_3 \quad X(-\epsilon \circ \lambda^2) = (f_{4+i})_3. \]
Note that
\[
\begin{pmatrix}
\varepsilon_1 \circ \lambda \\
\varepsilon_2 \circ \lambda \\
\varepsilon_3 \circ \lambda \\
\varepsilon_4 \circ \lambda
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\varepsilon_1 \circ \lambda^2 \\
\varepsilon_2 \circ \lambda^2 \\
\varepsilon_3 \circ \lambda^2 \\
\varepsilon_4 \circ \lambda^2
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4
\end{pmatrix}
\]

Let us give explicitly the commutation relations. Let $\delta$ be an element of $\mathfrak{g}$, of the form $X(\pm \epsilon_i \pm \epsilon_j)$ $(i \neq j)$. Then $\delta_1, \delta_2, \delta_3$ are of the form $\pm X(\pm \epsilon_i \pm \epsilon_j)$ by (1.6.3) and (1.6.4). Here $\delta_j f_i$ are of the form $\pm f_k$. By [F,4.9.4],

\[(1.6.8)\quad [\delta_i, (f_j)^\sim] = (\delta_j f_i)^\sim = \pm (f_k)^\sim \text{ or } 0.\]

The signature appeared in (1.6.8) can be easily determined by using (1.6.3), (1.6.4) and (1.6.7). A direct calculation shows that

\[(1.6.9)\quad [(f_i)^\sim, (f_j)^\sim] = 2(E_{ij'} - E_{j'i})\]

where

\[\iota' = \begin{cases} i + 4, & (i \leq 4) \\ i - 4, & (i > 4) \end{cases}\]

and

\[(1.6.10)\quad [(a)^\sim_i, (b)^\sim_j] = \left(- \frac{1}{2} a b \right)^\sim_k,\]
for each even permutation $(i,j,k)$ of $(1,2,3)$. Since $\frac{-1}{2}f_if_j$ is of the form $\pm f_k$ of 0, (1.6.8), (1.6.9) and (1.6.10) together with the results of (1.4), give the commutation relation among the Chevalley system given above. We may take as a root basis

$$\alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4, \quad \alpha_3 = \epsilon_4,$$
$$\alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

Then the Dynkin diagram is given by

$$\begin{array}{c}
\alpha_1 \Rightarrow \alpha_2 \quad \alpha_3 \Rightarrow \alpha_4
\end{array}$$

1.7. Type $E_6$.

In this paragraph, we use the notations of [F]. We may assume that

$$\mathfrak{g} = \mathfrak{e}_6 = \{\text{linear endomorphisms of } \mathfrak{f} \text{ which (infinitesimally)}$$
$$\text{preserves } \det(X,Y,Z)\}$$

[F,8.1]. The Lie algebra $\mathfrak{f}_4$ of infinitesimal automorphisms of $\mathfrak{f}$ is contained in $\mathfrak{g}$. Let $\mathfrak{h}_4$ be the Cartan subalgebra of $\mathfrak{f}_4$ which is given in (1.6). We may assume that

$$\mathfrak{h} = \mathfrak{h}_4 + \{ \left( \begin{array}{c} t_5 \\ t_6 \\ t_7 \end{array} \right) | t_5 + t_6 + t_7 = 0 \}.$$ 

Let

$$h(t_1, \ldots, t_7) = \text{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4) + \left( \begin{array}{c} t_5 \\ t_6 \\ t_7 \end{array} \right).$$
and
\[ \epsilon_i(h(t_1, \ldots, t_7)) = t_i. \]

Let us define endomorphisms \( \alpha_{ij} \) \((1 \leq i, j \leq 3)\) of \( \mathfrak{D}_4 \) by

\[
\begin{align*}
\alpha_{ii} &= 0 \quad (1 \leq i \leq 3), \\
\alpha_{23} &= 1, \\
\alpha_{31} &= \lambda, \\
\alpha_{32} &= \kappa, \\
\alpha_{12} &= \lambda^2, \\
\alpha_{13} &= \mathcal{K}\lambda, \\
\alpha_{21} &= \kappa\lambda^2
\end{align*}
\]

[\text{F},2.2]. Note that every \( \alpha_{ij} \) preserves \( \mathfrak{h}_4 \). Let

\[
\begin{align*}
A_{ii} &= 0 \quad (1 \leq i \leq 3), \\
A_{23} &= 1, \\
A_{31} &= \frac{1}{2}
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 \\
\end{pmatrix}, \\
A_{12} &= \frac{1}{2}
\begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}, \\
A_{32} &=
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \\
A_{13} &= \frac{1}{2}
\begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 \\
\end{pmatrix},
\end{align*}
\]
\[ A_{21} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \]

and
\[
\begin{pmatrix} t_{1}^{ij} \\ t_{2}^{ij} \\ t_{3}^{ij} \\ t_{4}^{ij} \end{pmatrix} = A_{ij} \begin{pmatrix} t_{1} \\ t_{2} \\ t_{3} \\ t_{4} \end{pmatrix}.
\]

Then
\[
\alpha_{ij}h(t_{1}, t_{2}, t_{3}, t_{4}, 0, 0, 0) = h(t_{1}, t_{2}, t_{3}, t_{4}, 0, 0, 0).
\]

We identify an element \( \delta \) of \( \mathfrak{D}_{4} \) with a linear endomorphism of \( \mathfrak{M}_{3} [F,4.1] \) as follows:
\[
\delta \left( \sum_{i,j=1}^{3} x_{ij}E^{(3)}_{ij} \right) = \sum_{i,j=1}^{3} (\delta_{ij}x_{ij})E^{(3)}_{ij},
\]

where \( \delta_{ij} = \alpha_{ij}(\delta) [F,4.9] \). Let
\[
R_{ij} = \{ \pm \epsilon_{k} \circ \alpha_{ij} \mid 1 \leq k \leq 4 \} \quad (1 \leq i, j \leq 3).
\]

Then
\[
R = \bigcup_{i \neq j} (R_{ij} + \frac{1}{2}(\epsilon_{4+i} - \epsilon_{4+j})) \cup \{ \pm \epsilon_{i} \pm \epsilon_{j} \mid 1 \leq i < j \leq 4 \}
\]
\[
= \{ \pm \epsilon_{i} \pm \frac{1}{2}(\epsilon_{6} - \epsilon_{7}) \mid 1 \leq i \leq 4 \},
\]
\[
\frac{1}{2} \sum_{i=1}^{4} s_{i} \epsilon_{i} = \frac{1}{2}(\epsilon_{5} - \epsilon_{7}) \quad (\prod_{i=1}^{4} s_{i} = -1),
\]
\[
\frac{1}{2} \sum_{i=1}^{4} s_{i} \epsilon_{i} = \frac{1}{2}(\epsilon_{5} - \epsilon_{6}) \quad (\prod_{i=1}^{4} s_{i} = 1),
\]
\[
\pm \epsilon_{i} \pm \epsilon_{j} \quad (1 \leq i < j \leq 4),
\]
where $s_i = \pm 1$. Define an order by

$$
\sum_{i=1}^{7} s_i \epsilon_i > 0,
$$

if $s_{\sigma(1)} = \cdots = s_{\sigma(k-1)} = 0$ and $s_{\sigma(k)} > 0$ for some $1 \leq k \leq 7$, where $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. Then the positive roots are

\[ \pm \epsilon_i + \frac{1}{2}(\epsilon_6 - \epsilon_7) \quad (1 \leq i \leq 4), \]
\[ \pm \frac{1}{2} \sum_{i=1}^{4} s_i \epsilon_i + \frac{1}{2}(\epsilon_5 - \epsilon_7) \quad (\prod_{i=1}^{4} s_i = -1), \]
\[ \pm \frac{1}{2} \sum_{i=1}^{4} s_i \epsilon_i + \frac{1}{2}(\epsilon_5 - \epsilon_6) \quad (\prod_{i=1}^{4} s_i = 1), \]
\[ \epsilon_i \pm \epsilon_j \quad (1 \leq i < j \leq 4), \]

and simple roots are

\[ r_1 = -\epsilon_1 + \frac{1}{2}(\epsilon_6 - \epsilon_7) = -\epsilon_1 \circ \alpha_{23} + \frac{1}{2}(\epsilon_6 - \epsilon_7) \]
\[ r_2 = \epsilon_3 - \epsilon_4 \]
\[ r_3 = \epsilon_1 - \epsilon_2 \]
\[ r_4 = \epsilon_2 - \epsilon_3 \]
\[ r_5 = \epsilon_3 + \epsilon_4 \]
\[ r_6 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) + \frac{1}{2}(\epsilon_5 - \epsilon_6) = \epsilon_1 \circ \alpha_{12} + \frac{1}{2}(\epsilon_5 - \epsilon_6). \]

Let $h_1 = h(1,0,0,0,0,0)$, $h_2 = h(0,1,0,0,0,0)$ etc. The coroots are given by

\[ H(\sum_{i=1}^{7} c_i \epsilon_i) = \sum_{i=1}^{4} c_i h_i + 2 \sum_{i=5}^{7} c_i h_i, \]
where $\sum_{i=1}^{7} c_i \epsilon_i \in R$. Especially

\[
H(r_1) = -h_1 + (h_6 - h_7),
\]
\[
H(r_2) = h_3 - h_4,
\]
\[
H(r_3) = h_1 - h_2,
\]
\[
H(r_4) = h_2 - h_3,
\]
\[
H(r_5) = h_3 + h_4,
\]
\[
H(r_6) = \frac{1}{2}(-h_1 - h_2 - h_3 - h_4) + (h_5 - h_6).
\]

Hence the Dynkin diagram is given by

\[ r_1 \rightarrow r_3 \rightarrow r_4 \rightarrow r_5 \rightarrow r_6 \]

\[ \quad \quad | \quad \quad \quad \quad \quad \quad \quad r_2 \]

Let

\[(a)_{ij} = aE_{ij}^{(3)} \quad (1 \leq i, j \leq 3, \ a \in \mathbb{C}),\]
\[\mathfrak{M}_3^r = \{T \in \mathfrak{M} \text{ with real diagonal elements}\},\]
\[\chi = x_{11} + x_{22} + x_{33}\]
\[\tilde{T}(X) = \frac{1}{2}(TX + XT^*) \quad (X \in \mathfrak{J}, \ T \in \mathfrak{M}_3)\]

Every element of $\mathfrak{g}$ can be uniquely expressed as

\[\delta + \tilde{T},\]
where $\delta \in \mathfrak{D}_4$, $T \in \mathfrak{M}_3$ and $\chi(T) = 0$ [F,8.1.1]. A Chevalley system is given by

\begin{align*}
X(\epsilon_i - \epsilon_j) &= E_{i,j} - E_{4+j,4+i} & (i \neq j) \\
X(\epsilon_i + \epsilon_j) &= E_{i,4+j} - E_{j,4+i} & (i < j) \\
X(-\epsilon_i - \epsilon_j) &= E_{4+j,i} - E_{4+i,j} & (i < j) \\
X(\epsilon_i \circ \alpha_{kl} + \frac{1}{2}(\epsilon_{4+k} - \epsilon_{4+l})) &= (f_{4+i})_{kl}^\sim & (1 \leq i \leq 4, 1 \leq k, l \leq 3, k \neq l).
\end{align*}

\section*{1.8. Type $E_7$.}

In this paragraph, we use the notations of [H]. Let

\[ X = \{(x, y) \mid x, y \text{ are alternating } 8 \times 8 \text{ matrices}\}. \]

Define linear endomorphisms of $X$ by

\begin{align*}
(1.8.1) & \quad (x, y) \rightarrow (px + x^t p, -y^t p - yp), \\
(1.8.2) & \quad ((x_{ij}), (y_{ij})) \rightarrow ((\sum_{m,n=1}^{8} \vartheta^{i j m n} y_{m n}), (-\sum_{m,n=1}^{8} \vartheta_{i j m n} x_{m n})),
\end{align*}

where $p$ is an $8 \times 8$ matrix with trace 0, and

where $\vartheta$ denotes a tensor, antisymmetric in its indices, and upper, lower indices satisfy the relation

\[ \vartheta_{i_1, \ldots, i_4} = \frac{1}{4!} \sum_{j_1, \ldots, j_4} I_{i_1, \ldots, i_4}^{j_1, \ldots, j_4} \vartheta_{j_1, \ldots, j_4}. \]
Here \( I_{k_{1},\ldots,k_{8}}^{1,\ldots,8} \) denotes the signature of the permutation \( \{1,\ldots,8\} \) if \( \{k_{1},\ldots,k_{8}\} = \{1,\ldots,8\} \), and 0 otherwise. Then, we may assume that \( \mathfrak{g} = \mathfrak{e}_{7} \) is the linear span of these linear endomorphisms, whose Lie algebra structure is given by

\[
[p, p'] = pp' - p'p, \quad \text{where } pp' \text{ denotes the matrix multiplication,}
\]

\[
[p, \vartheta] = \vartheta', \quad \text{where } (\vartheta')_{ijkl} = \sum_{m} (\vartheta^{mjk}p_{im} + \vartheta^{imkl}p_{jm} + \vartheta^{ijml}p_{km} + \vartheta^{ijkm}p_{lm}),
\]

\[
[\vartheta, \vartheta'] = p, \quad \text{where } p_{ij} = \frac{2}{3} \sum_{l,m,n} (\vartheta^{lmni}(\vartheta')_{lnmj} - \frac{1}{8} (\sum_{r} \vartheta^{lmnr}(\vartheta')_{lmnr})\delta_{ij}).
\]

Hereafter, we identify \( p \in \text{Lie}(SL_{8}(\mathbb{C})) \) with the element of \( \mathfrak{g} \) defined by (1.8.1). We may assume that

\[
\mathfrak{h} = \{ \text{diag}(t_{1},\ldots,t_{8}) | \sum_{i=1}^{8} t_{i} = 0 \}.
\]

Let

\[
\epsilon_{i}(\text{diag}(t_{1},\ldots,t_{8})) = t_{i}.
\]

Then

\[
R = \{ \epsilon_{i} - \epsilon_{j}, \quad 1 \leq i,j \leq 8, \quad i \neq j \},
\]

\[
\epsilon_{i} + \epsilon_{j} + \epsilon_{k} + \epsilon_{l}, \quad 1 \leq i < j < k < l \leq 8 \}.
\]

The coroots are given by

\[
H(\epsilon_{i} - \epsilon_{j}) = E_{ij},
\]

\[
H(\epsilon_{i} + \epsilon_{j} + \epsilon_{k} + \epsilon_{l}) = (E_{i} + E_{j} + E_{k} + E_{l}) - \frac{1}{2} \sum_{i=1}^{8} E_{m}.
\]

Let \( \vartheta(ijkl) \) be the tensor, with \( (ijkl) \)-coefficient = 1, all others zero (but to preserve the anti-symmetry of \( \vartheta \)), e.g., \( \vartheta(ijkl)_{ijkl} = 1 \). A Chevalley system is given by

\[
X(\epsilon_{i} - \epsilon_{j}) = E_{ij}, \quad (i \neq j)
\]

\[
X(\epsilon_{i} + \epsilon_{j} + \epsilon_{k} + \epsilon_{l}) = \frac{1}{2} \vartheta(ijkl), \quad (i \leq i < j < k < l \leq 8).
\]
In fact

\[ [X(\epsilon_i - \epsilon_j), X(\epsilon_j - \epsilon_k)] = X(\epsilon_i - \epsilon_k) \]
\[ [X(\epsilon_i - \epsilon_j), X(\epsilon_j + \epsilon_k + \epsilon_l + \epsilon_m)] = X(\epsilon_i + \epsilon_k + \epsilon_l + \epsilon_m) \]
\[ [X(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l), X(\epsilon_i + \epsilon_m + \epsilon_n + \epsilon_r)] = X(\epsilon_i - \epsilon_s), \]

where different letters indicates different numbers. (Note that if 7 indices \((ijklmnr)\) are given, then the remaining index, say \(s\), is uniquely determined.) The other commutators are all zero. By these commutation relations, we can show that there exists a unique involutory automorphism \(\iota\) of \(\mathfrak{C}_7\) such that

\[ \iota(p) = -^tp \quad (p \in \mathfrak{sl}_8(\mathbb{C})), \]

and

\[ \iota(\vartheta(ijkl)) = \vartheta(mnrs), \]

where \((mnrs)\) is chosen so that \(I_1^{12345678} = 1\). Let

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \]
\[ \alpha_8 = \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8. \]

Then we may take as a root basis

\[ \{\alpha_i \mid i \neq 1\} \quad \text{or} \quad \{\alpha_i \mid i \neq 7\}. \]

In fact, the extended Dynkin diagram is given by

\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \]
\[ \alpha_8 \]

The involutory automorphism \(\iota\) induces the unique non-trivial automorphism of the extended Dynkin diagram.
1.9. Type $E_8$.

In this paragraph, we use the notations of [VE]. Let us consider three kinds of tensors

\[
X = (x^i_j)_{1 \leq i,j \leq 9} \quad \text{with} \quad \sum_{i=1}^{9} x^i_i = 0, \\
X_* = (x_{ijk})_{1 \leq i,j,k \leq 9}, \\
X^* = (x^{ijk})_{1 \leq i,j,k \leq 9}.
\]

Here all the tensors are assumed to be antisymmetric in the covariant indices and in the contravariant indices. We may assume that $\mathfrak{g}$ is the vector space $\{X\} \oplus \{X_*\} \oplus \{X^*\}$, which is equipped with a Lie algebra structure by

\[
[X, Y] = Z, \quad z^i_j = x^i_j y^i_j - y^i_j x^i_j, \\
[X, Y_*] = Z_*, \quad z_{ijkl} = \frac{1}{2} I_{ijkl} x^i_j y^k_l - x^i_j y^k_l I_{ijkl}, \\
[X, Y^*] = Z^*, \quad z^{ijk} = -\frac{1}{2} I^{ijk} x^i_j y^k_l, \\
[X^*, Y] = Z, \quad z^i_j = \frac{1}{2} (x^{i..j} - \frac{1}{9} x^{i..} y^{j..} I^i_j), \\
[X^*, Y_*] = Z^*, \quad z^{ijk} = \frac{1}{36} I^{ijk} x^{i..} y^{j..}.
\]

Here we used the notations of the first two sections of [VE]. We may assume that $\mathfrak{h}$ is the set of the diagonal $X$'s. Let

\[
\epsilon_i(\sum_{j=1}^{9} t_i E_i) = t_i.
\]

The root system is given by

\[
R = \{\epsilon_i - \epsilon_j \ (1 \leq i,j \leq 9, \ i \neq j), \ \pm(\epsilon_i + \epsilon_j + \epsilon_k) \ (1 \leq i < j < k \leq 9)\}.
\]
The coroot are given by

\[ H(\epsilon_i - \epsilon_j) = E_i - E_j, \]
\[ H(\pm(\epsilon_i + \epsilon_j + \epsilon_k)) = \pm\{(E_i + E_j + E_k) - \frac{1}{3} \sum_{m=1}^{9} E_m\}. \]

Let \( X_*(ijk) \) (resp. \( X^*(ijk) \)) be the tensor of type \( X_* \) (resp. \( X^* \)), with \((ijk)\)-coefficient = 1, all others zero (but to preserve the anti-symmetry of \( X_* \) (resp. \( X^* \))). A Chevalley system is given by

\[ X(\epsilon_i - \epsilon_j) = E_{ij} \]
\[ X(\epsilon_i + \epsilon_j + \epsilon_k) = X_*(ijk), \]
\[ X(-\epsilon_i - \epsilon_j - \epsilon_k) = X^*(ijk). \]

Let

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq 8), \]
\[ \alpha_9 = -\epsilon_1 - \epsilon_2 - \epsilon_3. \]

Then we may take as a root basis

\[ \{\alpha_i \mid i \neq 8\}. \]

In fact, the extended Dynkin diagram is given by

\[ \begin{array}{cccccccc}
\alpha_1 & - & \alpha_2 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 & - & \alpha_8 \\
\alpha_9 \\
\end{array} \]
§2. Split \( Z \)-forms.

The purpose of this section is to classify and describe the split \( Z \)-forms of saturated, irreducible, prehomogeneous vector spaces \((G, \rho, V)\) over \( \mathbb{C} \). Here we use the definitions and the results of [G].

According to [G], first, we should choose highest weight vectors \( v_0 \) and \( v_0^\vee \) of \( V \) and \( V^\vee \) so that

\[
V_{\text{max}}(Z) \cap \mathbb{C}v_0 = V_{\text{min}}(Z) \cap \mathbb{C}v_0 = \mathbb{Z}v_0,
\]

where, by definition, \( V_{\text{min}}(Z) = \mathcal{U}_Z \cdot v_0 \) and \( V_{\text{max}}(Z) \) is the dual lattice of \( \mathcal{U}_Z \cdot v_0^\vee \) [G]. We shall describe \( V_{\text{min}}(Z) \) and \( V_{\text{max}}(Z) \) explicitly for each case. Our next task is to classify the graded \( \mathcal{U}_Z \)-modules \( V(Z) \) which are \( Z \)-lattices of \( V \) and

\[
V_{\text{min}}(Z) \subset V(Z) \subset V_{\text{max}}(Z).
\]

Fortunately, it will turn out that our second task is almost nothing. In fact, our calculation will show that such a \( V(Z) \) coincides with \( V_{\text{min}}(Z) \) or \( V_{\text{max}}(Z) \).

In course of our calculation, we need to fix a Chevalley system, a basis of a root system etc. In such a case, we always use those given in the first section. If a non-degenerate bilinear form \( \langle , \rangle \) is defined on \( V \), we identify the vector space \( V^\vee \) with the vector space \( V \) via the isomorphism \( I : V^\vee \xrightarrow{\sim} V \) defined by \( \langle v^\vee, v \rangle = \langle I(v^\vee), v \rangle \), where the left hand side is the natural pairing. (Note that \( I \) does not preserve the \( Z \)-structure.) For the sake of a convenience for later calculations, we will give a non-degenerate bilinear form such that \( \rho(G) = \rho^\vee(G) \), if we identify \( V^\vee \) with \( V \).

In (2.1)-(2.15), we shall treat reduced prehomogeneous vector spaces.

2.1. Type (1).
The representation space $V$ can be identified with the totality of $m \times m$ matrices $M_m(\mathbb{C})$. We may assume that $G = GL_m \times GL_m$. The action of $G$ is given by

$$\rho(g)X = g_1 X^t g_2 \quad (X \in M_m(\mathbb{C}), g = (g_1, g_2) \in G).$$

Then a highest weight vector is given by $v_0 = E_{11}$. By applying $U_Z$ to $v_0$, we have

$$V_{\text{min}}(Z) = M_m(Z).$$

We identify the dual space $V^\vee$ of $V$ with $V$ by $(X, Y) = \text{tr}(X^t Y)$ for $X, Y \in M_m(\mathbb{C})$. Then the action of $G$ on $V^\vee$ is given by

$$\rho^\vee(g)Y = g_1^{-1} Y g_2^{-1} \quad (Y \in M_m(\mathbb{C}), g = (g_1, g_2) \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via the above identification $V = V^\vee$. A highest weight vector of $V^\vee$ is given by

$$v_0^\vee = E_{mm}.$$

Hence $V_{\text{min}}^\vee(Z) = M_m(Z)$, and

$$V_{\text{max}}(Z) = M_m(Z).$$

Hence there is only one split $Z$-form. A $Z$-basis of $V_{\text{min}}(Z) = V_{\text{max}}(Z)$ is given by

$$E_{ij} \quad (1 \leq i, j \leq m)$$

and its dual is

$$E_{ij}^\vee = E_{ij} \quad (1 \leq i, j \leq m).$$
2.2. Type (2).

The representation space can be identified with the totality of $n \times n$ symmetric matrices $V = \{X \in M_n(\mathbb{C}) \mid ^tX = X\}$. We may assume that $G = GL_n$. The action is given by

$$\rho(g)X = gX^tg \quad (X \in V, g \in G).$$

A highest weight vector is given by $v_0 = E_{11}$. By applying $U_\mathbb{Z}$ to $v_0$, we have

$$V_{\min}(\mathbb{Z}) = \{X \in M_n(\mathbb{Z}) \mid ^tX = X\}.$$

We identify the dual space $V^\vee$ of $V$ with $V$ by $\langle X, Y \rangle = \text{tr} XY$. The action of $G$ on $V^\vee$ is given by

$$\rho^\vee(g)Y = t_g^{-1}g^{-1}Yg^{-1} \quad (Y \in V^\vee, g \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via the above identification $V = V^\vee$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = E_{nn}$. Since $V_{\min}^\vee = \{Y \in M_n(\mathbb{Z}) \mid ^tY = Y\}$,

$$V_{\max}(\mathbb{Z}) = \sum_{i=1}^{n} \mathbb{Z}E_i + \sum_{i<j} \frac{1}{2}(E_{ij} + E_{ji}).$$

We can show that $V_{\max}(\mathbb{Z})/V_{\min}(\mathbb{Z})$ is a simple graded $U_\mathbb{Z}$-module. (It is enough to consider the action of the Weyl group.) Hence there are exactly two split $\mathbb{Z}$-forms. A $\mathbb{Z}$-basis of $V_{\min}(\mathbb{Z})$ is given by

$$E_i \quad (1 \leq i \leq n), \quad E_{ij} + E_{ji} \quad (1 \leq i < j \leq n).$$

Its dual basis is given by

$$E_i^\vee = E_i \quad (1 \leq i \leq), \quad (E_{ij} + E_{ji})^\vee = \frac{1}{2}(E_{ij} + E_{ji}) \quad (1 \leq i < j \leq n),$$

which is a basis of $V_{\max}(\mathbb{Z})$. 
2.3. Type (3).

The representation space can be identified with the totality of $2m \times 2m$ skew-symmetric matrices $V = \{X \in M_{2m}(\mathbb{C}) \mid ^tX + X = 0\}$. We may assume that $G = GL_{2m}$. The action of $G$ is given by

$$\rho(g)X = gX^tg \quad (X \in V, g \in G).$$

A highest weight vector is given by $v_0 = E_{12} - E_{21}$. By applying $\mathcal{U}_Z$ to $v_0$, we have

$$V_{\min}(Z) = \{X \in M_{2m}(Z) \mid ^tX + X = 0\}.$$

We identify the dual space $V^\vee$ of $V$ with $V$ by $\langle X, Y \rangle = -\frac{1}{2} \text{tr} XY$. The action of $G$ on $V^\vee$ is given by

$$\rho^\vee(g)Y = ^t g^{-1} Y g^{-1} \quad (Y \in V^\vee, g \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via our identification. A highest weight vector of $v^\vee$ is given by $v_0^\vee = E_{2m-1,2m} - E_{2m,2m-1}$. Since $V_{\min}^\vee(Z) = \{Y \in M_{2m}(Z) \mid Y + ^tY = 0\}$,

$$V_{\max}(Z) = \{X \in M_{2m}(Z) \mid X + ^tX = 0\}.$$

Hence there is only one split $Z$-form. A $Z$-basis of $V_{\min}(Z) = V_{\max}(Z)$ is given by

$$E_{ij} - E_{ji} \quad (1 \leq i < j \leq 2m).$$

Its dual basis is

$$(E_{ij} - E_{ji})^\vee = E_{ij} - E_{ji} \quad (1 \leq i < j \leq 2m).$$
2.4. Type (4).

The representation space can be identified with the third symmetric product $S^3(\mathbb{C}^2)$ of a two dimensional vector space. We may assume that $G = GL_2 = GL(\mathbb{C}^2)$. Then $G$ acts naturally on $S^3(\mathbb{C}^2)$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. A highest weight vector is given by $v_0 = e_1^3$. By applying $\mathcal{U}$ to $v_0$, we have

$$V_{\min}(Z) = Z \cdot e_1^3 + Z \cdot 3e_1^2e_2 + Z \cdot 3e_1e_2^2 + Z \cdot e_2^3.$$  

We identify the dual space $V^\vee$ of $V$ with $V$ itself by

$$\langle e_1^a e_2^3 - a, e_1^b e_2^3 - b \rangle = \begin{cases} \binom{3}{a}^{-1} & (a = b) \\ 0 & (a \neq b). \end{cases}$$

If we denote the actions of $G$ on $V$ and $V^\vee$ by $\rho$ and $\rho^\vee$, respectively, then $\rho^\vee(g) = \rho(g^{-1})$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_2^3$. We have

$$V_{\min}^\vee(Z) = Z \cdot e_1^3 + Z \cdot 3e_1^2e_2 + Z \cdot 3e_1e_2^2 + Z \cdot e_2^3$$

and

$$V_{\max}(Z) = Z \cdot e_1^3 + Z \cdot e_1^2e_2 + Z \cdot e_1e_2^2 + Z \cdot e_2^3.$$  

We can show that $V_{\max}(Z)/V_{\min}(Z)$ is a simple graded $\mathcal{U}Z$-module. Hence there are exactly two split $\mathbb{Z}$-forms. A $\mathbb{Z}$-basis of $V_{\min}(Z)$ is given by

$$e_1^3, 3e_1^2e_2, 3e_1e_2^2, e_2^3.$$  

Its dual basis is given by

$$(e_1^3)^\vee = e_1^3, (3e_1^2e_2)^\vee = e_1^2e_2, (3e_1e_2^2)^\vee = e_1e_2^2, (e_2^3)^\vee = e_2^3,$$

which is a basis of $V_{\max}(Z)$. 
2.5. Types (5),(6),(7),(9),(10) and (11).

Let \((l, m, n) = (3, 6, 1), (3, 7, 1), (3, 8, 1), (2, 6, 2), (2, 5, 3) \) or \((2, 5, 4)\) for the prehomogeneous vector space of type \((5),(6),(7),(9),(10)\) or \((11)\), respectively. Then the representation space can be identified with \(V = \wedge^l(\mathbb{C}^m) \otimes \mathbb{C}^n\), where \(\wedge^l(\mathbb{C}^m)\) is the \(l\)-th Grassmann product of \(\mathbb{C}^m\). We may assume that \(G = GL(\mathbb{C}^m) \times GL(\mathbb{C}^n)\), which acts naturally on \(V\). Let \(\{e_i \mid 1 \leq i \leq m\}\) and \(\{f_j \mid 1 \leq j \leq n\}\) be the standard bases of \(\mathbb{C}^m\) and \(\mathbb{C}^n\), respectively. A highest weight vector is given by \(v_0 = (e_1 \wedge e_2 \wedge \cdots \wedge e_l) \otimes f_1\). By applying \(U_Z\) to \(v_0\), we have

\[
V_{\text{max}}(Z) = \sum_{1 \leq i_1 < \cdots < i_l \leq m, \ 1 \leq j \leq n} Z \cdot (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j.
\]

We identify the dual space \(V^\vee\) of \(V\) with \(V\) by

\[
\langle (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j, (e_{i_1'} \wedge \cdots \wedge e_{i_{\ell}'}) \otimes f_{j'} \rangle = \delta_{i_1i_1'} \cdots \delta_{i_{\ell}i_{\ell}'} \delta_{jj'},
\]

where \(i_1 < \cdots < i_l, \ i_1' < \cdots < i_{\ell}'\) and \(\delta\) is the Kronecker's delta. Denote the action of \(G\) on \(V\) and \(V^\vee\) by \(\rho\) and \(\rho^\vee\), respectively. Then \(\rho^\vee(g_1, g_2) = \rho(g_1^{-1}, g_2^{-1})\) for \((g_1, g_2) \in G\). In particular, \(\rho^\vee(G) = \rho(G)\). A highest weight vector of \(V^\vee\) is given by \(v_0^\vee = (e_{m-l+1} \wedge \cdots \wedge e_m) \otimes f_n\). Then we have \(V_{\text{min}}^\vee(Z) = V_{\text{max}}(Z)\) and \(V_{\text{max}}(Z) = V_{\text{min}}(Z)\). Hence there is exactly one split \(Z\)-form. A \(Z\)-basis of \(V_{\text{min}}(Z) = V_{\text{max}}(Z)\) is given by

\[
(e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j \quad (1 \leq i_1 < \cdots < i_l \leq m, 1 \leq j \leq n).
\]

Its dual basis if given by

\[
((e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j)^\vee = (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j.
\]
2.6. Type (8).

The representation space can be identified with \( V = S^2(\mathbb{C}^3) \otimes \mathbb{C}^2 \). We may assume that \( G = \text{GL}(\mathbb{C}^3) \times \text{GL}(\mathbb{C}^2) \), which acts naturally on \( V \). Let \( e_1 = \begin{pmatrix} 1, 0, 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0, 1, 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0, 0, 1 \end{pmatrix} \), \( F_1 = \begin{pmatrix} 1, 0 \end{pmatrix} \) and \( f_2 = \begin{pmatrix} 0, 1 \end{pmatrix} \). A highest weight vector of \( V \) is given by \( v_0 = e_1^2 \otimes f_1 \). By applying \( \mathcal{U}_\mathbb{Z} \) to \( v_0 \), we have

\[
V_{\text{min}}(\mathbb{Z}) = \left( \sum_{1 \leq i \leq 3} \mathbb{Z} \cdot e_i^2 + \sum_{1 \leq i < j \leq 3} \mathbb{Z} \cdot 2e_ie_j \right) \otimes (zf_1 + zf_2).
\]

We identify the dual space \( V^\vee \) of \( V \) with \( V \) by

\[
\langle e_1^{a_1}e_2^{a_2}e_3^{a_3} \otimes f_a, e_1^{b_1}e_2^{b_2}e_3^{b_3} \otimes f_b \rangle = \begin{cases} \frac{a_1!a_2!a_3!}{2!}, & \text{if } (a_1, a_2, a_3, a) = (b_1, b_2, b_3, b) \\ 0, & \text{otherwise.} \end{cases}
\]

Denote the actions of \( G \) on \( V \) and \( V^\vee \) by \( \rho \) and \( \rho^\vee \), respectively. Then \( \rho^\vee(g_1, g_2) = \rho(g_1^{-1}, g_2^{-1}) \) \((g_1, g_2) \in G\). In particular, \( \rho^\vee(G) = \rho(G) \). A highest weight vector of \( V^\vee \) is given by \( v_0^\vee = e_3^2 \otimes f_2 \). We have \( V_{\text{min}}^\vee(\mathbb{Z}) = V_{\text{min}}(\mathbb{Z}) \) and

\[
V_{\text{max}}(\mathbb{Z}) = \sum_{1 \leq i \leq j \leq 3, 1 \leq k \leq 2} \mathbb{Z} \cdot e_ie_j \otimes f_k.
\]

We can show that \( V_{\text{max}}(\mathbb{Z})/V_{\text{min}}(\mathbb{Z}) \) is a simple graded \( \mathcal{U}_\mathbb{Z} \)-module. Hence there are exactly two split \( \mathbb{Z} \)-forms. A \( \mathbb{Z} \)-basis of \( V_{\text{min}}(\mathbb{Z}) \) is given by

\[
e_i^2 \otimes f_k \quad (1 \leq i \leq 3, 1 \leq k \leq 2),
2e_ie_j \otimes f_k \quad (1 \leq i < j \leq 3, 1 \leq k \leq 2).
\]

Its dual basis is given by

\[
(e_i^2 \otimes f_k)^\vee = e_i^2 \otimes f_k \quad (1 \leq i \leq 3, 1 \leq k \leq 2),
(2e_ie_j \otimes f_k)^\vee = e_ie_j \otimes f_k \quad (1 \leq i < j \leq 3, 1 \leq k \leq 2),
\]

which is a basis of \( V_{\text{max}}(\mathbb{Z}) \).
2.7. Type (12).

The representation space can be identified with \( V = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \). We may assume that \( G = GL(\mathbb{C}^3) \times GL(\mathbb{C}^3) \times GL(\mathbb{C}^2) \). Let \( \{ e_i \mid 1 \leq i \leq 3 \} \) and \( \{ f_j \mid 1 \leq j \leq 2 \} \) be the standard bases of \( \mathbb{C}^3 \) and \( \mathbb{C}^2 \), respectively. A highest weight vector is given by \( v_0 = e_1 \otimes e_1 \otimes f_1 \). We have

\[
V_{\text{min}}(\mathbf{Z}) = \sum_{1 \leq i,j \leq 3, 1 \leq k \leq 2} \mathbf{Z} \cdot e_i \otimes e_j \otimes f_k.
\]

We identify \( V^\vee \) with \( V \) by

\[
(e_i \otimes e_j \otimes f_k, e_{i'} \otimes e_{j'} \otimes f_{k'}) = \delta_{ii'} \delta_{jj'} \delta_{kk'}.
\]

The action \( \rho^\vee \) of \( G \) on \( V^\vee \) is given by \( \rho^\vee (g_1, g_2, g_3) = \rho(t_{g_1}^{-1} t_{g_2}^{-1} t_3)g_3 \) for \( (g_1, g_2, g_2) \in G \). In particular, \( \rho^\vee (G) = \rho(G) \). A highest weight vector of \( V^\vee \) is given by \( v_0^\vee = e_1 \otimes e_1 \otimes f_2 \). Then we have \( V^\vee_{\text{min}}(\mathbf{Z}) = V_{\text{min}}(\mathbf{Z}) \) and \( V^\vee_{\text{max}}(\mathbf{C}) = V_{\text{min}}(\mathbf{Z}) \). Hence there is exactly one split \( \mathbf{Z} \)-form. A \( \mathbf{Z} \)-basis of \( V_{\text{min}}(\mathbf{Z}) = V_{\text{max}}(\mathbf{Z}) \) is given by

\[
e_i \otimes e_j \otimes f_k \quad (1 \leq i, j \leq 3, 1 \leq k \leq 2).
\]

Its dual basis is given by

\[
(e_i \otimes e_j \otimes f_k)^\vee = e_i \otimes e_j \otimes f_k.
\]

2.8. Type (13).

The representation space can be identified with \( V = \mathbb{C}^{2n} \otimes \mathbb{C}^{2m} \). We may assume that \( G = Sp_{2n}(\mathbb{C}) \times GL_{2m}(\mathbb{C}) \). Here we realize the symplectic group \( Sp_{2n}(\mathbb{C}) \) as in (1.3), i.e.,

\[
Sp_{2n}(\mathbb{C}) = \{ g \in GL_{2n}(\mathbb{C}) \mid g J t g = J \}.
\]
Then $G$ acts naturally on $V$. Let $\{e_i \mid 1 \leq i \leq 2n\}$ and $\{f_j \mid 1 \leq j \leq 2m\}$ be the standard bases of $\mathbb{C}^{2n}$ and $\mathbb{C}^{2m}$, respectively. A highest weight vector of $V$ is given by $v_0 = e_1 \otimes f_1$. We have

$$V_{\min}(Z) = \sum_{1 \leq i \leq 2n} \sum_{1 \leq j \leq 2m} Z \cdot e_i \otimes f_j.$$ 

We identify $V^\vee$ with $V$ by the skew-symmetric bilinear form defined by

$$(e_i \otimes f_j, e_{n+k} \otimes f_l) = \delta_{ik} \delta_{jl},$$

$$(e_i \otimes f_j, e_k \otimes f_l) = (e_{n+i} \otimes f_j, e_{n+k} \otimes f_l) = 0,$$

for $1 \leq i, k \leq n$ and $1 \leq j, l \leq 2m$. The action $\rho^\vee$ of $G$ on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(g_1, t_{g_2}^{-1})$ for $(g_1, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0 = e_1 \otimes f_{2m}$. Then we have $V_{\min}^\vee(Z) = V_{\min}(Z)$ and $V_{\max}(Z) = V_{\min}(Z)$. Hence there is exactly one split $Z$-form. A $Z$-basis of $V_{\min}(Z) = V_{\max}(Z)$ is given by

$$e_i \otimes f_j \quad (1 \leq i \leq 2n, 1 \leq j \leq 2m).$$

Its dual basis is given by

$$(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j,$$

where

$$i' = \begin{cases} i + n & (1 \leq i \leq n) \\ i - n & (n + 1 \leq i \leq 2n). \end{cases}$$

2.9. Type (14).

Let $\{e_i\}_{1 \leq i \leq 6}$ be the standard basis of $\mathbb{C}^6$. The representation space can be identified with

$$V = \{ \sum_{1 \leq i < j < k \leq 6} x_{ijk} e_i \wedge e_j \wedge e_k \mid x_{i14} + x_{i25} + x_{i36} = 0 \quad (1 \leq i \leq 6) \}.$$
where we regard \((x_{ijk})\) as an alternating tensor. We may assume that \(G = \C^\times \times \text{Sp}_6(\C)\), where \(\text{Sp}_6(\C)\) is realized as in (1.3). Then \(G\) acts naturally on \(V\). A highest weight vector is given by \(v_0 = e_1 \wedge e_2 \wedge e_3\). Let \(1' = 4\), \(2' = 5\), \(3' = 6\), \(4' = 1\), \(5' = 2\), \(6' = 3\), and \(ijk = e_i \wedge e_j \wedge e_k\). Then a \(\Z\)-basis of \(V_{\text{min}}(\Z)\) is given by

\[
123, \ 1'23, \ 12'3, \ 123', \ 1'23', \ 1'2'3', \ \quad (2.9.1)
\]

\[122' - 133', \ 211' - 233', \ 311' - 322', \]
\[1'22' - 1'33', \ 2'11' - 2'33', \ 3'11' - 3'22'. \]

Let us define a skew-symmetric bilinear form on \(\Lambda^3(\C^6)\) by

\[
\langle ijk, lmn \rangle = \text{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix},
\]

where \(\text{sgn}\) is the signature on the symmetric group \(S_6\) which is extended by

\[
\text{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix} = 0, \quad \text{if } \{ijklmn\} \neq \{123456\}.
\]

Note that \(X(r)\) acts on \(\Lambda^3(\C^6)\) as

(1) \(i \rightarrow j\), \(j' \rightarrow -i'\), \(k \rightarrow 0\) \(\quad (k \neq i, j')\)

(2) \(i' \rightarrow j\), \(j' \rightarrow i\), \(k \rightarrow 0\) \(\quad (k \neq i', j')\)

or

(3) \(i \rightarrow j', \quad j \rightarrow i', \quad k \rightarrow 0\) \(\quad (k \neq i, j),\)

where \(i, j \in \{1, 2, 3\}, 1 \leq k \leq 6, -i = -e_i\) and \(-i' = -e_{i'}\). Note also that

\[
\langle ijk, i'j'k' \rangle = 1 \quad \text{and} \quad \langle ijk', i'j'k \rangle = -1
\]
for \( i, j, k \in \{1, 2, 3\} \). By using these facts, we can show that our bilinear form is \( Sp_6(\mathbb{C}) \)-invariant. We identify \( V \) and \( V^\vee \) by this bilinear form. Hence the action \( \rho^\vee \) of \( G \) on \( V^\vee \) is given by \( \rho^\vee(g_1, g_2) = \rho(g_1^{-1}, g_2) \) for \((g_1, g_2) \in G = \mathbb{C}^\times \times Sp_6(\mathbb{C}) \). In particular, \( \rho^\vee(G) = \rho(G) \). A highest weight vector of \( V^\vee \) is given by \( v_0^\vee = 123 \). Then we have \( V_{\min}(\mathbb{Z}) = V_{\min}(\mathbb{Z}) \). The dual basis of (2.9.1) is

\[
\begin{align*}
(123)^\vee &= 1'2'3', \quad (1'23)^\vee = -12'3', \quad (12'3)^\vee = -1'23', \quad (123')^\vee = -1'2'3, \\
(12'3')^\vee &= -1'23, \quad (1'23')^\vee = -12'3, \quad (1'i2'3)^\vee = -123', \quad (1'i2'3')^\vee = 123 \\
(122' - 133')^\vee &= \frac{1}{2}(1'2'2 - 1'3'3) \quad \text{etc.} \\
(1'22' - 1'33')^\vee &= \frac{1}{2}(12'2 - 13'3) \quad \text{etc.}
\end{align*}
\]

Hence \( V_{\max}(\mathbb{Z}) \) is the free \( \mathbb{Z} \)-module generated by (2.9.2). We can show that \( V_{\max}(\mathbb{Z})/V_{\min}(\mathbb{Z}) \) is a simple graded \( \mathcal{U}_{\mathbb{Z}} \)-module. Hence, there are exactly two split \( \mathbb{Z} \)-forms.

2.10. Type (15B).

The representation space can be identified with \( V = \mathbb{C}^{2k+1} \otimes \mathbb{C}^m \). We may assume that \( G = SO_{2k+1}(\mathbb{C}) \times GL_m(\mathbb{C}) \). Here we realize the special orthogonal group \( SO_{2k+1}(\mathbb{C}) \) as in (1.2), i.e.,

\[
SO_{2k+1}(\mathbb{C}) = \{ g \in GL_{2k+1}(\mathbb{C}) \mid g J^t g = J \}.
\]

Then \( G \) acts naturally on \( V \). Let \( \{ e_i \mid 1 \leq i \leq 2k + 1 \} \) and \( \{ f_j \mid 1 \leq j \leq m \} \) be the standard bases of \( \mathbb{C}^{2k+1} \) and \( \mathbb{C}^m \), respectively. A highest weight vector of \( V \) is given by \( v_0 = e_1 \otimes f_1 \). We have

\[
V_{\min}(\mathbb{Z}) = \sum_{1 \leq i \leq 2k} \mathbb{Z} \cdot e_i \otimes f_j + \sum_{1 \leq j \leq m} \mathbb{Z} \cdot 2e_{2k+1} \otimes f_j.
\]
Let us identify $V$ with $M_{2k+1,m}(C)$ by

$$\sum_{p,q} a_{pq} e_p \otimes f_q \rightarrow (a_{pq}).$$

The induced $G$-action on $M_{2k+1,m}(C)$ is given by

$$v \rightarrow g_1 v^t g_2 \quad (g_1, g_2) \in G = SO_{2k+1} \times GL_m.$$

We identify $V^\vee$ with $V$ by the symmetric bilinear form defined by

$$\langle v_1, v_2 \rangle = \mathrm{tr}(v_1 J^{-1} v_2).$$

Then

$$\langle e_p \otimes f_q, e_{r} \otimes f_s \rangle = \begin{cases} 1 & (p = r' \neq 2k+1, q = s) \\ \frac{1}{2} & (p = r' = 2k+1, q = s) \\ 0 & \text{otherwise}, \end{cases}$$

where

$$i' = \begin{cases} i + k & (1 \leq i \leq k) \\ i - k & (k+1 \leq i \leq 2k) \\ 2k+1 & (i = 2k+1). \end{cases}$$

The action $\rho^\vee$ of $G$ on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(g_1, g_2^{-1})$ for $(g_1, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_1 \otimes f_m$.

Then we have $V_{\min}^\vee(Z) = V_{\min}(Z)$. A $Z$-basis of $V_{\min}(Z)$ is given by

$$e_i \otimes f_j \quad (1 \leq i \leq 2k, 1 \leq j \leq m),$$

$$2e_{2k+1} \otimes f_j \quad (1 \leq j \leq m).$$
Its dual basis is given by

\[
(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j \quad (1 \leq i \leq 2k, 1 \leq j \leq m),
\]

\[
(2e_{2k+1} \otimes f_j)^\vee = e_{2k+1} \otimes f_j \quad (1 \leq j \leq m).
\]

Hence

\[
V_{\max}(\mathbb{Z}) = \sum_{1 \leq i \leq 2k+1} \sum_{1 \leq j \leq m} \mathbb{Z} \cdot e_i \otimes f_j.
\]

We can show that \( V_{\max}(\mathbb{Z})/V_{\min}(\mathbb{Z}) \) is a simple graded \( \mathcal{U}_\mathbb{Z} \)-module. Hence, there are exactly two split \( \mathbb{Z} \)-forms.

2.11. Type (15D).

With a trivial modification of (2.10), we have

\[
(e_p \otimes f_q, e_r \otimes f_s) = \begin{cases} 
1 & (p = r', q = s) \\
0, & \text{otherwise}
\end{cases}
\]

\[v_0 = e_1 \otimes f_1,\]

\[v_0^\vee = e_1 \otimes f_m,\]

\[
V_{\min}(\mathbb{Z}) = V_{\max}(\mathbb{Z}) = \sum_{1 \leq i \leq 2k} \sum_{1 \leq j \leq m} \mathbb{Z} \cdot e_i \otimes f_j,
\]

and

\[
(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j.
\]

In particular, there is exactly one split \( \mathbb{Z} \)-form.
2.12. Types (20), (21), (23) and (24).

Let \((m, n) = (2, 5), (3, 5), (1, 6), (1, 7)\), if we are considering a prehomogeneous vector space of type (20), (21), (23), (24), respectively. Then the representation space can be identified with \(\wedge^{\text{even}}(\mathbb{C}^n) \otimes \mathbb{C}^m\). Here and below in this paragraph, we use the notations of (1.4). We may assume that \(G = \text{Spin}_{2n} \times \text{GL}_m\), which acts naturally on \(V\). Let \(\{e_i \mid 1 \leq i \leq n\}\) and \(\{u_j \mid 1 \leq j \leq m\}\) be the standard bases of \(\mathbb{C}^n\) and \(\mathbb{C}^m\), respectively. A highest weight vector is given by \(e_1 e_2 \cdots e_l \otimes u_1\), where \(l = 2\left\lfloor \frac{n}{2} \right\rfloor\). We have

\[
V_{\min}(Z) = \sum_{0 < k \leq l} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq l \text{ even} \atop 1 \leq j \leq m}} Z \cdot e_{i_1} e_{i_2} \cdots e_{i_k} \otimes u_j.
\]

We identify \(V^\vee\) with \(V\) by

\[
\langle e_{a_1} \cdots e_{a_k} \otimes u_i, e_{b_1} \cdots e_{b_l} \otimes u_j \rangle = \begin{cases} 1 & (\{a_1, \ldots, a_k\} = \{b_1, \ldots, b_l\}, i = j) \\ 0 & \text{(otherwise)}, \end{cases}
\]

where

\[
1 \leq a_1 < \cdots < a_k \leq n, \\
1 \leq b_1 < \cdots < b_l \leq n, \\
1 \leq i, j \leq m.
\]

Then the action \(\rho^\vee\) of \(G\) on \(V^\vee\) is given by \(\rho^\vee(g_1, g_2) = \rho(\iota(g_1), g_2^{-1})\) for \((g_1, g_2) \in G = \text{Spin}_{2n} \times \text{GL}_m\). Here \(\iota\) is the involutory automorphism of \(\text{Spin}_{2n}\) given in (1.4). In particular, \(\rho^\vee(G) = \rho(G)\). A highest weight vector is given by \(1 \otimes u_m\). Then we
have $V_{\min}^\vee(Z) = V_{\max}(Z)$ and $V_{\max}(Z) = V_{\min}(Z)$. Hence there is exactly one split $Z$-form. A $Z$-basis of $V_{\min}(Z)$ is given by

$$e_{i_k} \cdots e_{i_1} \otimes u_j \quad (1 \leq i_1 < \cdots < i_k \leq n, \ k : \text{even}, \ 1 \leq j \leq m),$$

and its dual basis is given by

$$(e_{i_1} \cdots e_{i_k} \otimes u_j)^\vee = e_{i_1} \cdots e_{i_k} \otimes u_j.$$

### 2.13. Types (27) and (28).

Let $n = 1, 2$ if we are considering a prehomogeneous vector space of type (27) or (28), respectively. The representation space can be identified with $\mathfrak{J} \otimes \mathbb{C}^n$. Here and below in this section, we use the notations of (1.6) and (1.7). We may assume that $G = G(E_6) \times GL_n$, where

$$G(E_6) = \{\text{linear automorphism of } \mathfrak{J} \text{ which preserves } det(X,Y,Z)\}.$$

See [F,8.1]. Then $G$ acts naturally on $V$. Let $\{u_i\}$ be the standard basis of $\mathbb{C}^n$. A highest weight vector of $V$ is given by $v_0 = E_{11}^{(3)} \otimes u_1$. We have

$$V_{\min}(Z) = \left( \sum_{1 \leq i \leq 3} Z \cdot E_{ii}^{(3)} + \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 8} Z \cdot (\frac{1}{2} f_j)_i \right) \otimes \sum_{1 \leq k \leq n} Z \cdot u_k.$$

See (1.6) for $(a)_i$. We identify $V^\vee$ with $V$ by the symmetric bilinear form defined by

$$(X \otimes u_j, Y \otimes u_k) = \delta_{jk} \cdot \chi(X \circ Y) \quad (X,Y \in \mathfrak{J}),$$
where $\chi$ is the trace function of $\mathfrak{J}$ (see (1.7)), and $X \circ Y = \frac{1}{2}(XY + YX)$. A direct calculation shows

$$\left\langle \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y}_2 \\ \overline{y}_3 & \eta_2 & y_1 \\ y_2 & \overline{y}_1 & \eta_3 \end{pmatrix} \right\rangle = \sum_{i=1}^{3} \{\xi_i \eta_i + 2(x_i, y_i)\},$$

where

$$(x, y) = \frac{1}{2}(x \overline{y} + y \overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x) \quad (x, y \in \mathbb{C}).$$

Let $\rho^\vee$ be the dual of $\rho$. Since $\chi(X \circ Y)$ is $S_4$-invariant [F, 4.5.13],

$$\rho^\vee(g_1, g_2) = \rho(g_1, g_2^{-1})$$

for $(g_1, g_2) \in G(F_4) \times GL_2$. Here $G(F_4)$ is the subgroup of $G(E_6)$ which corresponds to the Lie subalgebra $\mathfrak{f}_4(\subset \mathfrak{e}_6)$ of the infinitesimal automorphisms of the Jordan algebra $\mathfrak{J}$. A direct calculation shows that

$$\chi(a_{ij}^\sim X \circ Y) + \chi(X \circ (-\overline{a})_{ji}^\sim Y) = 0 \quad (i \neq j, a \in \mathfrak{C}, X \in \mathfrak{J}, Y \in \mathfrak{J}).$$

Hence we can define an involutory automorphism $\iota$ of $\mathfrak{C}_6$ by

$$\iota((a)_{ij}^\sim) = (-\overline{a})_{ji}^\sim \quad (i \neq j, a \in \mathfrak{C})$$

and

$$\iota|\mathfrak{f}_4 \equiv \text{identity.}$$

Since $G(E_6)(\supset \{\omega | \omega^3 = 1\})$ is simply connected, $\iota$ induces an automorphism of $G(E_6)$, which we shall denote by the same letter $\iota$. Then we have

$$\chi(gX \circ Y) = \chi(X \circ \iota(g)Y) \quad (g \in G(E_6)).$$
Hence \( \rho^\vee(g_1, g_2) = \rho(\iota(g_1), t_{g_2}^{-1}) \) for \((g_1, g_2) \in G = G(E_6) \times GL_n\). In particular, \( \rho^\vee(G) = \rho(G) \). A highest weight vector of \( V^\vee \) is given by \( v_0^\vee = E_{33}^{(2)} \otimes u_n \). Then we have \( V_{\min}^\vee(Z) = V_{\min}(Z) \). A \( Z \)-basis of \( V_{\min}(Z) \) is given by
\[
E_{ii}^{(2)} \otimes u_k \quad (1 \leq i \leq 3, 1 \leq k \leq n)
\]
\[
\left(\frac{1}{2}f_j\right)_i \otimes u_k \quad (1 \leq i \leq 3, 1 \leq j \leq 8, 1 \leq k \leq n)
\]
Its dual basis is given by
\[
(E_{ii}^{(3)} \otimes u_k)^\vee = E_{ii}^{(3)} \otimes u_k
\]
\[
(\left(\frac{1}{2}f_j\right)_i \otimes u_k)^\vee = -\left(\frac{1}{2}f_{\sigma(j)}\right)_i \otimes u_k,
\]
where
\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4
\end{pmatrix}
\]
Hence \( V_{\max}(Z) = V_{\min}(Z) \). Hence there is exactly one split \( Z \)-form.

2.14. Type (29).

In this paragraph, we use the notations of (1.8). The representation space can be identified with \( X \). We may assume that \( G = G(E_7) \times GL_1 \), where \( G(E_7) \) is the subgroup of \( GL(X) \) which corresponds to the Lie subalgebra \( \mathfrak{e}_7 \) of \( \mathfrak{gl}(X) \). A highest weight vector is given be \( v_0 = (0, E_{18} - E_{81}) \). Here and below, we choose \( \{\alpha_2, \ldots \alpha_8\} \) as a basis of \( R \). We have
\[
V_{\min}(Z) = \sum_{1 \leq i < j \leq 8} Z \cdot (E_{ij} - E_{ji}, 0) + Z \cdot (0, E_{ij} - E_{ji})).
\]
We identify \( V^\vee \) with \( V \) by the symmetric bilinear form defined by
\[
((x_1, x_2), (y_1, y_2)) = -\frac{1}{2}(\text{tr}(x_1y_1) + \text{tr}(x_2y_2)), \quad (x_1, x_2), (y_1, y_2) \in X.
\]
Since \( G(E_7)(\supset \{\pm 1\}) \) is simply connected, the involutory automorphism \( \iota \) defined in (1.8) induces an involutory automorphism of \( G(E_7) \), which we shall denote by the same letter \( \iota \). Then the action \( \rho^\vee \) of \( G \) on \( V^\vee \) is given by \( \rho^\vee(g_1, g_2) = \rho(\iota(g_1), g_2^{-1}) \) for \( (g_1, g_2) \in G = G(E_7) \times GL_1 \). In particular, \( \rho^\vee(G) = \rho(G) \). A highest weight vector of \( V^\vee \) is given by \( v_0^\vee = (E_{18} - E_{81}, 0) \). We have \( V_{\min}^\vee(Z) = V_{\min}(Z) \). A \( Z \)-basis of \( V_{\min}(Z) \) is given by

\[
(E_{ij} - E_{ji}, 0), (0, E_{ij} - E_{ji}) \quad (1 \leq i < j \leq 8)
\]

and its dual basis is given by

\[
(E_{ij} - E_{ji}, 0)^\vee = (E_{ij} - E_{ji}, 0), \text{ and } (0, E_{ij} - E_{ji})^\vee = (0, E_{ij} - E_{ji}).
\]

Hence \( V_{\max}(Z) = V_{\min}(Z) \). Hence there is exactly one split \( Z \)-form.

2.15. Non-regular prehomogeneous vector space with a relative invariant.

There is a unique non-regular irreducible reduced prehomogeneous vector space which has a non-trivial relative invariant, which we refer to as the type (NR) (= non-regular) provisionally in this paper. The representation space can be identified with \( V = \mathbb{C}^{2n} \times S^2(\mathbb{C}^2) \). We may assume that \( G = \mathbb{C}^\times \times Sp_{2n}(\mathbb{C}) \times SL_2(\mathbb{C}) \), where \( Sp_{2n}(\mathbb{C}) \) is realized as in (1.3). (Note that \( SL_2(\mathbb{C})/\{\pm\} = SO_3(\mathbb{C}) \).) The first factor \( \mathbb{C}^\times \) acts on \( V \) as scalar multiplications, \( Sp_{2n}(\mathbb{C}) \) (resp. \( SL_2(\mathbb{C}) \)) acts naturally on \( \mathbb{C}^{2n} \) (resp. \( S^2(\mathbb{C}^2) \)), and hence we get a \( G \)-action \( \rho \) on \( V \). Let \( \{e_i\}_{1 \leq i \leq 2n} \) (resp. \( \{f_1, f_2\} \)) be the standard basis of \( \mathbb{C}^{2n} \) (resp. \( \mathbb{C}^2 \)). A highest weight vector is given by \( v_0 = e_1 \otimes f_1^2 \). A \( Z \)-basis of \( V_{\min}(Z) \) is given by

\[
e_i \otimes f_1^2, \quad e_i \otimes 2f_1f_2, \quad e_i \otimes f_2^2, \quad (1 \leq i \leq 2n).
\]
We identify $V^\vee$ with $V$ by the skew-symmetric bilinear form on $V$ defined by

$$
\langle e_i \otimes f_p q, e_j \otimes f_r s \rangle = \langle e_i, e_j \rangle \langle f_p q, f_r s \rangle,
$$

$$
\langle e_i, e_{n+j} \rangle = -\langle e_i, e_{n+j} \rangle = \delta_{ij},
$$

$$
\langle e_i, e_j \rangle = \langle f_{1}^{2} f_{1}^{2}, f_{1} f_{2} \rangle = \frac{1}{2},
$$

$$
\langle f_{p} f_{q}, f_{r} f_{s} \rangle = 0 \quad \text{for the other cases},
$$

for $1 \leq i, j \leq 2n$ and $1 \leq p, q, r, s \leq 2$. Then the action $\rho^\vee$ of $G$ on $V^\vee$ is given by $\rho^\vee(g_{1}, g_{2}, g_{3}) = \rho(g_{1}^{-1}, g_{2}, g_{3}^{-1}) \in G = \mathbb{C}^n \times Sp_{2n}(\mathbb{C}) \times SL_{2}(\mathbb{C})$. In particular $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v^\vee_0 = e_i \otimes f_2^2$. Then we have $V^\vee_{\min}(Z) = V_{\min}(Z)$. A $Z$-basis of $V_{\max}(Z)$ is given by $e_i \otimes f_1^2$, $e_i \otimes f_1 f_2$, $e_i \otimes f_2^2$, $(1 \leq i \leq 2n)$.

We can show that $V^\vee_{\max}(Z)/V^\vee_{\min}(Z)$ is a simple graded $\mathcal{U}_Z$-module. Hence there are exactly two split $Z$-forms.

**2.16.** Let $(G_i, \rho_i, V_i)$ $(i = 1, 2)$ be two irreducible representations and $(G_i, \rho_i^\vee, V_i^\vee)$ their duals. We assume that a Borel subgroup of each $G_i$ is given. Let $v_i$ and $v_i^\vee$ be highest root vectors of $V_i$ and $V_i^\vee$, respectively. Assume that a non-degenerate bilinear form $\langle, \rangle$ is given for each $V_i$ and that $\rho_i(G_i) = \rho_i^\vee(G_i)$, if we identify $V_i^\vee$ with $V_i$ via this bilinear form.

Let us consider the irreducible representation $(G, \rho, V) = (G_1 \times G_2, \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ and its dual $(G, \rho^\vee, V^\vee) = (G_1 \times G_2, \rho_1^\vee \otimes \rho_2^\vee, V_1^\vee \otimes V_2^\vee)$. Highest weight vectors of $V_1 \otimes V_2$ and $V_1^\vee \otimes V_2^\vee$ are given by $v_1 \otimes v_2$ and $v_1^\vee \otimes v_2^\vee$. Then we have

$$
V_{\min}(Z) = V_{1, \min}(Z) \otimes V_{2, \min}(Z),
$$

$$
V_{\max}(Z) = V_{1, \max}(Z) \otimes V_{2, \max}(Z).
$$
A non-degenerate bilinear form on $V$ is given by

$$\langle v'_1 \otimes v'_2, v''_1 \otimes v''_2 \rangle = \langle v'_1, v'_1 \rangle \langle v'_2, v''_2 \rangle$$

for $v'_i, v''_i \in V_i$ ($i = 1, 2$). Then $V^\vee$ can be identified with $V$ and $\rho^\vee(G) = \rho(G)$.

Combining this fact with the calculations in (2.1)-(2.15), we have the following theorem.

2.17. **Theorem.** Let $(G, \rho, V)$ be an irreducible prehomogeneous vector space. Then there are at most two split $\mathbf{Z}$-forms which are given by $V_{\max}(\mathbf{Z})$ and $V_{\max}(\mathbf{Z})$. The exact number of split $\mathbf{Z}$-forms of each $(G, \rho, V)$ is given in the following table. (The first row indicates the type of $(G, \rho, V)$ and the second row indicates the number of split $\mathbf{Z}$-forms.)

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References


