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An explicit formula for zeta functions associated with quadratic forms[1]

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Here we shall consider the following problem taking up the space of quadratic forms.

Problem. *Explicitly write down the zeta functions of prehomogeneous vector spaces.*

§1.

1.1. Let

$V=V_n:=\{x\in M_n\mid x={}^tx\}$,

$G:=GL_n$,

$\rho(g)x:=gx\,{}^tg\quad(g\in G,\ x\in V)$,

$V_n(\mathbb{R})\setminus\{\det x=0\}=V_n^n\cup V_n^{n-1}\cup\cdots\cup V_n^0$, where $x\in V_n^i$ if and only if x has i positive and $n-i$ negative eigenvalues,

$L\subset V_n(\mathbb{R})$ an $SL_n(\mathbb{Z})$-invariant lattice,

$L^{(i)}:=L\cap V_n^i$,

$c_n:=\frac{2\prod_{k=1}^n\Gamma(\frac{k}{2})}{\pi^{n(n+1)/4}}$,

$\mu(x)$:='size' of $\rho(SL_n(\mathbb{Z}))x$ for $x\in L$ with $\det x\neq 0$ (see [**Sa**, (1.5)] and (1.2) below),

$\zeta_i(s,L):=c_n\sum_{x\in L_n^{(i)}/SL_n(\mathbb{Z})}\frac{\mu(x)}{|\det x|^s}$.

[1]Notes by Akihiko Gyoja.

1.2. Example. For $x \in L^{(n)}$, we have

$$c_n \mu(x) = \varepsilon(x)^{-1},$$

where

$$\varepsilon(x) := |\{\gamma \in SL_n(\mathbb{Z}) \mid \gamma x\, {}^t\gamma = x\}|.$$

In particular,

$$\zeta_n(s, L) = \sum_{x \in L^{(n)}/SL_n(\mathbb{Z})} \frac{1}{|\det x|^s \varepsilon(x)}.$$

1.3. Remark. If $n \geq 3$, there are exactly 2 possibilities of the choice of L (up to constant multiple), i.e.,

$$\begin{aligned} L_n :&= \{(x_{ij}) \in V_n \mid x_{ij} \in \mathbb{Z}\} \\ &= \text{the integral lattice,} \end{aligned}$$

and

$$\begin{aligned} L_n^* :&= \{(x_{ij}) \in V_n \mid x_{ii} \in \mathbb{Z}, \quad x_{ij} \in \tfrac{1}{2}\mathbb{Z}\ (i \neq j)\} \\ &= \text{the half integral lattice.} \end{aligned}$$

Cf. [**IS2**]. Note that L_n^* is the dual lattice of L_n with respect to the bilinear form $\mathrm{tr}(xy)$, and its elements can be identified with the integral quadratic forms.

If $n = 2$, there are 4 lattices, i.e., besides L_2 and L_2^*, we have the lattices

$$M := \{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_2 \mid a + b + c \equiv 0 \mod 2\}, \text{ and}$$
$$N := \{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_2^* \mid a \equiv b \equiv c \mod 2\}.$$

Cf. [**IS3**]. (Since $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we can see that M and N are $SL_2(\mathbb{Z})$-invariant simply by noting that they are invariant under these two matrices.)

§2. Review of quadratic forms (1).

2.1. Genera and classes. For $x_1, x_2 \in V_n(\mathbb{Q})$, we say that

(2.1.1) x_1 and x_2 belong to the *same genus* if $x_1 \underset{GL_n(\mathbb{Z}_v)}{\sim} x_2$ for all places $v \leq +\infty$, where $\mathbb{Z}_v = \mathbb{Z}_p$ if $v = p$ and $\mathbb{Z}_\infty = \mathbb{R}$, and

(2.1.2) x_1 and x_2 belong to the *same class* if $x_1 \underset{SL_n(\mathbb{Z})}{\sim} x_2$.

Then each genus consists of several classes, whose cardinality is known to be finite, and is called the *class number.* The class number measures the difference between the local theory and the global theory. It is important but rarely calculable.

2.1.3. Remark. If x_1 and x_2 belong to the same genus, then $\det x_1 = \det x_2$. (In fact, $c := \det x_1 / \det x_2 \in \mathbb{Q}^\times$ belongs to $\mathbb{Z}_v^{\times 2}$ for all $v \leq +\infty$. Hence $c = 1$.)

2.2. Siegel Mass formula. For $x \in L_n^*$,

$$2^{-1} \cdot p^{-\frac{n(n-1)\nu}{2}} \times (\#O(x \mod p^\nu)) \tag{2.2.1}$$

becomes stable as $\nu \to +\infty$, where $O(x \mod p^\nu)$ is the orthogonal group contained in $GL_n(\mathbb{Z}/p^\nu\mathbb{Z})$. We put $\alpha_p(x) := \lim_{\nu \to +\infty} (2.2.1)$, and call it the *local density.* An explicit formula for $\alpha_p(x)$ will be given in (3.3).

Let $\mathcal{L}$ be a genus, and d the common value of $\det x$ ($x \in \mathcal{L}$). Cf. (2.1.3). The Siegel Mass formula says

$$\sum_{\substack{x \in \mathcal{L}/SL_n(\mathbb{Z}) \\ \det x = d}} \mu(x) = \frac{2|d|^{\frac{n+1}{2}}}{\prod_p \alpha_p(x)}. \tag{2.2.2}$$

§3. Review of quadratic forms (2).

3.1. Equivalence of quadratic forms over fields.

3.1.1. Discriminant. For $x \in V_n(\mathbb{Q}_v)$, if $x \underset{GL_n(\mathbb{Q}_v)}{\sim} \mathrm{diag}(a_1, \cdots, a_m, 0, \cdots, 0)$ with $a_i \in \mathbb{Q}_v^\times$, put

$$\Delta_v(x) := \prod_{i=1}^{m} a_i.$$

Then $\Delta_v(x)$ is well-defined as an element of $\mathbb{Q}_v^\times/\mathbb{Q}_v^{\times 2}$ and called the *discriminant* of x.

3.1.2. Hasse invariant. For $0 \neq a, b \in \mathbb{Z}_v$ $(v \leq +\infty)$, the Hilbert symbol $(a,b)_v = \pm 1$ is defined so that $= +1$ iff $ax^2 + by^2 = 1$ has a solution (x,y) in $\mathbb{Q}_v$.[2] The Hilbert symbol defines a symmetric bilinear form on $\mathbb{Q}_v^\times/\mathbb{Q}_v^{\times 2}$ (i.e., $(aa', b)_v = (a,b)_v(a',b)_v$ and $(a,b)_v = (b,a)_v$). Moreover it is non-degenerate and satisfies $(a,-a)_v = (a, 1-a)_v = 1$. For any $x \in V_n(\mathbb{Q}_v)$ there exists $g \in GL_n(\mathbb{Q}_v)$ such that

$$gx\,{}^tg = \mathrm{diag}(a_1, \cdots, a_m, 0, \cdots, 0) \quad (a_i \in \mathbb{Q}_v^\times)$$

(Cf. [**Se**, Chapter 4, Theorem 1].) Put

$$S_v(x) := \prod_{1 \leq i \leq j \leq m} (a_i, a_j)_v.$$

Then $S_v(x)$ depends only on x, and is independent of the choice of g.[3] The invariant $S_v(x)$ is called the *Hasse invariant* of x.

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[2]Note that $(a,b)_v = +1$ iff $ax^2 + by^2 = z^2$ has a solution $(x,y,z) \neq (0,0,0)$ in $\mathbb{Q}_v$. In fact, if it has a solution (x_0, y_0, z_0) with $z_0 \neq 0$ (resp. $z_0 = 0$), then $(x,y) = (x_0/z_0, y_0/z_0)$ (resp. $(x,y) = (\frac{1}{2}x_0((ax_0^2)^{-1}+1), \frac{1}{2}y_0((ax_0^2)^{-1}-1)$) is a solution of $ax^2 + by^2 = 1$. Then the above definition is equivalent to the one given in [**Se**, Chapter 3].

[3]In fact, the well-definedness of $\varepsilon_v(x) := \prod_{i<j}(a_i, a_j)_v$ is proved in [**Se**, Chapter 4, Theorem 5]. Since

$$\prod_i (a_i, a_i)_v = \prod_i (a_i, -a_i)_v (a_i, -1)_v = (\Delta_v(x), -1)_v,$$

we have $S_v(x) = \varepsilon_v(x)(\Delta_v(x), -1)_v$, and we get the well-definedness.

Remark. As we have seen above, our definition of the Hasse invariant is different from the invariant $\varepsilon(x)$ studied in [**Se**]. Our convention is the same as [**O**] and [**Kit**].

Example. Assume that $p \neq 2$, $d_1, d_2 \in \mathbb{Z}_p^\times$ and $\sigma_1, \sigma_2 \in \mathbb{Z}$. Then

$$(p^{\sigma_1} d_1, p^{\sigma_2} d_2)_p = \left(\frac{-1}{p}\right)^{\sigma_1 \sigma_2} \left(\frac{d_1}{p}\right)^{\sigma_2} \cdot \left(\frac{d_2}{p}\right)^{\sigma_1},$$

where$\left(\frac{*}{p}\right)$ is the Legendre symbol. [**Se**, Chapter 3, Theorem 1].

Local Theory.

3.1.3. (1) A $GL_n(\mathbb{Q}_p)$-isomorphism class of $x \in V_n(\mathbb{Q}_p)$ is determined by

(a) rank x,

(b) $\Delta_p(x) \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$, and

(c) $S_p(x) =$ Hasse invariant.

Cf. [**Se**, Chapter 4, Theorem 7]. (Note that if $\det x \neq 0$, $\Delta_p(x) = (\det x \mod \mathbb{Q}_p^{\times 2})$.)

(2) A $GL_n(\mathbb{R})$-isomorphism class of $x \in V_n(\mathbb{R})$ is determined by the signature of x.

Global theory.

The determination of the equivalence relation of quadratic forms over $\mathbb{Q}$ consists of the following two steps.

3.1.4. Hasse principle. The diagonal mapping

$$\delta : V_n(\mathbb{Q})/GL_n(\mathbb{Q}) \to \prod_{v \leq +\infty} V_n(\mathbb{Q}_v)/GL_n(\mathbb{Q}_v)$$

is injective. Cf. [**Se**, Chapter 4, Theorem 9].

3.1.5. Image of δ. Assume that $(x_v)_v \in \prod_{v \leq +\infty} V_n(\mathbb{Q}_v)$ is given. In order that the equivalence class of $(x_v)_v$ belongs to the image of δ, it is necessary and sufficient that the following four conditions are satisfied.

(0) For all $v \leq +\infty$, $\mathrm{rad}(x_v) := \{a \in, \mathbb{Q}_v^n \mid a x_v b = 0 \text{ for all } b \in \mathbb{Q}_v^n\}$ are defined over $\mathbb{Q}$, and $\mathrm{rad}(x_v) \cap \mathbb{Q}^n$ are independent of v.

(1) There exists $d \in \mathbb{Q}^\times$ such that $d \in \Delta_v(x_v) \cdot \mathbb{Q}_v^{\times 2}$ for all $v \le +\infty$;

(2) $S_v(x_v) = 1$ for almost all v;

(3) $\prod_{v \le +\infty} S_v(x_v) = 1$.

Cf. [**Se**, Chapter 4, Prop. 7].

3.2. Equivalence of quadratic forms over rings. Here and below, we assume that the $\mathbb{Z}$-structure of V_n is given by $V_n(\mathbb{Z}) = L_n^*$.

3.2.1. Local theory. First assume that $p \neq 2$. A complete list of representatives of $\{x \in V_n(\mathbb{Z}_p) \mid \det x \neq 0\}/GL_n(\mathbb{Z}_p)$ is given by

$$\text{(1)} \qquad \operatorname{diag}(p^{\sigma_1} \operatorname{diag}(\underbrace{1, \cdots, 1, d_1}_{n_1}), \cdots, p^{\sigma_s} \operatorname{diag}(\underbrace{1, \cdots, 1, d_s}_{n_s}))$$

with $d_i \in \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$ $(1 \le i \le s)$. The complete set of invariants is given by[4]

$$\left\{ \begin{matrix} 0 \le \sigma_1 < \sigma_2 < \cdots < \sigma_s \\ n_i \in \mathbb{Z}_{>0}, \ \left(\dfrac{d_i}{p}\right) = \pm 1, \ (1 \le i \le s) \end{matrix} \right\},$$

i.e.,

(a) two elements x and y are $GL_n(\mathbb{Z}_p)$-quivalent iff these invariants for x and y are the same and

(b) every such $\{\sigma_i, n_i, \pm 1\}$ are given by some x. ((b) is trivial.)

3.2.2. Global theory. Let us consider when

(1) a collection $(x_v)_{v \le +\infty} \in \prod_{v \le +\infty} V_n(\mathbb{Z}_v)/GL_n(\mathbb{Z}_v)$ comes from an element of $V_n(\mathbb{Z})$,

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[4] See[**O**] for the detail. It is enough to read pp. 81–89 in order to understand basic concepts. Then read pp.227-233 and pp.243–247 in order to understand the above result. In the terminology of [**O**], the decomposition into the direct sum of $p^{\sigma_i} \operatorname{diag}(1, \cdots, 1, d_i)$ is the Jordan splitting. Each block gives a p^{σ_i}-modular lattice. Concerning the isomorphism class of each block, see [**O**, 92:1], which, in fact, can be proved directly. Note that we are assuming $p \neq 2$. Glance over pp. 250– 279 and see how complicated the similar result in the case $p = 2$.

assuming that $d \in \mathbb{Q}^\times$ is given, and that $x_v \in V_n(\mathbb{Z}_v)$ and $\det(x_v) = d$ for all $v \le +\infty$.

As we have seen in (3.1.5), it is necessary that (3.1.5, (1)–(3)) are satisfied. By our assumption, (3.1.5, (1)) is satisfied. Note that $d \in \mathbb{Z}_p^\times$ for almost all p. If $x_v \underset{GL_n(\mathbb{Z}_v)}{\sim} \mathrm{diag}(1, \cdots, 1, \varepsilon)$, then $\varepsilon \in d \cdot \mathbb{Z}_v^{\times 2}$ and hence $S_v(x_v) = (\varepsilon, \varepsilon)_v = (d, d)_v = 1$ for almost all $v \le +\infty$ by [**Se**, Chapter 3, Theorem 3], i.e., (3.1.5, (2)) is satisfied. Then only (3.1.5, (3)) is essential. In fact, we can show the following.

For $d \in \mathbb{Q}^\times$, the diagonal mapping

$$\begin{aligned}\delta : \{x \in V_n(\mathbb{Z}) \mid \det(x) = d\}/ \approx \\ \to \{(x_v)_v \in \prod_{v \le +\infty} V_n(\mathbb{Z}_v) \mid \det(x_v) = d\mathbb{Z}_v^{\times 2} \textit{ for all } v, \\ \textit{and} \prod_{v \le +\infty} S_v(x_v) = 1\}/ \prod_{v \le +\infty} GL_n(\mathbb{Z}_v)\end{aligned}$$

is a bijection, where $x_1 \approx x_2$ means that x_1 and x_2 belong to the same genus.[5]

Proof. The injectivity is trivial. Let us prove that δ is surjective. By (3.1.5), there exists $y \in V_n(\mathbb{Q})$ such that $y \underset{GL_n(\mathbb{Q}_v)}{\sim} x_v$ for all $v \le +\infty$. Note that $y \in V_n(\mathbb{Z}_p)$ and $\det(y) \in \mathbb{Z}_p^\times$ for almost all $p < \infty$. For such $p \ne 2$, $y \underset{GL_n(\mathbb{Z}_p)}{\sim} x_p$ by (3.2.1), i.e., there exists $g \in GL_n(\mathbb{Q}_A)$ such that $y = g(x_v)_v {}^t g$. Decompose $g = \gamma^{-1} g'$ according to the decomposition $GL_n(\mathbb{Q}_A) = GL_n(\mathbb{Q}) \times \prod_{v \le +\infty} GL_n(\mathbb{Z}_v)$.[6] Put $x := \gamma y {}^t\gamma$. Then $x = g'(x_v)_v {}^t g'$ with $g' \in \prod_{v \le +\infty} GL_n(\mathbb{Z}_v)$, and hence $x = \gamma y {}^t \gamma \in V_n(\mathbb{Q}) \cap \prod_v V_n(\mathbb{Z}_v) = V_n(\mathbb{Z})$. ∎

We record a modification of (3.2.2).

[5]By the same argument as above, we can see that, if $\det(x_v) \in d\mathbb{Z}_v^{\times 2}$ for all v, then $S_v(x_v) = 1$ for almost all v, and hence $\prod_v S_v(x_v)$ is a finite product.

[6]In fact, $|GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{Q}_A) / \prod_{v \le +\infty} GL_n(\mathbb{Z}_v)|$ =(class number of $\mathbb{Q}$)= 1.

3.2.3. In the same notation as (3.2.2), for $0 < d \in 2^{-n}\mathbb{Z}$, the diagonal mapping

$$\begin{aligned}&\delta : \{x \in L_n^{*(n)} \mid \det(x) = d\}/\approx \\ (1)\qquad &\to \{(x_p)_p \in \prod_{p<\infty} V_n(\mathbb{Z}_p) \mid \det(x_p) \in d\mathbb{Z}_p^{\times 2} \text{ for all } p < +\infty, \text{ and} \\ &\prod_{p<+\infty} S_p(x_p) = 1\}/\prod_{p<\infty} GL_n(\mathbb{Z}_p).\end{aligned}$$

is a bijection.

3.3. Local density $\alpha_p(x)$ for x as (3.2.1, (1)) is given by

$\alpha_p(x) = 2^{s-1} \cdot p^w \cdot P \cdot E$, where

$w := \sum_{i=1}^{s} \frac{n_i(n_i+1)}{2}\sigma_i + \sum_{j<i} n_i n_j \sigma_j$,

$P := \prod_{i=1}^{s} P\left(\left[\frac{n_i}{2}\right]\right)$ with $P(m) := \prod_{i=1}^{m}\left(1 - \frac{1}{p^{2i}}\right)$,

$E := \prod_{n_i\,:\,\text{even}} \left(1 + \frac{1}{p^{n_i/2}}\left(\frac{(-1)^{n_i/2} d_i}{p}\right)\right)^{-1}$.

See [**Ko**] and [**Kit**].

§4. Calculation in some simple cases.

4.1. Now we calculate

$$(4.1.1) \qquad \zeta_n(s, L_n^*) = c_n \sum_{x \in L_n^{*(n)}/SL_n(\mathbb{Z})} \frac{\mu(x)}{|\det x|^x}.$$

(For the sake of simplicity, we restrict ourselves to $\zeta_i(s, L)$ with $i = n$ and $L = L_n^*$. For general i with $L = L_n$ or $L = L_n^*$, see [**IS1**]. For $L = M$ or $L = N$, see [**IS3**].) By (3.2.3), we have

$$(4.1.2) \qquad \zeta_n(s, L_n^*) = c_n \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s} \left\{ \sum_{(x_p)_p} \left(\sum_{\substack{x \in L_n^{*(n)}/SL_n(\mathbb{Z}) \\ x \underset{GL_n(\mathbb{Z}_p)}{\sim} x_p, \text{ for all } p}} \mu(x) \right) \right\},$$

where in the second summation on the right hand side, $(x_p)_p$ runs over the right hand side of (3.2.3, (1)). By Siegel's Mass formula (2.2.2), the inside of { } of (4.1.2) is equal to

$$(4.1.3)\qquad \begin{aligned} &2d^{\frac{n+1}{2}} \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2} \\ \prod_p S_p(x_p)=1}} \frac{1}{\prod_p \alpha_p(x)} \\ &= d^{\frac{n+1}{2}} \left\{ \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \frac{1}{\prod_p \alpha_p(x_p)} + \sum_{\substack{(x_p)_p \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \frac{1}{\prod_p \alpha_p(x_p) S_p(x_p)} \right\} \end{aligned}$$

By (4.1.2) and (4.1.3), we get

$$(4.1.4)\qquad \begin{aligned} &c_n^{-1}\zeta_n(s, L_n^*) \\ &= \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s+\frac{n+1}{2}} \prod_p \left(\sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \alpha_p(x_p)^{-1} \right) \\ &+ \sum_{d \in 2^{-n}\mathbb{Z}_{>0}} d^{-s+\frac{n+1}{2}} \prod_p \left(\sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} (\alpha_p(x_p) S_p(x_p))^{-1} \right). \end{aligned}$$

4.2. We proceed to the remaining calculation taking up as an example the case $n = 3$. Then the complete list of representatives x as in (3.2.1, (1)) with $\det(x) = d \in 2^{-n}\mathbb{Z}_{>0}$ ($d = p^t d_0$, $(d_0, p) = 1$) is given by

$(1,1,1)$: $x = \mathrm{diag}(p^{\sigma_1}d_1, p^{\sigma_2}d_2, p^{\sigma_3}d_3)$, $t = \sigma_1 + \sigma_2 + \sigma_3$, $d_0 = d_1 d_2 d_3$,

$(1,2)$: $x = \mathrm{diag}(p^{\sigma_1}d_1, p^{\sigma_2}, p^{\sigma_2}d_2)$, $t = \sigma_1 + 2\sigma_2$, $d_0 = d_1 d_2$,

$(2,1)$: $x = \mathrm{diag}(p^{\sigma_1}, p^{\sigma_1}d_1, p^{\sigma_2}d_2)$, $t = 2\sigma_1 + \sigma_2$, $d_0 = d_1 d_2$,

(3) : $x = \mathrm{diag}(p^{\sigma_1}, p^{\sigma_1}, p^{\sigma_1}d_1)$, $t = 3\sigma_1$, $d_0 = d_1$,

where $(1,1,1)$ etc. indicates $(n_1, n_2, \cdots)$ in (3.2.1, (1)).

Let $p \neq 2$. Fix an element $\varepsilon \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^{\times 2}$. For $d = p^t d_0$ with $(d_0, p) = 1$, we get using (3.3) that

$$\sum_{\substack{x_p \in V(\mathbb{Z}_p)/GL_n(\mathbb{Z}_p) \\ \det x_p \in d\mathbb{Z}_p^{\times 2}}} \alpha_p(x_p)^{-1} \tag{4.2.1}$$

$$= \sum_{\substack{0 \le \sigma_1 < \sigma_2 < \sigma_3 \\ d_1, d_2, d_3 \in \{1, \varepsilon\} \\ \sigma_1 + \sigma_2 + \sigma_3 = t \\ d_1 d_2 d_3 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^2 \cdot p^{3\sigma_1 + 2\sigma_2 + \sigma_3}\right)^{-1} \quad \text{contribution of } (1,1,1)$$

$$+ \sum_{\substack{0 \le \sigma_1 < \sigma_2 \\ d_1, d_2 \in \{1, \varepsilon\} \\ \sigma_1 + 2\sigma_2 = t \\ d_1 d_2 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^1 \cdot p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \left(1 + \frac{1}{p}\left(\frac{-d_2}{p}\right)\right) \quad \text{contribution of } (1,2)$$

$$+ \sum_{\substack{0 \le \sigma_1 < \sigma_2 \\ d_1, d_2 \in \{1, \varepsilon\} \\ 2\sigma_1 + \sigma_2 = t \\ d_1 d_2 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^1 \cdot p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \left(1 + \frac{1}{p}\left(\frac{-d_1}{p}\right)\right) \quad \text{contribution of } (2,1)$$

$$+ \sum_{\substack{0 \le \sigma_1 \\ d_1 \in \{1, \varepsilon\} \\ 3\sigma_1 = t \\ d_1 \in d_0 \mathbb{Z}_p^{\times 2}}} \left(2^0 \cdot p^{6\sigma_1} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \quad \text{contribution of } (3)$$

$$= \sum_{\substack{0 \le \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_1 + \sigma_2 + \sigma_3 = t}} \left(p^{3\sigma_1 + 2\sigma_2 + \sigma_3}\right)^{-1} \tag{4.2.2}$$

$$+ \sum_{\substack{0 \le \sigma_1 < \sigma_2 \\ \sigma_1 + 2\sigma_2 = t}} \left(p^{3\sigma_1 + 3\sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1}$$

$$+ \sum_{\substack{0 \le \sigma_1 < \sigma_2 \\ 2\sigma_1 + \sigma_2 = t}} \left(p^{5\sigma_1 + \sigma_2} \left(1 - \frac{1}{p^2}\right)\right)^{-1}$$

$$+ \sum_{\substack{0 \le \sigma_1 \\ 3\sigma_1 = t}} \left(p^{6\sigma_1} \left(1 - \frac{1}{p^2}\right)\right)^{-1}$$

What is important concerning the last expression is that it does not contain d any more. Similar expression can be obtained also for $p = 2$. (This case is much more difficult. See [**IS2**, pp.17–24].) Hence we can express the first summation of (4.1.4) as a product, for all prime numbers p, of

$$\begin{aligned}
&\sum_{t\geq 0} p^{t(-s+2)} \left\{ \sum_{\substack{0\leq\sigma_1<\sigma_2<\sigma_3 \\ \sigma_1+\sigma_2+\sigma_3=t}} p^{-3\sigma_1-2\sigma_2-\sigma_3} \right. \\
&\quad + \left(1-\frac{1}{p^2}\right)^{-1} \sum_{\substack{0\leq\sigma_1<\sigma_2 \\ \sigma_1+2\sigma_2=t}} p^{-3\sigma_1-3\sigma_2} \\
&\quad + \left(1-\frac{1}{p^2}\right)^{-1} \sum_{\substack{0\leq\sigma_1<\sigma_2 \\ 2\sigma_1+\sigma_2=t}} p^{-5\sigma_1-\sigma_2} \\
&\quad \left. + \left(1-\frac{1}{p^2}\right)^{-1} \sum_{\substack{0\leq\sigma_1 \\ 3\sigma_1=t}} p^{-6\sigma_1} \right\} \\
&= \sum_{0\leq\sigma_1<\sigma_2<\sigma_3} u^{\sigma_1+\sigma_2+\sigma_3} p^{-\sigma_1+\sigma_3} + \cdots
\end{aligned}$$

$$\begin{aligned}
&= \frac{p^2u^3}{(1-u^3)(1-pu^2)(1-pu)} \\
&\quad + \left(1-\frac{1}{p^2}\right)^{-1} \frac{pu^2}{(1-u^3)(1-pu^2)} \\
&\quad + \left(1-\frac{1}{p^2}\right)^{-1} \frac{pu}{(1-u^3)(1-pu)} \\
&\quad + \left(1-\frac{1}{p^2}\right)^{-1} \frac{1}{1-u^3}
\end{aligned} \tag{4.2.3}$$

$$= \left(1-\frac{1}{p^2}\right)^{-1} \frac{1}{(1-pu)(1-pu^2)}, \tag{4.2.4}$$

where $u = p^{-s}$. In the case $p = 2$, we have in place of (4.2.4)

$$\frac{1}{3}\frac{u^{-2}}{(1-2u)(1-2u^2)} = \left(1-\frac{1}{p^2}\right)^{-1} \frac{1}{(1-pu)(1-pu^2)} \cdot \frac{2^{2s}}{4}. \tag{4.2.5}$$

Next we calculate the second summand of (4.1.4). For x of type $(1,1,1)$, $(1,2)$, $(2,1)$ and (3), the respective Hasse invariants are given as follows.

$(1,1,1): S_p(x) = \left(\frac{-1}{p}\right)^{\sum_{i<j}\sigma_i\sigma_j}\left(\frac{d_0}{p}\right)^{\sigma_1+\sigma_2}\left(\frac{d_1}{p}\right)^{\sigma_1+\sigma_3}\left(\frac{d_2}{p}\right)^{\sigma_2+\sigma_3}$

$(1,2): S_p(x) = \left(\frac{-d_2}{p}\right)^{\sigma_1+\sigma_2}$

$(2,1): S_p(x) = \left(\frac{-d_1}{p}\right)^{\sigma_1+\sigma_2}$

$(3): S_p(x) = 1$

Hence replacing $\alpha_p(x_p)^{-1}$ in (4.2.1) with $\alpha_p(x_p)^{-1}\cdot S_p(x_p)$, we get in place of (4.2.2)

$$\sum_{\substack{0\le\sigma_1<\sigma_2<\sigma_3\\ \sigma_1+\sigma_2+\sigma_3=t\\ \sigma_1\equiv\sigma_2\equiv\sigma_3}} \left(p^{3\sigma_1+2\sigma_2+\sigma_3}\right)^{-1}$$
$$+\sum_{\substack{0\le\sigma_1<\sigma_2\\ \sigma_1+2\sigma_2=t\\ \sigma_1\equiv\sigma_2}} \left(p^{3\sigma_1+3\sigma_2}\left(1-\frac{1}{p^2}\right)\right)^{-1} + \sum_{\substack{0\le\sigma_1<\sigma_2\\ \sigma_1+2\sigma_2=t\\ \sigma_1\equiv\sigma_2+1}} \left(p^{3\sigma_1+3\sigma_2}\left(1-\frac{1}{p^2}\right)\right)^{-1}\frac{1}{p}$$
$$+\sum_{\substack{0\le\sigma_1<\sigma_2\\ 2\sigma_1+\sigma_2=t\\ \sigma_1\equiv\sigma_2}} \left(p^{5\sigma_1+\sigma_2}\left(1-\frac{1}{p^2}\right)\right)^{-1} + \sum_{\substack{0\le\sigma_1<\sigma_2\\ 2\sigma_1+\sigma_2=t\\ \sigma_1\equiv\sigma_2+1}} \left(p^{5\sigma_1+\sigma_2}\left(1-\frac{1}{p^2}\right)\right)^{-1}\frac{1}{p}$$
$$+\sum_{\substack{0\le\sigma_1\\ 3\sigma_1=t}} \left(p^{6\sigma_1}\left(1-\frac{1}{p^2}\right)\right)^{-1},$$

where the congruence relation is considered modulo 2. Therefore we get in place of (4.2.3)

$$\begin{aligned}
&\frac{p^4u^6}{(1-u^3)(1-p^2u^4)(1-p^2u^2)}\\
&+\left(1-\frac{1}{p^2}\right)^{-1}\frac{p^2u^4+u^2}{(1-u^3)(1-p^2u^4)}\\
&+\left(1-\frac{1}{p^2}\right)^{-1}\frac{p^2u^2+u}{(1-u^3)(1-p^2u^2)}\\
&+\left(1-\frac{1}{p^2}\right)^{-1}\frac{1}{1-u^3}\\
&=\left(1-\frac{1}{p^2}\right)^{-1}\frac{1}{(1-u)(1-p^2u^2)}.
\end{aligned}\tag{4.2.6}$$

In the case $p = 2$, we have in place of (4.2.6)

$$-\frac{1}{3}\frac{u^{-2}}{(1-u)(1-4u^2)} = -\left(1-\frac{1}{p^2}\right)^{-1}\frac{1}{(1-u)(1-p^2u^2)}\cdot\frac{2^{2s}}{4}. \tag{4.2.7}$$

Summing up (4.2.4)–(4.2.7), we get[7]

$$\zeta_3(s, L_3^*) = \frac{2^{2s}}{24}(\zeta(s-1)\zeta(2s-1) - \zeta(s)\zeta(2s-2)).$$

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[7] If n is odd, $\zeta_n(s, L_n^*)$ can be expressed in terms of the Riemann zeta function $\zeta(s)$ in a similar way. If n is even , then d does not disappear in (4.2.2), and in the final expression of $\zeta_n(s, L_n^*)$ appears an infinite sum of Dirchlet L-functions for the quadratic Dirichlet characters associated with the quadratic fields. Cf. [**IS1**] and [**IS2**]. In the case of the space of quadratic forms, this infinite sum incidentally coincides with the Mellin transform of some Eisenstein series of one variable of half-integral weight $(n+1)/2$ belonging to $\Gamma_0(4)$. It is not clear whether this is of general feature, when we consider more general prehomogeneous vector spaces.

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