ON FUNCTIONAL EQUATIONS OF PREHOMOGENEOUS ZETA DISTRIBUTIONS
OVER A LOCAL FIELD OF CHARACTERISTIC P

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ABSTRACT. For a local field of characteristic 0, the functional equations of zeta distributions of prehomogeneous vector spaces are obtained by M.Sato, T.Shintani, J.Igusa and F.Sato (See [17], [9], [13], [15].) In this paper, we shall consider the case of local fields of characteristic $p > 0$.

§1. K-regular P.V.’s

We fix a local field $K$ of characteristic $p > 0$. Let $G$ be a connected linear algebraic group, $\rho$ its rational representation of $G$ on a finite-dimensional vector space $V$, all defined over an algebraic closure $\overline{K}$ of $K$. We call a triplet $(G, \rho, V)$ a prehomogeneous vector space (abbrev. P.V.) if $V$ has a Zariski-dense $G$-orbit $Y$.

Any point of $Y$ is called a generic point and the isotropy subgroup

$$G_y = \{g \in G; \rho(g)y = y\}$$

of a generic point $y$ is called a generic isotropy subgroup. Note that we have $\dim G_y = \dim G - \dim V$ if and only if $y \in Y$. A non-zero rational function $f(x)$ on $V$ is called a relative invariant of $(G, \rho, V)$ if $f(\rho(g)x) = \chi(g)f(x)$ holds for any $g \in G$ and $x \in Y$ where $\chi : G \to GL_1$ is a rational character of $G$.

The complement $S$ of $Y$ is a Zariski-closed set which is called the singular set of the P.V. $(G, \rho, V)$. Now we assume that $(G, \rho, V)$ is defined over $K$, i.e., $G, \rho, V$ are all defined over $K$. Let $S_i = \{x \in V; f_i(x) = 0\}$ $(i = 1, \ldots, l)$ be the $K$-irreducible component of the $K$-rational points $S_K$ of $S$ of codimension one defined by a $K$-irreducible (not necessarily absolutely irreducible) polynomial $f_i(x)$ $(i = 1, \cdots, l)$.

Then $f_1(x), \ldots, f_l(x)$ are algebraically independent relative invariants and any relative invariant $f(x)$ in $K(V)$ is of the form $f(x) = c \cdot f_1(x)^{m_1} \cdots f_l(x)^{m_l}$ $(c \in K^\times, (m_1, \ldots, m_l) \in \mathbb{Z}^l)$. We call $f_1(x), \cdots, f_l(x)$ the basic $K$-relative invariants of $(G, \rho, V)$. Let $\chi_i$ be the rational character of $G$ corresponding to $f_i(i = 1, \ldots, l)$. Let $X(G)_K$ be the group of $K$-rational characters of $G$, $X_1(G)_K$ its subgroup corresponding to $K$-relative invariants. Then $X_1(G)_K$ is a free abelian group of rank $l$ generated by $\chi_1, \ldots, \chi_l$. 
Let $G_1$ be a subgroup of $G$ generated by the commutator subgroup $[G,G]$ and a generic isotropy subgroup. This does not depend on a choice of a generic point. For $\chi \in X(G)_K$, it is in $X_1(G)_K$ if and only if $\chi_{1,G_1} = 1$. For a relative invariant $f(x)$ of $(G,\rho,V)$, we can define a rational map $\varphi_f : Y \to V^*$ by

$$\varphi_f(x) = t \left( \frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_1}(x), \ldots, \frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_n}(x) \right)$$

where $V^*$ is the dual vector space of $V$. We sometimes denote $\varphi_f(x)$ by $\text{grad log} f(x)$. By a direct calculation, we have

(1) $\varphi_f(\rho(g)x) = \rho^*(g)\varphi_f(x)$ for $g \in G$ and $x \in Y$ where $\rho^*$ denotes the contragradient representation of $\rho$,

and

(2) $\langle d\rho(A)x, \varphi_f(x) \rangle = \delta \chi(A) \text{ for } x \in Y \text{ and } A \in \text{Lie}(G)$ where $d\rho$ (resp. $\delta \chi$) is the infinitesimal representation of $\rho$ (resp. the infinitesimal character of $\chi$) of the Lie algebra $\text{Lie}(G)$ of $G$.

A relative invariant $f(x)$ is called non-degenerate if $\varphi_f : Y \to V^*$ is dominant and the Hessian $H_f(x) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$ is not identically zero. In this case, a rational function $F(x) = \frac{f(x)^n}{H_f(x)} (n = \text{dim } V)$ is a relative invariant corresponding to the character $\chi_0(g) = \det(\rho(g))^2$.

If there exists a non-degenerate relative invariant $f(x)$ in $K(V)$, we say that $(G,\rho,V)$ is a $K$-regular P.V. Then we have $\det(\rho(g))^2 \in X_1(G)_K$. In general, we denote by $Y_K, S_K, \ldots$ $K$-rational points of $Y, S, \ldots$. We write $X_1^*(G)_K$ (resp. $X^*(G)_K, Y^*, S^*, \ldots$) for $(G,\rho^*,V^*)$ which corresponds to $X_1(G)_K$ (resp. $X(G)_K, Y, S, \ldots$) for $(G,\rho,V)$.

**Proposition 1.1**

Assume that $(G,\rho,V)$ and $(G,\rho^*,V^*)$ are $K$-regular P.V.'s. Then we have the following assertion.

(1) $X_1(G)_K = X_1^*(G)_K$.

(2) For a non-degenerate $K$-relative invariant $f$, the map $\varphi = \text{grad log} f : Y \to Y^*$ is bijective.

[Proof]

Since $\varphi(Y)$ is a Zariski-dense $G$-orbit in $V^*$, we have $\varphi(Y) = Y^*$, i.e., $\varphi = \text{surjective}$. Since $\rho^*(g)\varphi(x) = \varphi(\rho(g)x)$, we have $G_x \subset G_{\varphi(x)}$ for $x \in Y$. Now let $f^*$ be a non-degenerate relative invariant in $K(V^*)$, and put $\varphi^* = \text{grad log} f^* : Y^* \to Y$. Similarly we have $G_y \subset G_{\varphi^*(y)} = G_y \subset G_{\varphi^*(y)}$ for $y = \varphi(x)$ and $x' = \varphi^*(y)$, and hence $G_x \subset G_y \subset G_{x'}$. Since $x' = \rho(g_0)x$ for some $g_0 \in G$, we have $G_{x'} = g_0G_xg_0^{-1} \supset G_x$. Since $\text{dim } G_{x'} = \text{dim } G_x$, the algebraic group $G_{x'}$ and $G_x$ have the same connected component $H$ of the identity. Since $G_{x'}$ is isomorphic to $G_x$, the numbers of their connected components coincide, i.e., $[G_{x'} : H] = [G_x : H]$ with $G_{x'} \supset G_x$. This implies $G_{x'} = G_x$, and hence $G_x = G_y$ with $y = \varphi(x)$. 
Thus we have $G_1 = G_1^*$ and hence $X_1(G)_K = X_1^*(G)_K$. Note that $X_1(G)_K = \{ \chi \in X(G)_K; \chi|_{G_1} = 1 \}$. Now assume that $\varphi(x_1) = \varphi(x_2)$ with $x_2 = \rho(g)x_1$ for some $g \in G$. Then we have $\varphi(x_1) = \varphi(x_2) = \varphi(\rho(g)x_1) = \rho^*(g)\varphi(x_1)$ and hence $g \in G_{\varphi(x_1)} = G_{x_1}$, i.e., $x_2 = \rho(g)x_1 = x_1$. Thus $\varphi$ is injective. \hfill \Box

Now assume that $(G, \rho, V)$ is a $K$-regular P.V. Then, as we have seen above, the dual triplet $(G, \rho^*, V^*)$ is a P.V. For a generic point $y \in Y^*$, a dominant morphism $\psi : G \to V^*$ defined by $\psi(g) = \rho^*(g)y$ is called an open orbit morphism.

**Proposition 1.2**

Assume that $(G, \rho, V)$ is a $K$-regular P.V. and an open orbit morphism $\psi : G \to V^*$ is a separable morphism. Then there exists a $K$-relative invariant $f^*$ such that $\text{grad } \log f^* : Y^* \to V$ is dominant.

[Proof]

Let $f$ be a non-degenerate relative invariant in $K(V)$ and put $\varphi = \text{grad } \log f : Y \to Y^*$. First we show that $\varphi$ is injective. Assume that $\varphi(x) = \varphi(x')$. Since $\delta \chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$, we have $\langle x - x', d\rho^*(A)\varphi(x) \rangle = 0$ for all $A \in \text{Lie}(G)$. Since $\psi : G \to V^*$ with $\psi(g) = \rho^*(g)\varphi(x)$ is separable, we have $\{d\rho^*(A)\varphi(x); A \in \text{Lie}(G)\} = V^*$, and hence $x - x' = 0$, i.e., $x = x'$. For any $g \in G_{\varphi(x)}(\supset G_x)$, we have $\varphi(\rho(g)x) = \rho^*(g)\varphi(x) = \varphi(x)$. As $\varphi$ is injective, we have $\rho(g)x = x$, i.e., $g \in G_x$. This implies that $G_x = G_{\varphi(x)}$ and hence $X_1(G)_K = X_1^*(G)_K$. A rational character $\chi$ corresponding to $f$ is in $X_1(G)_K$ and hence $\chi^{-1} \in X_1^*(G)_K$.

This implies that there exists a relative invariant $f^*$ in $K(V^*)$ satisfying $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$ for $g \in G$ and $y \in Y^*$.

Put $\varphi = \text{grad } \log f^*$. Then we have $\langle \varphi^*(y), d\rho^*(A)y \rangle = -\delta \chi(A)$. Since $\delta \chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$, we have $\langle x - \varphi^*(y), d\rho^*(A)y \rangle = 0$ for $y = \varphi(x)$ and all $A \in \text{Lie}(G)$.

Since the open orbit morphism $\psi$ is separable, we have $\{d\rho^*(A)y; A \in \text{Lie}(G)\} = V^*$, and hence $\varphi^*(y) = x \in Y$, i.e., $\varphi^*(Y^*) = Y$. \hfill \Box

Note that in the case of $\text{ch}(K) = 0$, the proof of Proposition 1.2 gives the equivalence between $K$-regularity of $(G, \rho, V)$ and that of $(G, \rho^*, V^*)$.

**Proposition 1.3**

Assume that $(G, \rho, V)$ and $(G, \rho^*, V^*)$ are $K$-regular P.V.'s. Then we have ${\# \rho(G)_K \backslash Y_K = \# \rho^*(G)_K \backslash Y_K}$.

[Proof]
Let $f$ be a non-degenerate relative invariant in $K(V)$ and put $\varphi = \text{grad log } f$. Then for any $x \in Y_K$, we have
\[ \varphi(\rho(G)_K \cdot x) = \rho^*(G)_K \cdot \varphi(x) \subset Y_K^\ast, \text{ i.e., } \varphi \text{ maps an orbit in } Y_K \to \text{ an orbit in } Y_K^\ast. \]

By Proposition 1.1, this map $\varphi$ is injective, and hence $\# \rho(G)_K \backslash Y_K \leq \# \rho^*(G)_K \backslash Y_K^\ast$. Similarly we have $\# \rho^*(G)_K \backslash Y_K^\ast \leq \# \rho(G)_K \backslash Y_K$. $\square$

Now we shall consider a sufficient condition that $\# \rho(G)_K \backslash Y_K$ is finite.
Professor J.P.Serre kindly let us know the following theorem with the proof which was explained by Tits to him.

**Theorem 1.4**

Let $K$ be a local field of characteristic $p > 0$ (or more generally let $K$ be a field complete with respect to a discrete valuation, and with the residue field $k$ of type $(F)$ in the sense of Serre [18]. Let $G$ be a connected smooth reductive group over $K$. Then $H^1(K, G)$ is finite.

[Proof]( after Serre's letter on 9th. September 1992.)

Let $K'$ be the maximal unramified extension of $K$. The field $K'$ is known to be of $\text{dim.} \leq 1$ (in the sense of CG, II, §3). By a theorem of Steinberg (for $K'$ perfect) and of Borel-Springer (for $K'$ imperfect - see Borel Col. Papers II, p.761) we have $H^1(K', G) = 0$. Hence the Galois cohomology of $G$ over $K$ is killed by $K'$, i.e., it is equal to $H^1(K'/K, G)$. We may now apply a theorem of Bruhat-Tits (J.Fac.Sci.Tokyo, 34 (1987), p.693, th.3.12); this says that $H^1(K'/K, G)$ is contained in a finite union of cohomology sets $H^1(k, G_i)$, where the $G_i$'s are algebraic linear groups (non necessarily connected) over $k$. Since $k$ is type $(F)$, each $H^1(k, G_i)$ is finite (see e.g. Borel, Col.Papers II, p.404, th.6.2, or Coh. Gal. III-30, th.4). Hence $H^1(K, G)$ is finite. $\square$

**Proposition 1.5**

Let $(G, \rho, V)$ be a P.V. defined over $K$ with a reductive generic isotropy subgroup. Then $\# \rho(G)_K \backslash Y_K$ is finite.

[Proof]

Let $H$ be a generic isotropy subgroup of a point in $Y_K$. Then there exists a bijection between $\rho(G)_K \backslash Y_K$ and $\text{Ker}(H^1(K, H) \to H^1(K, G))$ (see Serre [18]).

By Theorem 1.4, $H^1(K, H)$ is finite, and hence $\rho(G)_K \backslash Y_K$ is a finite set. $\square$

**Example 1.6**

Let $G$ be the subgroup of $GL_n$ consisting of all lower triangular matrices. Let $V$ be the totality of symmetric $n \times n$ matrices and define $\rho$ by $\rho(g)x = gx^tg$ for all $g \in G$.
and } x \in V. \text{ Since } \dim G = \dim V, \text{ a generic isotropy subgroup is a finite subgroup and hence we have } \# \rho(G) \setminus Y_K = \nu < +\infty \text{ by Proposition } 1.5.

Moreover det } x \text{ is a non-degenerate } K\text{-relative invariant. By } \text{ tr}(xy) (x, y \in V), \text{ we identify } V \text{ with its dual } V^*.

Then } (G, \rho, V) \text{ and } (G, \rho^*, V^*) \text{ are } K\text{-regular P.V.'s. Hence, by Proposition } 1.3, \text{ we have } \# \rho^*(G) \setminus Y_K^* = \nu < +\infty.

\textbf{Proposition 1.7}

Let } (G, \rho, V) \text{ be an irreducible regular P.V. defined over } K. \text{ Then we have } \# \rho(G) \setminus Y_K < +\infty.

[Proof]

By a classification of irreducible P.V.'s (see Z. Chen [4]), we know that a generic isotropy subgroup is reductive.

\square

\S 2. Zeta distributions

Let } K \text{ be a local field of characteristic } p > 0. \text{ Assume that } (G, \rho, V) \text{ and its dual } (G, \rho^*, V^*) \text{ are } K\text{-regular P.V.'s. Moreover we shall assume that } Y_K = Y_1 \cup \cdots \cup Y_\nu \text{ decomposes into a finite union of } \rho(G) \setminus Y_K = \nu < +\infty. \text{ Then by Proposition } 1.3, \text{ we have } Y_K^* = Y_1^* \cup \cdots \cup Y_\nu^*.

Let } f_1(x), \ldots, f_\ell(x) \text{ (resp. } f_1^*(y), \ldots, f_\ell^*(y)) \text{ be basic } K\text{-relative invariants of } (G, \rho, V) \text{ (resp. } (G, \rho^*, V^*)). \text{ Let } \chi_i \text{ (resp. } \chi_i^* \text{) be the corresponding character of } f_i \text{ (resp. } f_i^* \text{). Then we have}

\[ X_1(G)_K = \langle \chi_1, \ldots, \chi_\ell \rangle \text{ and } X_1^*(G)_K = \langle \chi^*_1, \ldots, \chi^*_\ell \rangle. \]

By Proposition 1.1, we have } X_1(G)_K = X_1^*(G)_K \text{ so that there exists uniquely a matrix}

\[ U = (u_{ij}) \in GL_l(\mathbb{Z}) \]

satisfying } \chi_i = \prod_{j=1}^l \chi^*_j u_{ij}. \text{ Since } \det \rho(g)^2 \in X_1(G)_K, \text{ we have } \det \rho(g)^2 = \chi_1^{2\lambda_1} \cdots \chi_\ell^{2\lambda_\ell} \text{ for some } \lambda = (\lambda_1, \cdots, \lambda_\ell) \in (\frac{1}{2} \mathbb{Z})^l \text{ and } \det \rho^*(g)^2 = \chi_1^{*2\lambda_1} \cdots \chi_\ell^{*2\lambda_\ell} \text{ for some } \lambda^* = (\lambda_1^*, \cdots, \lambda_\ell^*) \in (\frac{1}{2} \mathbb{Z})^l. \text{ Since } \det \rho^*(g) = \det \rho(g)^{-1}, \text{ we have } \lambda^* = -\lambda U.

\textbf{Example 2.1}

For simplicity, we deal with the case } n = 2 \text{ in Example 1.6. Then we have}

\[ G = \{ g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}; ab \neq 0 \} \]

and
The basic $K$-relative invariants of $(G, \rho, V)$ (resp. $(G, \rho^*, V^*)$) are $f_1(X) = x$ and $f_2(X) = \det X$ (resp. $f_1^*(X) = x$, $f_2^*(X) = \det X$) corresponding to $\chi_1(g) = a^2$, $\chi_2(g) = a^2b^2$ (resp. $\chi_1^*(g) = b^{-2}$, $\chi_2^*(g) = a^{-2}b^{-2}$) for

$$g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$$

in $G$.

Hence $\chi_1 = \chi_1^* \chi_2^* - 1$ and $\chi_2 = \chi_2^{-1}$ so that we have

$$U = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$  

Since

$$\det \rho \left( \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \right) = a^3b^3,$$

we have $\lambda = \lambda^* = (0, \frac{3}{2})$.

Let $\{\varepsilon_1, \cdots, \varepsilon_\nu\}$ be the complete representatives of $K^\times/K^\times 2$ in $K^\times$. Then we have $Y_K = Y_1 \cup \cdots \cup Y_\nu$ with

$$Y_i = \{ y \in Y_K; f_2(y) \equiv \varepsilon_i \mod K^\times 2\} \ (i = 1, \cdots, \nu).$$

Let $\omega^{(i)} : K^\times \to \mathbb{C}^\times$ (i = 1, \cdots, l) be a quasicharacter, i.e., a continuous homomorphism.

For $\omega = (\omega^{(1)}, \cdots, \omega^{(l)})$ and the basic $K$-relative invariants $f(x) = (f_1(x), \cdots, f_l(x))$, we write $\omega(f(x))$ instead of $\Pi_{i=1}^{l}\omega^{(i)}(f_i(x))$ for simplicity of notations.

Let $| |$ be the absolute value of $K$ normalized by $|q| = q^{-1}$ for a prime element $q$ where $q$ is the module of $K$. For $s = (s_1, \cdots, s_l)$, we write $\omega_s = (|s_1|, \cdots, |s_l|)$ so that $\omega_s(f(x)) = \Pi_{i=1}^{l}|f_i(x)|^{s_i}$.

Let $dx$ be the Haar measure on $V_K = K^n$ normalized by $\int_{R^n} dx = 1$ where $R$ is the maximal compact subring of $K$. Since $d(\rho(g)x) = |\det \rho(g)|dx$ and $\omega_\lambda(f(\rho(g)x)) = |\det \rho(g)|\omega_\lambda(f(x))$, the measure $dy(x) = \frac{dx}{\omega_\lambda(f(x))}$ is a $G$-invariant measure on $Y$.

For $\Phi \in \mathcal{S}(V_K)$ where $\mathcal{S}(V_K)$ denotes the Schwartz-Bruhat space of $V_K$, we define an integral

$$Z_i(\omega, \Phi) = \int_{Y_i} \omega(f(x))\Phi(x)dY(x) \ (i = 1, \cdots, \nu).$$

Now any quasi-character $\omega^{(i)} : K^\times \to \mathbb{C}^\times = \{ z \in \mathbb{C}; z \neq 0 \}$ can be written uniquely as $\omega^{(i)} = |s_i| \cdot \phi_i$ for some $s_i \in \mathbb{C}$ and $\phi_i : R^\times \to C_1^\times = \{ z \in \mathbb{C}; |z| = 1 \}$ where $R^\times$ is the units of $R$. Put $Re \omega^{(i)} = Re s_i \ (i = 1, \cdots, l)$. The following lemma is easy to prove and we omit the proof (cf. F.Sato [15]).

**Lemma 2.2**
If $\text{Re} \, \omega^{(i)} > \lambda_i \ (i = 1, \cdots, l)$, the integral $Z_i(\omega, \Phi)$ is absolutely convergent and holomorphic with respect to $s = (s_1, \cdots, s_l) \in (\mathbb{C}/(\frac{2\pi i}{\log q}\mathbb{Z}))^l \cong \mathbb{C} \times \mathbb{C}$ for $\omega = (|s_1 \cdot \phi_1, \cdots, |s_l \cdot \phi_l)$.

Let $\mathcal{S}'(V_K) = \{z : \mathcal{S}(V_K) \rightarrow \mathbb{C}, \mathbb{C} - \text{linear mapping}\}$ be the space of distributions on $V_K$. By Lemma 2.2, the mapping $\Phi \mapsto Z_i(\omega, \Phi)$ defines a distribution on $V_K$ when $\text{Re} \, \omega^{(i)} > \lambda_i \ (i = 1, \ldots, l)$.

For $(G, \rho^*, V^*)$, we can define similar distribution $Z^*_j(\omega) \ (j = 1, \cdots, \nu)$ given by

$$Z^*_j(\omega, \Phi^*) = \int_{V^*_K} \omega(f^*(y))\Phi^*(y)dy.$$

Now we fix a non-trivial additive character $\psi : K \rightarrow \mathbb{C}_1^\times$ and define the Fourier transformation $\mathcal{S}(V_K^*) \ni \Phi^* \mapsto \hat{\Phi}^* \in \mathcal{S}(V_K)$ by

$$\hat{\Phi}^*(x) = \int_{V_K^*} \Phi^*(y)\psi(\langle x, y \rangle)dy$$

where $dy$ is a Haar measure on $V_K^*$ dual to a fixed Haar measure on $V_K$.

For $\omega = (\omega^{(1)}, \cdots, \omega^{(l)})$, put $\omega^* = \omega^U = (\prod_{i=1}^l \omega^{(i)u_{i1}}, \cdots, \prod_{i=1}^l \omega^{(i)u_{il}})$.

Our purpose is to show that $Z_i(\omega)$ and $Z^*_j(\omega)$ are continued analytically to all $\omega$ and satisfy the functional equation:

$$\hat{Z}_i(\omega) = \sum_{j=1}^\nu \Gamma_{ij}(\omega)Z^*_j(\omega^* \omega_{\lambda^*}) \ (i = 1, \cdots, \nu)$$

under some additional conditions where

$$\hat{Z}_i(\omega)(\Phi^*) = Z_i(\omega, \Phi^*).$$

Recall that $\omega_{\lambda^*} = (|\lambda_1^*, \cdots, |\lambda_l^*)$ with $\det \rho^*(g)^2 = \chi_1^{2\lambda_1^*} \cdots \chi_l^{2\lambda_l^*}$.

Actually when $K$ is a local field of $ch(K) = 0$, then (2.1) is obtained under some conditions and it is called "the fundamental theorem of P.V. over $K$".

§3. Rationality for almost all $p$

For a rational prime $p$, let $K_p$ denotes the local field with the constant field $\mathbb{F}_p$.

For $f \in \mathbb{Z}[x_1, \cdots, x_n]$, we denote $f \mod p \in \mathbb{F}_p[x_1, \cdots, x_n]$ by $f_p$. Then we have the following theorem which is suggested by Professor M.Kashiwara.
Theorem 3.1
For almost all $p$, the integral
\[ Z_p(s, \Phi_p) = \int_{K_p^n} |f_p(x)|^s \Phi_p(x) d_p x \]
is a rational function of $t = p^{-s}$ where $\Phi_p \in \mathfrak{S}(K_p^n)$ and $d_p x$ is a Haar measure on $K_p^n$.

[Proof]
Let $K = \mathbb{Q}((t))$ be a field of formal power series over $\mathbb{Q}$, $X = \Omega^n$ the affine space and $X_K = K^n$. Let $f$ denote the morphism $X \to \Omega$ defined by $f(x)$; then there exists a nonsingular algebraic variety $Y$ and a projective morphism $h : Y \to X$ both defined over $K$ with the following property: let $b$ denote an arbitrary point of $Y_K$, $\mathfrak{M}_K$ the ideal of non-units of $\mathcal{O}_K$; then there exists an ideal basis $(y_1, \cdots, y_n)$ of $\mathfrak{M}_K$, elements $u, v$ of $\mathcal{O}_K - \mathfrak{M}_K$, and integers $N_i \geq 0, \nu_i \geq 1$ for $1 \leq i \leq n$ such that
\[ f \circ h = u \prod_{i=1}^{n} y_i^{N_i}, \quad h^*(dx) = v \prod_{i=1}^{n} y_i^{\nu_i-1} dy. \]
The existence of such a pair $(Y, h)$ is guaranteed by Hironaka's theorem [5] p.109 -p.326. Then for almost all $p$, the reduction modulo $p$ is well-defined and we have similar results for $K_p, f_p, \cdots$ etc. Then by just similar argument as in Appendix of Igusa [11], we obtain our result. □

Remark 3.2
Let $K$ be a number field. For $f \in \mathcal{O}_K[x_1, \cdots, x_n]$, we have a similar result as Theorem 3.1 for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$.

§4. Functional equations

Lemma 4.1
Let $G$ denote a locally compact totally disconnected group, $H$ a closed subgroup of $G$, $X = H \backslash G$, and $\omega : G \to \mathbb{C}^\times$ a quasicharacter. Put
\[ \xi_X(\omega) = \{ T \in \mathfrak{S}(X); gT = \omega(g)^{-1} T \text{ for all } g \in G \}. \]
Then we have $\dim_{\mathbb{C}} \xi_X(\omega) \leq 1$. Moreover $\dim_{\mathbb{C}} \xi_X(\omega) = 1$ if and only if $\Delta_G \cdot \omega|_H = \Delta_H$ where $\Delta_G, \Delta_H$ denotes the module of $G, H$ respectively.

[Proof]
Let \((G, \rho, V)\) and its dual \((G, \rho^*, V^*)\) be \(K\)-regular P.V.'s with
\[
\# \rho(G)_K \backslash Y_K = \nu < +\infty
\]
where \(K\) is a local field of characteristic \(p\). Then, by Proposition 1.3, we have
\[
Y_K = Y_1 \cup \cdots \cup Y_\nu \quad \text{and} \quad Y_K^* = Y_1^* \cup \cdots \cup Y_\nu^*.
\]
i.e., \(\# \rho^*(G)_K \backslash Y_K^* = \nu\).
As in §2, we can define the zeta distribution \(Z_i(\omega, \Phi)\) (resp. \(Z_i^*(\omega, \Phi^*)\)) which is convergent when \(\text{Re } \omega^{(j)} > \lambda_j\) (resp. \(\text{Re } \omega^{(j)} > \lambda_j^*\)) \((1 \leq i \leq \nu, 1 \leq j \leq l)\).

We denote by \(Z_i(\omega)\) the distribution defined by \(\Phi \mapsto Z_i(\omega, \Phi)\) etc.

**Proposition 4.2**

We have

\[
(1) \quad Z_j^*(\omega^* \omega_{\lambda^*}) \in \xi_{\lambda^*}^{(j)}(\omega^* \omega_{\lambda^*})
\]
and

\[
(2) \quad Z_i(\omega) \in \xi_{\lambda_i}^{(j)}(\omega^* \omega_{\lambda^*}).
\]

\[(i, j = 1, \ldots, \nu)\]

[Proof]
By a direct calculation, we obtain our results. \(\square\)

**Proposition 4.3**

Let \(K\) be a local field of characteristic \(p > 0\) with the module \(q\). For \(\omega = (\omega^{(1)}, \cdots, \omega^{(l)})\) with \(\omega^{(i)} = \omega_s \cdot \phi_i\) (\(\phi_i(\pi) = 1\) for a prime element \(\pi\)), assume that \(Z_i(\omega, \Phi)\) and \(Z_j^*(\omega, \Phi^*)\) are rational functions of \(q^{-s_1}, \cdots, q^{-s_l}\). Then for all \(\Phi^* \in \mathfrak{S}(Y_K^*)\), we have
\[
Z_i(\omega, \hat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)
\]
for \(i, j = 1, \cdots, \nu\).

[Proof]
Since \(Z_i(\omega, \Phi)\) and \(Z_j(\omega, \Phi^*)\) are rational functions, it is defined for all \(\omega\) except poles and hence by Lemma 4.1 and Proposition 4.2, we have our result. \(\square\)

**Theorem 4.4**

Let \((G, \rho, V)\) be a \(K\)-regular P.V. satisfying the following conditions:

\((C1)\) its dual \((G, \rho^*, V^*)\) is a \(K\)-regular P.V. such that
\[
\# \rho^*(G)_K \backslash V_K^* < +\infty,
\]
\((C2)\) for \(x \in S_K^*\), there exists \(\chi \in X_1(G)_K\) satisfying \(\chi(G, K) \not\subset R^x\) where \(R^x\) is the units of the maximal compact subring \(R\) of \(K\) and
\((C3)\) \(Z_j(\omega, \Phi)\) is a rational function of \(q^{-s_1}, \cdots, q^{-s_l}\) where
\[
\omega = (\omega^{(1)}, \cdots, \omega^{(l)}) \quad \text{with} \quad \omega^{(i)} = \omega_s \quad (1 \leq i \leq l).
\]
Then we have the functional equation...
\[ Z_i(\omega, \hat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*}, \Phi^*) \]

for all \( \Phi^* \in \mathcal{S}(V_K^*) \) for \( i, j = 1, \ldots, \nu \) where \( \nu = \# \rho^*(G)K \backslash \mathrm{Y}_K \).

**Proof**

The condition \((G2)\) corresponds to Lemma 2.2 in F.Sato [15] p.474 for the case of \( ch(K) = 0 \). Then the proof is just similar as the case of \( ch(K) = 0 \) (using Proposition 4.3) (See Igusa [9] and F.Sato [15] p.477).

Now let \((G, \rho, V)\) be a reductive \(\mathbb{Q}\)-regular P.V. Then for almost all \( p \), we have a reduction modulo \( p \) and we obtain \( K_p \)-regular P.V. \((G_p, \rho_p, V_p)\) where \( K_p \) is a local field with the constant field \( \mathbb{F}_p \).

**Assumption A**

Assume that \( \# \rho_p(G)K \backslash S_{K_p} = +\infty \) and for \( x \in S_{K_p} \), there exists \( \chi \in X_1(G_p)_{K_p} \), satisfying \( \chi(G_{p,x,K}) \neq \mathbb{R}_p^\times \) for almost all \( p \).

Let \((G, \rho, V)\) be a reductive \(\mathbb{Q}\)-regular P.V. with \(\text{Assumption A} \). Let \( f_1, \ldots, f_l \) be basic \(\mathbb{Q}\)-relative invariants with \( \mathbb{Z}\)-coefficients. Denote \( |f_1 \mod p|_{K_p}^s, \ldots, |f_l \mod p|_{K_p}^s \) by \( |f^{(p)}(x)|_{K_p}^s \), \( Z_i^p(s, \Phi_p) = \int_{(Y_{K_p} \cap \mathrm{Y}_1)} \int_{(Y_{K_p} \cap \mathrm{Y}_1)} |f^{(p)}(x)|_{K_p}^s \Phi_p(x) dY_p(x) \)

for \( \Phi_p \in \mathcal{S}(V_{K_p}) \).

**Theorem 4.5**

Let \((G, \rho, V)\) be a reductive \(\mathbb{Q}\)-regular P.V. with \(\text{Assumption A} \). Then for almost all rational prime \( p \), the integral \( Z_i^p(s, \Phi_p) \) \(( i = 1, \ldots, \nu_p, Y_{K_p} = Y_1 \cup \cdots \cup Y_{\nu_p} \) is a rational function and satisfies the functional equation:

\[ Z_i^p(s, \hat{\Phi}_p) = \sum_{j=1}^{\nu_p} \Gamma_{ij}(s)Z_j^p(s^*, \Phi_p) \]

\(( i = 1, \ldots, \nu_p \).

When \( l = 1 \), we have \( s^* = \frac{d}{2} - s \) with \( n = \dim V \) and \( d = \deg f \). In general, for \( \omega = \omega_s = \omega_{s_1} \cdots \omega_{s_l} \), we have \( \omega_{s^*} = \omega^*\omega_{\lambda^*} \).

**Proof**

By Theorem 4.4 and using the results of §1 and §3, we obtain our result.
REFERENCES

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