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# Prehomogeneous Vector Spaces over Finite Fields

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## §0. Introduction.

Let $G$ be a complex reductive group which acts linearly and prehomogeneously on $V=\mathbb{C}^n$, $V^\vee$ the dual space, and $f$ (resp. $f^\vee$) a relative invariant on $V$ (resp. $V^\vee$) such that $f(v)f^\vee(v^\vee)$ ($(v,v^\vee)\in V\times V^\vee$) is absolutely $G$-invariant.

Roughly, the fundamental theorem of the theory of prehomogeneous vector spaces due to M.Sato says that

$$\text{(1)}\qquad (\text{Fourier transform of } f^s)=(f^\vee)^{-s}\times(*)$$

for $s\in\mathbb{C}$, with some factor $(*)$. One of the main problems is to determine $(*)$ explicitly.

In this note, we consider an analogue of (1) over a finite field $\mathbb{F}_q$. Especially, we give a closed formula for $(*)$ of (1) in the $\mathbb{F}_q$-case assuming $\operatorname{char}\mathbb{F}_q\gg 0$.

This work was started as a joint work with N.Kawanaka around 1984, and has been completed recently (in 1994) as a joint work with J.Denef.

## §1. Review of prehomogeneous vector spaces.

Let $G$ be a linear algebraic group over an algebraically closed field $k$, and $\rho: G\to GL(V)$ a finite dimensional rational representation. For $\phi\in\operatorname{Hom}(G,k^\times)$, put

$$k[V]_\phi:=\{f\in k[V]\mid f(gv)=\phi(g)f(v)\}.$$

**1.1. Lemma.** *The following conditions are mutually equivalent.*

*(1) There exists an open $G$-orbit in $V$.*

*(2) For any $\phi \in \mathrm{Hom}(G, k^\times)$, $\dim_k k[V]_\phi \leq 1$.*

*(3)* $\mathrm{tr.deg}_k\, k(V)^G = 0$.

*(4)* $k(V)^G = k$.

*Moreover, in this case, every $f \in k[V]_\phi$ is a homogeneous polynomial.*

*Proof.* (1) $\Rightarrow$ (2) Cf. the lecture of T.Kimura in the same volume.

(2) $\Rightarrow$ (4) Assume that $f_1/f_2 \in k(V)^G \setminus k$, $f_1, f_2 \in k[V]$ and $(f_1, f_2) = 1$. Then it is easy to see that $f_1, f_2 \in k[V]_\phi$ for some $\phi$. Hence $\dim_k k[V]_\phi \geq 2$.

Since $k$ is algebraically closed, (3) $\Leftrightarrow$ (4).

The implication (4) $\Rightarrow$ (1) follows from [**R**, Theorem 2]. (Cf. [**G**].)

Since $f(cx)$ ($c \in k^\times$) belongs to $k[V]_\phi$, it is a constant multiple of $f$ and hence we get the last assertion.

**1.2. Definition.** If the above conditions are satisfied, $(G, \rho, V)$ is called *a prehomogeneous vector space.*

Let $\rho^\vee : G \to GL(V^\vee)$ be the dual of $\rho$.

**1.3. Lemma.** *Assume that the $G$-module $k[V]$ is completely reducible. Then*

*(1) $k[V^\vee]$ is also completely reducible,*

*(2)* $\dim k[V]_\phi = \dim k[V^\vee]_{\phi^{-1}}$ *for any $\phi$, and*

*(3) if $(G, \rho, V)$ is a prehomogeneous vector space, then $(G, \rho^\vee, V^\vee)$ is also a prehomogeneous vector space.*

*Proof.* Since $k[V^\vee]$ (resp. $k[V^\vee]_{\phi^{-1}}$) is the dual $G$-module of $k[V]$ (resp. $k[V]_\phi$), we get (1) and (2). Then (3) follows from (1.1).

**1.4.** Henceforth, we assume the following.

**Assumptions.**

(1) base field = $\mathbb{C}$.

(2) $G$ = reductive.

(3) $(G, \rho, V)$ = prehomogeneous vector space.

**Notations etc.**

**prehomogeneous vector spaces**

(4) $V = \mathbb{C}^n$

(5) $V^\vee$ = dual space

(6) $\langle\ \ \rangle : V^\vee \times V \to \mathbb{C}$ pairing

(7) $\rho : G \to GL(V)$

(8) $\rho^\vee : G \to GL(V^\vee)$

(9) $(G, \rho^\vee, V^\vee)$ = prehomogeneous vector space by (1.3, (3)).

**relative invariant**

(10) $f \in \mathbb{C}[V]$, $f(gv) = \phi(g)f(v)$ $(g \in G,\ v \in V)$

(11) $f^\vee \in \mathbb{C}[V^\vee]$, $f(gv^\vee) = \phi(g)^{-1}f^\vee(v^\vee)$ $(g \in G,\ v^\vee \in V^\vee)$ (The existence is guaranteed by (1.3, (2)).)

(12) $\Omega := V \setminus f^{-1}(0)$

(13) $\Omega^\vee := V^\vee \setminus f^{\vee-1}(0)$

(14) $O_1 \subset \Omega$ : unique closed $G$-orbit

(15) $O_1^\vee \subset \Omega^\vee$ : unique closed $G$-orbit

(16) $F := \operatorname{grad}\log f$, $(F(\Omega) = O_1^\vee)$

(17) $F^\vee := \operatorname{grad}\log f^\vee$, $(F^\vee(\Omega^\vee) = O_1)$

(18) $m := \dim O_1 = \dim O_1^\vee$,

(19) $d := \deg f = \deg f^\vee$.

(20) $n := \dim V = \dim V^\vee$,

**$b$-function**

(21) $f^\vee(\mathrm{grad}_x)f(x)^{s+1} = b(s)f(x)^s$

(22) $f(\mathrm{grad}_y)f^\vee(y)^{s+1} = b(s)f^\vee(y)^s$

(23) $b(s) = b_0\prod_{j=1}^d(s+\alpha_j)$, $(b_0 \in \mathbb{C}^\times,\ \alpha_j \in \mathbb{Q}_{>0})$

(24) $b^{\exp}(t) := \prod_{j=1}^d(t - e^{2\pi\sqrt{-1}\alpha_j}) = \prod_{j\geq 1}(t^j - 1)^{e(j)}$, $(e(j) \in \mathbb{Z})$

**finite field**

(25) $\mathrm{char}\,\mathbb{F}_q = p \gg 0$

(26) $\psi \in \mathrm{Hom}(\mathbb{F}_q, \mathbb{C}^\times)$, $\not\equiv 1$

(27) $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$

(28) $G(\chi,\psi) = \sum_{x\in\mathbb{F}_q^\times}\chi(x)\psi(x)$

**rank** : notation for Theorem B.

(29) $r := \mathrm{card}\{j \mid \alpha_j \in \mathbb{Z}\} = \sum_{j\geq 1} e(j)$.

(30) $r(\ ) :=$ rank $=$ dimension of a maximal torus.

(31) $s(\ ) :=$ split rank $=$ dimension of a maximal split torus.

(32) $G_{v^\vee} =$ isotropy group at $v^\vee \in V^\vee(\mathbb{F}_q)$.

(33) $r(v^\vee) := r(G) - r(G_{v^\vee})$.

(34) $s(v^\vee) := s(G) - s(G_{v^\vee})$.

**quadratic form** ($\mathrm{char}\,\mathbb{F}_q \neq 2$) : notation for Theorem C.

(35) $\chi_{1/2} \in \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$, $\neq 1$, $(\chi_{1/2})^2 = 1$ (Legendre symbol)

(36) $h^\vee(v^\vee) =$ discriminant of $\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right)$ $(v^\vee \in O_1^\vee(\mathbb{F}_q))$

(For a symmetric matrix $A$, (discriminant of $A$) $= \prod_{i=1}^m a_i$ if $A \sim (\mathrm{diag}(a_1, \cdots$

$\cdots, a_m, 0, \cdots, 0)$.)

(37) $(m+r)/2 \in \mathbb{Z}$

## §2. Main Theorems.

**Theorem A1.** *Assume that* $\mathrm{char}(\mathbb{F}_q) \gg 0$. *Then*

$$q^{-n} \sum_{v \in \Omega(\mathbb{F}_q)} \chi(f(v))\psi(\langle v^\vee, v \rangle) = \begin{cases} q^{-m/2} \prod_{j \geq 1} \left( \frac{G(\chi^j, \psi)}{\sqrt{q}} \right)^{e(j)} \cdot \chi\left( \frac{b_0 f^\vee(v^\vee)^{-1}}{\prod_{j\geq 1} (j^j)^{e(j)}} \right) \cdot \kappa^\vee(v^\vee) & \text{if } v^\vee \in O_1^\vee(\mathbb{F}_q) \\ 0 & \text{if } v^\vee \in (\Omega^\vee \setminus O_1^\vee)(\mathbb{F}_q), \end{cases}$$

*where* $\kappa^\vee(v^\vee) = \pm 1$ *depends on* $v^\vee$ *but not on* $\chi$.

**Theorem A2.** *Assume that* $\mathrm{char}(\mathbb{F}_q) \gg 0$. *Then*

$$q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \chi(f^\vee(v^\vee))\psi(\langle v^\vee, v \rangle) = q^{-m/2} \prod_{j \geq 1} \left( \frac{G(\chi^j, \psi)}{\sqrt{q}} \right)^{e(j)} \cdot \chi\left( \frac{b_0 f(v)^{-1}}{\prod_{j\geq 1} (j^j)^{e(j)}} \right) \cdot \kappa^\vee(F(v))$$

*for* $v \in \Omega(\mathbb{F}_q)$, *with* $\kappa^\vee$ *the same as in Theorem A1.*

**Theorem B.** *Assume that* $\mathrm{char}(\mathbb{F}_q) \gg 0$. *Then*

$$\kappa^\vee(v^\vee) = (-1)^{r(v^\vee) - s(v^\vee)}$$

*for* $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

**Theorem C.** *Assume that* $\mathrm{char}(\mathbb{F}_q) \gg 0$. *Then*

$$\kappa^\vee(v^\vee) = \chi_{1/2}\left( (-1)^{(m+r)/2} \prod_{j \geq 1} j^{e(j)} \cdot h^\vee(v^\vee) \right)$$

*for* $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

**Remark.** Note that $e(j)$'s appear in many places in the above theorems. In other words, $b^{\exp}(t)$ often appears. (See (1.4, (24).) The function $b^{\exp}(t)$, which was constructed using the $b$-function $b(s)$, is known to be the minimal polynomial of the Picard-Lefschetz monodromy of 'the vanishing cycle sheaf' $R\psi_f(\mathbb{C})$. Thus (at least at present), I can not expect that these theorems could be proved in an elementary way, completely in the framework of finite fields, without using the algebraic geometry such as $l$-adic étale sheaves, or the $L$-functions etc.

**§3. Example.**

**Notations etc.**

(1) $V(R) = \{(x_{ij})_{1\le i,j\le n} \mid x_{ij} = x_{ji} \in R \ (i \le j)\}$, where $R = \mathbb{Z}, \mathbb{C}, \mathbb{F}_q$.

(2) $V^\vee(R) = \{(y'_{ij})_{1\le i,j\le n} \mid y'_{ij} = y'_{ji}, y'_{ii} \in R, 2y'_{ij} \in R \ (i < j)\}$.

(3)
$$y_{ij} = \begin{cases} y'_{ii} & (i = j) \\ 2y'_{ij} & (i < j) \end{cases}$$

(4) $f(v) = \det v$.

(5)
$$f^\vee(v^\vee) = \begin{cases} \frac{1}{2}\det(2v^\vee) & (n = \text{odd}) \\ \det(2v^\vee) & (n = \text{even}) \end{cases}$$

(Then $f \in \mathbb{Z}[V]$ and the coefficients have no common divisor, and similarly for $f^\vee$.)

(6) $\langle\, v, v^\vee \,\rangle = \operatorname{trace}(vv^\vee)$.

(7) $\rho(g)v = gv\,{}^tg$ for $g \in G = GL_n$.

(8) $\rho^\vee(g)v^\vee = {}^tg^{-1} \cdot v^\vee \cdot g^{-1}$ for $g \in G = GL_n$.

(9) $\phi(g) = \det(g)^2$.

(10) $b(s) = b_0(s+1)(s+\frac{3}{2})(s+\frac{4}{2})\cdots(s+\frac{n+1}{2})$.

(11) $b_0 = (2^2)^l$ if $n = 2l+1$ or $2l$.

$$(12) \qquad b^{\exp}(t) = \begin{cases} (t-1)(t^2-1)^l & (n = 2l+1) \\ (t^2-1)^l & (n = 2l). \end{cases}$$

(13) $b_0/\prod_j (j^j)^{e(j)} = 1$.

Let $O(v^\vee)$ $(v^\vee \in V^\vee)$ be the orthogonal group with respect to the symmetric matrix $v^\vee$.

$$(14) \qquad r(v^\vee) = r(GL_n) - r(O(v^\vee)) = r = \begin{cases} l+1 & (n = 2l+1) \\ l & (n = 2l) \end{cases}$$

$$(15) \qquad r(v^\vee) - s(v^\vee) = \begin{cases} 0 & (O(v^\vee) = \text{split type}) \\ -1 & (O(v^\vee) = \text{non-split type}) \end{cases}$$

$$(16) \qquad (-1)^{(m+r)/2} = \begin{cases} (-1)^{(l+1)^2} = (-1)^{l+1} & (n = 2l+1) \\ (-1)^{l^2+l} = 1 & (n = 2l) \end{cases}$$

$$(17) \qquad \prod_{j\geq 1} j^{e(j)} = 2^l \text{ if } n = 2l+1 \text{ or } 2l.$$

$$(18) \qquad h(v) = \det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}(v)\right) = (-1)^{n(n+1)/2} 2^{n(n-1)/2} f(v)^{-n-1}.$$

$$(19) \qquad \begin{aligned} h^\vee(v^\vee) &= \det\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right) \\ &= \begin{cases} (-1)^{n(n+1)/2} 2^{(n-1)(n+2)/2} f^\vee(v^\vee)^{-n-1}, & (n = 2l+1), \\ (-1)^{n(n+1)/2} 2^{n(n+3)/2} f^\vee(v^\vee)^{-n-1}, & (n = 2l). \end{cases} \end{aligned}$$

Since $O_1 = \Omega$ and $O_1^\vee = \Omega^\vee$, Theorem A1 and Theorem A2 become the same. In such a case, we refer to them simply as 'Theorem A'.

Now Theorem A implies that

$$q^{-n(n+1)/4} \sum_{\substack{v\in V(\mathbb{F}_q)\\ \det v\neq 0}} \chi(\det v)\,\mathrm{trace}(v^\vee v) \tag{20}$$
$$= \begin{cases} G(\chi,\psi)G(\chi^2,\psi)^l \chi(f^\vee(v^\vee)^{-1})\kappa^\vee(v^\vee) & (n=2l+1) \\ G(\chi^2,\psi)^l \chi(f^\vee(v^\vee)^{-1})\kappa^\vee(v^\vee) & (n=2l) \end{cases}$$

if $v^\vee \in V^\vee(\mathbb{F}_q)$ and $\det v^\vee \neq 0$.

Theorem B implies that

$$\kappa^\vee(v^\vee) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even and } O(v^\vee) \text{ is of split type} \\ -1 & \text{if } n \text{ is even and } O(v^\vee) \text{ is of non-split type} \end{cases} \tag{21}$$

Theorem C implies

$$\kappa^\vee(v^\vee) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \chi_{1/2}((-1)^l f^\vee(v^\vee)) & \text{if } n \text{ is even} \end{cases} \tag{22}$$

(It is easy to see that (21) and (22) are equivalent.)

## §4. History.

Around 1981, Z.Chen started to study an $\mathbb{F}_q$-analogue of the theory of prehomogeneous vector spaces. Especially, he explicitly calculated the character sum

$$q^{-n/2} \sum_{v\in V(\mathbb{F}_q)\setminus f^{-1}(0)} \chi(f(v))\psi(\langle v^\vee, v\rangle) \quad (v^\vee \in V^\vee(\mathbb{F}_q)\setminus f^{\vee-1}(0)) \tag{1}$$

for several prehomogeneous vector spaces. (This character sum is an $\mathbb{F}_q$-analogue of the Fourier transform of $f^s$. See §0.)

Around 1983, N.Kawanaka has taken up the same problem, independently of Z.Chen. His motivation lies in the theory of complex linear representations of $G(\mathbb{F}_q)$ with $G$ reductive; he found that character sums of type (1) actually appears in the character table of $G(\mathbb{F}_q)$. (The explicit determination of the character table is the main open problem in the representation theory of $G(\mathbb{F}_q)$ (in April, 1995).)

Later, Kawanaka formulated the following conjecture concerning (1), which is now contained in Theorems A and C.

**4.1. 'Conjecture' of Kawanaka.** *Assume that $(G, \rho, V)$ is an irreducible regular prehomogeneous vector space, and* $\mathrm{char}(\mathbb{F}_q) \gg 0$. *Then*

$$\text{(K1)} \qquad \begin{aligned} &q^{-n/2} \sum_{v \in \Omega(\mathbb{F}_q)} \chi(f(v))\psi(\langle v^\vee, v \rangle) \\ &= \varepsilon(\chi, \psi)\chi(f^\vee(v^\vee)^{-1})\chi_{1/2}(f^\vee(v^\vee)^{-2n/d}) \end{aligned}$$

*with some constant $\varepsilon(\chi, \psi)$.*
*(K2) Fix an isomorphism* $\left(\frac{1}{1-q}\mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\simeq} \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$, $\alpha \mapsto \chi$. *Put*

$$m(\alpha) := \mathrm{card}\{j \mid \alpha + \alpha_j \equiv 0 \mod \mathbb{Z}\}.$$

*Then*

$$|\varepsilon(\chi, \psi)| = q^{-m(\alpha)/2}.$$

**4.2. Remark.** In the irreducible regular case, $h^\vee$ is a non-zero constant multiple of $f^{\vee -2n/d}$. Hence

$$\chi_{1/2}(f^\vee(v^\vee)^{-2n/d}) \sim \chi_{1/2}(h^\vee(v^\vee)) \sim \kappa^\vee(v^\vee)$$

up to $\mathbb{C}^\times$. Thus (4.1) implies 'Theorem A + Theorem C up to a constant, say $C(\chi)$, of absolute value one' in the irreducible regular case..

**4.3. Lefschetz principle.** One of the motivations for the conjecture of Kawanaka was what I would like to call the Lefschetz principle, after Harish-Chandra; *in the context of prehomogeneous vector spaces, whatever is true for the* $\mathbb{R}$*-case or the* $\mathbb{C}$*-case is also true for the* $\mathbb{F}_q$*-case.* In fact, in the $\mathbb{R}$-case, it is known that

$$\text{(1)} \quad \begin{aligned} &(\text{Fourier transform of } f(x)^s) \\ &= (\text{some function of } s) \times \prod_{j=1}^{d} \Gamma(s+\alpha_j) \times f^\vee(y)^{-s} \times \kappa^\vee(y) \end{aligned}$$

($s \in \mathbb{C}$), with some $\kappa^\vee(y)$ independent of $s$. (Cf. the lecture of T.Kimura in the same volume.) In (4.1, (K1)), the constant $\varepsilon(\chi,\psi)$ is the counterpart of '(some function of $s$) $\times \prod_{j=1}^{d} \Gamma(s+\alpha_j)$'.

## §5. Idea of proof (1).

The study of finite field is 'easy' in the sense that we do not need to worry about difficulty such as the convergence problem. However it is same to say that the deep analytic tools are not at our disposal. Therefore, once we can not deal with some problem within the framework of finite field, we need to recover the continuity at the cost of the finiteness.

**5.1. How to recover the continuity.**

**– A rapid but insufficient course in $l$-adic étale sheaves.**

**(1) Vector space.** Let $V = \mathbb{C}^n$ and $F \in GL(V)$. Then we get a complex number as follows:

$$F \curvearrowright V \Rightarrow \mathrm{trace}(F, V) \in \mathbb{C}.$$

**(2) Complex.** Next, let us consider the following sequence of vector spaces over $\mathbb{C}$ and linear mappings between them.

$$V^{\cdot} = (\cdots \to V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \to \cdots).$$

Assume that $d^i \cdot d^{i-1} = 0$, i.e., $\ker(d^i) \supset \mathrm{image}(d^{i-1})$. In this case, $V^{\cdot}$ is called a *complex*. Put $H^i(V^{\cdot}) := \ker(d^i)/\,\mathrm{image}(d^{i-1})$, which is called the *i-th cohomology*. Assume that an operator $F$ acts on each $V^i$ as a linear automorphism and compatibly with $d^i$'s. Then $F$ linearly acts on each $H^i(V^{\cdot})$. Assume further that

$$\dim H^i(V^{\cdot}) < \infty \text{ for all } i, \text{ and}$$

$$H^i(V^{\cdot}) = 0 \text{ for almost all } i$$

Then we get a complex number as follows:

$$F \curvearrowright V^{\cdot} \Rightarrow \sum_i (-1)^i \operatorname{trace}(F, H^i(V)) \in \mathbb{C}.$$

We identify a vector space $V$ with the complex

$$V^i = \begin{cases} V & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

Thus (2) can be regarded as a generalization of (1).

**(3) Sheaf.** Third, let us consider a mapping $\pi : L \to X$ between two sets, on which an operator $F$ acts compatibly with $\pi$. Assume that $L_x := \pi^{-1}(x)$ $(x \in X)$ are finite dimensional vector spaces over $\mathbb{C}$, and $F : L_x \to L_{F(x)}$ $(x \in X)$ are linear mappings. Then we get a $\mathbb{C}$-valued function on $X^F := \{x \in X \mid Fx = x\}$ as follows:

$$F \curvearrowright \begin{pmatrix} L \\ \downarrow \pi \\ X \end{pmatrix} \Rightarrow \mathrm{trace}_L(x) := \mathrm{trace}(F, L_x) \quad (x \in X^F).$$

**(4) Complex of sheaves.** Note that $L = \bigcup_{x \in X} L_x$ (disjoint union) can be regarded as a family of vector spaces parametrized by $X$. In this sense, (3) can be regarded as a generalization of (1).

Thus we get two generalizations of (1) (cf. the remark at the end of (2)). Consider a family $L^\cdot_x$ $(x \in X)$ of complexes as a generalization of (2) and (3). Then we get a $\mathbb{C}$-valued function on $X^F$ as follows:

$$F \curvearrowright \begin{pmatrix} L^\cdot \\ \downarrow \pi \\ X \end{pmatrix} \Rightarrow \mathrm{trace}_{L^\cdot}(x) := \sum_i (-1)^i \,\mathrm{trace}(F, H^i(L^\cdot)_x) \quad (x \in X^F).$$

**(5) Constructible sheaf.** If $X$ is a complex algebraic variety (with the Hausdorff topology), we need to claim some kind of 'continuity' for the correspondence $X \ni x \mapsto L^\cdot_x$. In fact, we can define a category $D^b_c(X, \mathbb{C})$ whose object gives some $L^\cdot \xrightarrow{\pi} X$ as in (4).

$$D^b_c(X, \mathbb{C}) \ni \blacksquare \mapsto \begin{pmatrix} L^\cdot \\ \downarrow \pi \\ X \end{pmatrix}.$$

(Here I omit the definition of $D^b_c(X, \mathbb{C})$, and so $\blacksquare$ is a black box.)

**(6) Etale $\overline{\mathbb{Q}}_l$-sheaf.** Let $X$ be an algebraic variety over $\mathbb{F}_q$. (If $X$ is an affine variety, this means that

$$X = X(k) = \{(x_1, \cdots, x_N) \in k^N \mid P_i(x_1, \cdots, x_N) = 0 \ (i = 1, \cdots, M)\}$$

with some $P_i \in \mathbb{F}_q[x_1, \cdots, x_N]$. Here and below, $k$ denotes an algebraic closure of $\mathbb{F}_q$, and we identify $X$ with the set of rational points $X(k)$.) Let $F \in \mathrm{Gal}(k/\mathbb{F}_q)$ be the Frobenius endomorphism; $Fx = x^q$ ($x \in k$). Then

$$F \curvearrowright X \quad (F \text{ acts on } X),$$

$$X^F = X(\mathbb{F}_q), \text{ and}$$

$$D_c^b(X, \overline{\mathbb{Q}}_l) \ni \blacksquare \mapsto \left[ F \curvearrowright \begin{pmatrix} L^{\cdot} \\ \downarrow \pi \\ X \end{pmatrix} \right].$$

(Exercise: $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$.) Therefore, we can obtain a $\mathbb{C}$-valued function on $X(\mathbb{F}_q)$

$$\mathrm{trace}_{L^{\cdot}}(x) := \sum_i (-1)^i \, \mathrm{trace}(F^*, H^i(L^{\cdot})_x) \quad (x \in X(\mathbb{F}_q)),$$

where $F^* = F^{-1}$.

**5.2. Example 1.**

**(1) Multiplicative character $\chi$ of $\mathbb{F}_q^\times$ ($\mathbb{C}$-valued function on $\mathbb{F}_q^\times$).** Let us explain how to find $L^{\cdot} \xrightarrow{\pi} k^\times$ which gives $\chi$ as in (5.1, (6)).

**(2) Lang torsor $L_\chi$ on $k^\times$ ($\overline{\mathbb{Q}}_l$-sheaf on $k^\times$).** Put $\lambda(x) = x^{q-1}$ ($x \in k^\times$). For $a \in \mathbb{F}_q^\times$, fix $a' \in \lambda^{-1}(a)$. Then $\lambda^{-1}(a) = a'\mathbb{F}_q^\times$. Thus $\lambda : k^\times \to k^\times$ is something like a principal $\mathbb{F}_q^\times$-bundle on $k^\times$, from which we construct a 'line bundle' $\pi : k^\times \times_{\mathbb{F}_q^\times} \mathbb{C} \to k^\times$ as follows.

Let $\mathbb{F}_q^\times$ act on $k^\times \times \mathbb{C}$ by $c \cdot (x,t) = (cx, \chi(c) \cdot t)$ ($c \in \mathbb{F}_q^\times$, $x \in k^\times$, $t \in \mathbb{C}$). Put $L_\chi := k^\times \times_{\mathbb{F}_q^\times} \mathbb{C} := (k^\times \times \mathbb{C})/\mathbb{F}_q^\times$. Denote the image of $(x,t)$ in $L_\chi$ by $[x,t]$. Then we can define $\pi : L_\chi \to k^\times$ by $\pi([x,t]) := \lambda(x)$, and the $F$-action on $L_\chi$ by $F([x,t]) := [x^q, t]$. It is easy to see that $\pi^{-1}(a) = \{[a', t] \mid t \in \mathbb{C}\}$, which we shall identify with $\mathbb{C}$ by $[a', t] = t$.

Now, let us calculate the $F$-action on $\pi^{-1}(a) =: (L_\chi)_a$ ($a \in \mathbb{F}_q^\times$). Since $a'^{q-1} = a$, $a'^q = aa'$. Hence

$$t = [a', t] \xrightarrow{F} [a'^q, t] = [aa', t] = [a', \chi(a)^{-1}t] = \chi(a)^{-1}t,$$

and, finally we get

$$\mathrm{trace}_{L_\chi}(x) = \chi(x) \quad (x \in \mathbb{F}_q^\times).$$

(Recall that we have used $F^* = F^{-1}$ instead of $F$ to define $\mathrm{trace}_{L'}(x)$ in (5.1, (6).)

**(3) Similar sheaf on $\mathbb{C}^\times$.** Note that $\lambda(x) = x^{q-1}$ can be considered also for $x \in \mathbb{C}^\times$. Then $\lambda : \mathbb{C}^\times \to \mathbb{C}^\times$ is a principal $\mathbb{F}_q^\times$-bundle, and we can define $\pi : L_\chi \to \mathbb{C}^\times$ in the same way as in (2). The 'principal $\mathbb{F}_q^\times$-bundle' $\lambda : k^\times \to k^\times$ considered in (2) is obtained from $\lambda : \mathbb{C}^\times \to \mathbb{C}^\times$ by the 'reduction modulo $p$'. (More precisely, first note that $\lambda$ is defined over $\mathbb{Q}$. Then, in general, regard the coefficients of the defining equations as $p$-adic integers, if $p \gg 0$, and then as elements of $\mathbb{F}_q$ ($p := \mathrm{char}\,\mathbb{F}_q$). The resulting geometric object is called the *'reduction modulo p'*.

**(4) Another description of (3).** Fix an isomorphism

$$\left(\frac{1}{1-q}\mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\simeq} \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times), \quad \alpha \mapsto \chi.$$

Then $L_\chi \simeq \mathbb{C}x^\alpha$. (Although $x^\alpha$ is multivalued, the ambiguity is multiplication by some root of unity. Hence the totality of its scalar multiple is globally well-defined.)

**(5) Differential equation of Fuchsian type.** Note that $u = x^\alpha$ is characterized by the differential equation

$$x\frac{du}{dx} = \alpha u,$$

which is of Fuchsian type, i.e., its singularities at 0 and $\infty$ are regular. In other words, $L_\chi \simeq \mathbb{C}x^\alpha$ in (4) can be regarded as the totality of local solutions of the above equation.

**5.3. Example 2 (additive character).** Let $1 \neq \psi \in \mathrm{Hom}(\mathbb{F}_q, \mathbb{C}^\times)$. Put $\alpha(x) = x - x^q$ $(x \in k)$. Since $\alpha'(x) = 1 - qx^{q-1} = 1$ $(q = 0$ in $k)$, $\alpha : k \to k$ is an unramified covering of the affine line $k$. It is easy to see that $\alpha : k \to k$ is a 'principal $\mathbb{F}_q$-bundle', and hence we can define $\pi : L_\psi \to K$ such that $\mathrm{trace}_{L_\psi}(x) = \psi(x)$ $(x \in \mathbb{F}_q)$ in the same way as in (5.2). This $L_\psi$ is called the *Artin-Schreier torsor.*

Obviously, $L_\psi$ is not compatible with the 'reduction modulo $p$'.

**5.4. Example 3 (substitution).**

Let us come back to the situation of (5.1, (3)). Let $\pi : L \to X$ be a family of vector spaces $\{L_x \mid x \in X\}$ on which $F$ acts as in (5.1, (3)). If a $F$-mapping $f : Y \to X$ is given, we can define a new family of vector spaces $\{(f^*L)_y \mid y \in Y\}$ (i.e, $f^*L \to Y$) by

$$(f^*L)_y := L_{f(y)}$$

and we call $f^*L$ the *pull-back* of $L$ by $f$. Then the $F$-action on $L \to X$ induces one on $f^*L$ by

$$(f^*L)_y = L_{f(y)} \xrightarrow{F} L_{F(f(y))} = L_{f(F(y))} = (f^*L)_{F(y)}.$$

Then

$$\mathrm{trace}_{f^*L}(y) = \mathrm{trace}_L(f(y)) \quad (y \in Y^F).$$

Therefore, we can say that the 'pull-back' is the geometric counterpart of the 'substitution'. It would be obvious how to generalize the content of this paragraph to the situation of (5.1, (4)).

**5.5. Example 4 (multiplication).** If $L \to X$ and $L' \to X$ are given as in (5.1, (3)), define $L \otimes L' \to X$ by

$$(L \otimes L')_x = L_x \otimes L'_x \text{ (tensor product).}$$

If $F$-actions on $L$ and $L'$ are given, then we can naturally define the $F$-action on $L \otimes L'$, and we get

$$\text{trace}_{L\otimes L'}(x) = \text{trace}_L(x) \times \text{trace}_{L'}(x) \quad (x \in X^F).$$

Therefore, we can say that the 'tensor product' is the geometric counterpart of the 'multiplication'.

**5.6. Example 5 (summation).**

**Theorem (Grothendieck-Lefschetz trace formula).** *Let $X$ be an algebraic variety over $\mathbb{F}_q$. Then for any $L^\cdot \in D^b_c(X, \overline{\mathbb{Q}}_l)$,*

$$\sum_i (-1)^i \operatorname{trace}(F^*, H^i_c(X, L^\cdot)) = \sum_{x \in X(\mathbb{F}_q)} \text{trace}_{L^\cdot}(x).$$

Here $H^i_c$ denotes the ($l$-adic étale) cohomology with compact supports, whose definition we do not give here. Anyway, we can say that the 'cohomology with compact supports' is the geometric counterpart of the 'summation'.

**5.7. Proof of (K1).** (1) Summing up (5.2)–(5.6), we can find the geometric counterpart of the both sides of (4.1, (K1)) (= étale perverse $\overline{\mathbb{Q}}_l$-sheaves on $V^\vee(k)$).

(2) Using the fact that $f$ is a homogeneous polynomial (cf. (1.1)), we can eliminate (5.3), which is not compatible with the reduction modulo $p$. Since all the remaining (i.e., (5.2) and (5.4)–(5.6)) are compatible with the reduction modulo $p$, we can lift the geometric object over $\mathbb{F}_q$ obtained in (1) to a geometric object over $\mathbb{C}$ (= perverse $\mathbb{C}$-sheaves on $V^\vee(\mathbb{C})$).

(3) The last geometric objects over $\mathbb{C}$ are in one-to-one correspondence with the regular holonomic $D$-modules on $V(\mathbb{C})$ via the Riemann-Hilbert correspondence (M.Kashiwara and Z.Mebkout).

$$\begin{array}{c}
\boxed{\text{regular holonomic } D\text{-modules on } V(\mathbb{C})} \\
(3)\downarrow \text{ Riemann-Hilbert correspondence} \\
\boxed{\text{perverse } \mathbb{C}\text{-sheaves on } V(\mathbb{C})} \\
(2)\downarrow \text{ reduction modulo } p \\
\boxed{\text{étale perverse } \overline{\mathbb{Q}}_l\text{-sheaves on } V(k)} \\
(1)\downarrow \text{ trace} \\
\boxed{\text{function on } V(\mathbb{F}_q)}
\end{array}$$

Now let us consider the identity (4.1, (K1)), which we are now going to prove. This identity belongs to the bottom of the above diagram. After lifting the identity to the top by the procedure explained so far, we can prove it as the identity between two $D$-modules.

Note that $D$-module is a system of *linear* differential equations. Thus we can say intuitively that we have characterized functions by linear differential equations. Thus in our characterization, the ambiguity of multiplication by scalar is inevitable. This ambiguity causes the ambiguity of '*some* constant $\varepsilon(\chi, \psi)$' at the end.

**5.8. Proof of (K2).** After the famous work of P.Deligne on the Weil conjecture (= the Riemann hypothesis for zeta functions of algebraic varieties), we can study geometrically such an arithmetic problem as the determination of (archimedean) absolute value of, say, a character sum. This procedure is called the *Weil estimate.* More precisely, we can determine such an absolute value by calculating the weight filtration.

In fact, we can start from the mixed Hodge module of M.Saito in the top of the diagram of (5.7), and we get the *Weil estimate* at the bottom, and we get a proof of (4.1, (K2)).

**5.9. Remark.** (1) When I have first considered the procedure of (5.8) in 1986, the mixed Hodge theory of M.Saito was not yet available, and here I obtained a substantial help from M.Kashiwara.

(2) Our main theorems in §2 can be now proved without using the mixed Hodge theory.

## §6.

**6.1.** Although we have used many deep results, the ambiguity of $\arg(\varepsilon(\chi, \psi))$ remained at the end of §5. (Without to say, this ambiguity will disappear at the very end.) Furthermore, in the $\mathbb{R}$-case, we do not know a closed formula for '(some function of $s$)' in (4.3, (1)). Thus, once, there was a doubt about the existence of a closed formula in the $\mathbb{F}_q$-case. (Conversely speaking, since we have now obtained a closed formula in the $\mathbb{F}_q$-case, it becomes a realistic problem to find a closed formula in the case of the other fields.)

**6.2.** In 1989, I have succeeded to conjecture a closed formula for $\varepsilon(\chi, \psi)$. The basic motivation was the fact that the left hand side of Theorem A1 is transformed in a simple way under the castling transformation. (In general, I want to call a quantity associated to a prehomogeneous vector space *a castling invariant* if it is transformed 'in a simple way' under the castling transformation, although this is not a definition in the usual sense.) Therefore, it would be natural to assume that $\varepsilon(\chi, \psi)$ would be expressed in terms of castling invariants. In fact, this assumption turned out to be very definitive, and I have obtained conjectures, which are now Theorems A C.

I expect that the same procedure would be useful in considering the problem stated at the end of (6.1).

## §7. Idea of proof (2).

**7.1.** In April of 1993, I have learnt a crucial idea from J.Denef. I will explain it here.

Let us come back to the situation of (5.1, (6)). There we have considered a $\mathbb{C}$-valued function

$$\text{(1)} \qquad \operatorname{trace}_{E^\cdot}(x) = \sum_i (-1)^i \operatorname{trace}(F^*, H^i(E^\cdot)_x) \quad (x \in X^F)$$

for $E^\cdot \in D^b_c(X, \overline{\mathbb{Q}}_l)$. Instead of such an additive theory, we can consider the multiplicative theory:

$$\text{(2)} \qquad \varepsilon_{E^\cdot}(x) := \prod_i \det(-F^*, H^i(E^\cdot)_x)^{(-1)^{i+1}} \quad (x \in X^F).$$

**7.2. From additive theory to multiplicative theory.** Assume that there exists $i_0 \in \mathbb{Z}$ such that

$$\text{(1)} \qquad \begin{aligned} &\dim H^{i_0}(E^\cdot)_x = 1 \text{ for all } x \in X, \text{ and} \\ &H^i(E^\cdot) = 0 \text{ if } i \neq i_0. \end{aligned}$$

Then

$\operatorname{trace}_{E^\cdot}(x)$,

eigenvalue of $F^*$ on $H^{i_0}(E^\cdot)_x$, and

$\varepsilon_{E^\cdot}(x)$

are essentially the same. Hence if (1) occurs in some place, we can move from the additive theory to the multiplicative theory, and vice versa. In fact, we can do in the situation of our present concern.

Next, let us explain an advantage of the multiplicative theory, i.e., the product formula of G.Laumon.

**7.3. Grothendieck theory on Artin L.** Let $k$ be a field and $k_s$ its separable closure. Then

$$(1)\qquad \begin{aligned} &\{E \mid \text{ étale } \overline{\mathbb{Q}}_l\text{-sheaf of rank } n \text{ on } X = \operatorname{Spec} k\} \\ &= \{E' \mid n\text{-dimensional } (\overline{\mathbb{Q}}_l, \operatorname{Gal}(k_s/k))\text{-module}\} \end{aligned}$$

and

$$(2)\qquad H^i_{\text{étale}}(X, E) = H^i_{\text{Galois}}(\operatorname{Gal}(k_s/k), E').$$

For a general scheme $X$, $D^b_c(X, \overline{\mathbb{Q}}_l)$ which appeared in (5.1, (6)) (without definition) is a certain generalization of (1), and the $l$-adic étale cohomology is a generalization of (2).

Coming back to a variety $X$ over $\mathbb{F}_q$ and $E^{\cdot} \in D^b_c(X, \overline{\mathbb{Q}}_l)$, let $H^i(X, E^{\cdot})$ be the $l$-adic étale cohomology. After A.Grothendieck, we define the Artin $L$-function by

$$(3)\qquad L(X, E^{\cdot}, s) := \prod_i \det(1 - q^{-s}F^*, H^i(X, E^{\cdot}))^{(-1)^i}.$$

(This is a generalization of the usual Artin $L$-functions associated with $(\overline{\mathbb{Q}}_l, \operatorname{Gal}(k_s/k))$-modules.) Then the following functional equation follows from the Poincaré duality.

$$(4) \qquad L(X, E^{\cdot}, s) = \varepsilon(X, E^{\cdot}) q^{-sa(X,E^{\cdot})} L(X, \mathbb{D}E^{\cdot}, -s),$$

where $a(X, E^{\cdot})$ is the Euler characteristic, $\mathbb{D}(\ )$ denote the Verdier duality, and

$$(5) \qquad \varepsilon(X, E^{\cdot}) = \prod_i \det(-F^*, H^i(X, E^{\cdot}))^{(-1)^{i+1}}.$$

Note that here appears a similar product as (7.1, (2)). In fact, in essence, our task is to calculate (5).

**7.4. Langlands theory on Hecke L.**

**Notation.**

$X$ : a connected smooth projective curve over $\mathbb{F}_q$.

$K$ : the function field of $X$.

$K_A$ : the adèlization of $K$.

$\pi$ : an automorphic cuspical representation of $GL_n(K_A)$.

$\pi^{\vee}$ : the contragradient representation of $\pi$.

R.P.Langlands defined a Hecke $L$-function $L(X, \pi, s)$, and showed the following functional equation.

$$(1) \qquad L(X, \pi, s) = \varepsilon(X, \pi) q^{-sa(X,\pi)} L(X, \pi^{\vee}, 1-s).$$

Since this functional equation is obtained as a product of local functional equations (like the Tate theory),

$$(2) \qquad \varepsilon(X,\pi) = C \cdot \prod_{v \in |X|} (\text{local } \varepsilon\text{-factor at } v)$$

with a certain constant $C$, where $|X|$ denotes the set of places of $K$, i.e, the set of closed points of $X$. See [**L**, 3.1.3.5].

**7.5. Langlands conjecture** (a generalization of the reciprocity law of E.Artin).

*There would exist a correspondence*

$$\{E \mid \textit{smooth irreducible } \overline{\mathbb{Q}}_l\textit{-sheaf of rank } n \textit{ on some open dense } U \subset X\}$$
$$\to \{\pi_E \mid \textit{cuspidal automorphic representation of } GL_n(K_A)\}$$

*such that*

$$(1) \qquad L(X,\pi_E,s) = L(X,E,s).$$

**7.6. Local constant.** Summing up (7.3)–(7.5), we get

$$\begin{aligned}
&\prod_i \det(F^*, H^i(X,E))^{(-1)^{i+1}} = \varepsilon(X,E) \quad \text{by (7.3, (5))} \\
(1) \qquad &\stackrel{?}{=} C' \cdot \varepsilon(X,\pi_E) \quad \text{by (7.5, (1))} \\
&= CC' \cdot \prod_{v \in |X|} (\text{local } \varepsilon\text{-factor at } v) \quad \text{by (7.4, (2))},
\end{aligned}$$

with a certain constant $C'$. (Here '?' means that it is based on a conjecture.)

A candidate for such 'local $\varepsilon$-factor' associated to such $E$ was first constructed by B.Dwork up to sign, and then by R.P.Langlands unconditionally. P.Deligne simplified the proof by a global argument, and proved that their product (multiplied by a certain constant) is actually equal to the most left hand side of (1) under a certain assumption. G.Laumon [**L**, (3.2.1.1)] succeeded to prove the product formula in general.

**7.7.** Although the most basic ingredient is the product formula of Laumon, we need the following materials as well in an actual argument (= a joint work with J.Denef).

(1) A work by F.Loeser and C.Sabbah [**LS**], and independently by G.W.Anderson [**A**] on Aomoto complexes.

(2) A work by J.Denef and F.Loeser [**DL**], and independently by T.Saito [**S**] on global $\varepsilon$-factors.

In fact, comparing (1) and (2), we get Theorem A1.

(3) An idea of N.Kawanaka [**K**, 3.1].

In fact, we get Theorem B using this idea.

The proof of Theorem C depends on a comparatively direct calculation of local $\varepsilon$-factors.

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