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Brauer Group of $\mathbb{R}(X)$ and Eichler Type Theorem

By
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Abstract

The Brauer group of $\mathbb{R}(X)$, the rational function field over the real field, is isomorphic to the continuous direct sum of $\mathbb{Z}/2\mathbb{Z}$. A central division algebra over $\mathbb{R}(X)$ has strong approximation property for $\mathbb{R}[X]$ if and only if it is trivial at the place not corresponding to a prime ideal of $\mathbb{R}[X]$. This is a generalization of Eichler theorem.

We discuss similar problems for algebraic function fields over $\mathbb{R}$ and obtain partial solutions for some cases.

1. Brauer groups of $\mathbb{R}((X))$ and $\mathbb{R}(X)$.

Let $\mathbb{R}((X))$ be the field of formal power series over $\mathbb{R}$. It is a complete valuation field with the residue field $\mathbb{R}$. By J.P.Serre “Corps locaux” Chap 12, we have

$$Br(\mathbb{R}((X))) \simeq Gal(C/\mathbb{R}) \times Br(\mathbb{R}) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$  

($Br(K)$ denotes the Brauer group of $K$). We shall determine it more concretely.
Let $D$ be a central division algebra over $\mathbb{R}((X))$. Since $Br(\mathbb{C}((X)))$ is trivial, $D$ splits over $\mathbb{C}((X))$, so that $D$ contains a maximal subfield isomorphic to $\mathbb{C}((X))$. Thus we have

$$D = K + Ki + Kj + Kij, \quad K = \mathbb{R}((X)),$$

$$i^2 = -1, j^2 = f \in K^\times, ji = -ij.$$

We shall denote this $D$ by $\{-1, f\}$.

Since $\{-1, f\} \cong \{-1, f'\} \iff ff'^{-1} \in N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times) = (K^2 + K^2)^\times$, we have $Br(K) \cong K^\times/(K^2 + K^2)^\times$, whose complete representative system is given by

$$\{1, -1, X, -X\} \text{ so that } Br(\mathbb{R}((X))) = \{\mathbb{R}((X)), \{-1, f\}, \{-1, X\}, \{-1, -X\}\}.$$
For \( f = \pm(X - a_1) \cdots (X - a_n) \), \( D = \{-1, f\} \) is trivial at \( a \) such that \( f(a) > 0 \). It is ramified at \( a_i \) and at the non-prime place (which will be denoted by \( \infty \)) if the degree of \( f \) is odd. Since \( \{-1, f_1\} \otimes_{\mathbb{R}(X)} \{-1, f_2\} \sim \{-1, f_1f_2\} \), the multiplication in \( Br(\mathbb{R}(X)) \) corresponds to the symmetric difference of the sets of ramified places. Thus we have obtained the desired result (2).

**Remark** A discrete valuation is called real (or imaginary) if its residue field is \( \mathbb{R} \) (or \( \mathbb{C} \)). The set of all real places will be denoted by \( \mathbb{RP}(K) \). For \( K = \mathbb{R}(X) \), we have \( \mathbb{RP}(K) = \mathbb{R} \coprod \{\infty\} \).

Then, we have \( Br(\mathbb{R}(X)) \simeq (\mathbb{Z}/2\mathbb{Z})_{0}^{\mathbb{RP}(\mathbb{R}(X))} \). The isomorphism is given as follows. Suppose that a central division algebra \( D \) over \( \mathbb{R}(X) \) corresponds to a finite subset \( A \) of \( \mathbb{RP}(\mathbb{R}(X)) = \mathbb{R} \coprod \{\infty\} \). \( D \) is ramified at every \( a \in A \setminus \{\infty\} \), and at \( \infty \) if \( |A \setminus \{\infty\}| \) is odd. There are two \( D \)s which are ramified at no place. They are attributed to \( \mathbb{Z}/2\mathbb{Z} \) at \( \infty \).

**Corollary** \( \mathbb{R}(X) \) satisfies Hasse’s principle.

\( \{-1, -1\} \) is unramified but non-trivial at every place. All other non-trivial \( \{-1, f\} \) are ramified at some places.

### 2. Brauer group of \( \mathbb{R}(X, Y) \).

Let \( K \) be a finite extension of \( \mathbb{R}(X) \), namely an algebraic function field of one variable over \( \mathbb{R} \). In other words, \( K = \mathbb{R}(X, Y) \), \( Y \) is algebraic over \( \mathbb{R}(X) \).

If \( \sqrt{-1} \in K \), then \( K \) is an algebraic function field of one variable over \( \mathbb{C} \), so that \( Br(K) \) is trivial.

Hereafter we shall assume that \( \sqrt{-1} \notin K \). Since \( Br(K(\sqrt{-1})) \) is trivial, a central division algebra \( D \) over \( K \), splits over \( K(\sqrt{-1}) \). This implies that \( D \) is a quaternion algebra and \( D = \{-1, f\} \) for some \( f \in K^\times \). From this we see that \( Br(K) \) has the exponent 2, and \( Br(K) \simeq K^\times/(K^2 + K^2)^\times \).

A valuation on \( K \) which is trivial on \( R^\times \) is called a place. The residue field of a place \( v \) is \( \mathbb{R} \) or \( \mathbb{C} \), according to which \( v \) is called real or imaginary. (Note that this terminology
differs from the ones used for algebraic number fields).

For an imaginary place $v$, $D_v$ is trivial over $K_v$. For a real place $v$, $D_v$ is one of four algebras over $K_v$. The one is trivial, another one is an unramified quaternion, and the other two are ramified quaternions. See the results in §1.

Let $RP(K)$ be the set of all real places. Since the place of $K(\sqrt{-1}) = C(X, Y)$ are in one-to-one correspondence with points of a compact Riemann surface $\mathfrak{R}$, and since a real place $v$ of $K$ does not decompose in $K(\sqrt{-1})$, $RP(K)$ is identified with a subset of $\mathfrak{R}$.

For a real place $v$ of $K$, we have $3\varphi \in K$, $\text{ord}_v(\varphi) = 1$. Then, $\varphi(z)$ is a local coordinate in a neighbourhood of the corresponding $z_v \in \mathfrak{R}$. Since $z \in RP(K)$ is equivalent to $\varphi(z) \in \mathfrak{R}$ in this neighbourhood, $RP(K)$ is a one-dimensional real manifold. Since $\mathfrak{R}$ is compact, $RP(K)$ consists of $\nu$ closed curves, where $\nu$ is the number of connected components of $RP(K)$.

**Theorem 2** We have $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{RP(K)}$.

The isomorphism is given as follows: Fix a point $z_i$ ($1 \leq i \leq \nu$) from each connected component of $RP(K)$. Suppose that $Br(K) \ni D$ corresponds to a finite subset $A$ of $RP(K)$. Then, $D$ is ramified at $A \setminus \{z_1, \cdots, z_{\nu}\}$ and possibly at $z_i$. The ramification at $z_i$ is determined by the rule that $D$ is ramified at even number of places on each connected component of $RP(K)$.

There are $2^\nu$ different division algebras which are ramified at no real place. They are attributed to $(\mathbb{Z}/2\mathbb{Z})^{\{z_1, \cdots, z_{\nu}\}}$.

**Proof** Let $Br_1(K)$ be the group of all division algebras which are ramified at no real place. Then, $D = \{-1, f\} \in Br_1(K)$ is equivalent to that $\text{ord}_z(f)$ is even for every $z \in RP(K)$, namely that $f(z)$ has definite sign on each connected component of $RP(K)$.

As shown later, Hasse's principle holds for $K = R(X, Y)$. Therefore, $D = \{-1, f\}$ is trivial if and only if $f$ is non-negative on $RP(K)$, so that we have $|Br_1(K)| \leq 2^\nu$. The equality holds if for any connected component $C$ of $RP(K)$, there exists $f \in K^\times$ such that $f(z) \leq 0$ on $C$ but $f(z) \geq 0$ on $RP(K) \setminus C$. Since $RP(K)$ is mapped homeomorphically into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1 \leq i \leq 4}$, $T_1(z) = \frac{X(z)}{X(z)^2 + 1}, T_2(z) = \frac{1}{X(z)^2 + 1}, T_3(z) = \frac{Y(z)}{Y(z)^2 + 1}, T_4(z) = \frac{1}{Y(z)^2 + 1}$, and since the function $F$ defined by $F(z) = -1$ on $C$ and $F(z) = 1$ on $RP(K) \setminus C$ is continuous on $RP(K)$, the polynomial approximation theorem of Weierstrass assures
that there exists a polynomial $P(T_i)$ such that $P(T_i(z)) < 0$ on $C$ but $P(T_i(z)) > 0$ on $RP(K) \setminus C$. This completes the proof of $Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{\{z_1, \ldots, z_n\}}$.

Take any $f \in K^\times$. If $\text{ord}_{z_0}(f)$ is odd for $z_0 \in RP(K)$, then $f(z)$ changes its sign when $z$ crosses $z_0$. Since a connected component $C$ of $RP(K)$ is a closed curve, $f(z)$ must change its sign even times on $C$, therefore $D = \{-1, f\}$ is ramified at even number of places on $C$.

Now, we shall show that for any two points $\zeta$ and $\zeta'$ on $C$, there exists $f \in K^\times$ such that $D = \{-1, f\}$ is ramified at $\zeta$ and $\zeta'$, but not ramified at other real places. Again we shall map $RP(K)$ into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1 \leq i \leq 4}$. Since $C$ is a closed analytic curve, there are $\zeta = \zeta_0$, $\zeta_1$, $\ldots$, $\zeta_n = \zeta'$ ($\zeta_i \in C$) and small spheres $S_j : \sum_{i=1}^{4} (T_i - a_{ij})^2 = r_j^2$ such that $S_j \cap RP(K) = \{\zeta_{j-1}, \zeta_j\}$.

Then, $f = \prod_{j=1}^{n} \left\{ \sum_{i=1}^{4} (T_i(z) - a_{ij})^2 - r_j^2 \right\}$ satisfies $\text{ord}_\zeta(f) = \text{ord}_{\zeta'}(f) = 1$, $\text{ord}_{\zeta_i}(f) = 2$ ($1 \leq i \leq n - 1$), and $\text{ord}_z(f) = 0$ for $z \in RP(K) \setminus \{\zeta_i\}$. This $f$ is the desired element of $K^\times$.

Thus we have proved $Br(K)/Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})_{0}^{RP(K)}\{z_1, \ldots, z_n\}$, so combining with the result for $Br_1(K)$, we get $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})_{0}^{RP(K)}$. 

![Diagram of a closed analytic curve with points labeled ζ and ζ']
Remark  $K$ satisfies Hasse's principle as a result of the following lemma.

Let $\square K$ be the set of all sums of squares, $\square K = \{\sum x_i^2 | x_i \in K\}$.

Lemma  Let $K = \mathbb{R}(X, Y)$ be an algebraic function field over $\mathbb{R}$.

(1) For $f \in K^\times$, $f \in \square K$ if and only if $f(z) \geq 0$ for $\forall z \in RP(K)$. Especially, if $RP(K) = \emptyset$ then $\square K = K$.

(2) Every element of $\square K$ can be written as a sum of two squares.

We shall omit the proof here, and refer to [6], Th.3.2, Chap.3 and Th.2.1, Chap.4.

Corollary  $K = \mathbb{R}(X, Y)$ satisfies Hasse's principle.

Proof  $D = \{-1, f\}$ is locally trivial if and only if $f(z) \geq 0$ for $\forall z \in RP(K)$, which is equivalent to $f \in K^2 + K^2 = N_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times)$, hence $D = \{-1, f\}$ is trivial.

3. Approximation in idele groups.

Let $R$ be a Dedekind domain, and $K$ be its quotient field. Every prime ideal $p$ of $R$ defines the $p$-adic valuation on $K$. This is called a prime valuation. Besides $p$-adic valuations, we often consider some others, which are called non-prime valuations. For instance, all valuations trivial on $k^\times$ when $K$ is an algebraic function field over the constant field $k$.

We define the adele ring $R_\mathbb{A}$ of $R$ by $R_\mathbb{A} = \prod_p R_p$, where $p$ runs over all prime valuations and $R_p$ denotes the completion of $R$ at the place $p$. Also we define the adele ring $K_\mathbb{A}$ of $K$ by $K_\mathbb{A} = K \otimes_R R_\mathbb{A} \simeq \bigcup_{S \in \text{prime}} \prod_{p \in S} K_p \times \prod_{p \notin S} R_p$ where $S$ runs over all finite set of prime valuations. The idele group $K_\mathbb{A}^\times$ is defined as the group of invertible elements of $K_\mathbb{A}$. It is written in the form of $K_\mathbb{A}^\times = \bigcup_{S \in \text{prime}} (\prod_{p \in S} K_p^\times \times \prod_{p \notin S} R_p^\times)$.

The fundamental system of neighbourhoods of 0 in $K_\mathbb{A}$ is given by $\{V(S, n)\}$, where

$$V(S, n) = \prod_{p \in S} p^n R_p \times \prod_{p \notin S} R_p.$$ 

Similarly, the fundamental system of neighbourhoods of 1 in $K_\mathbb{A}^\times$ is given by $\{U(S, n)\}$,
where
\[
U(S, n) = \prod_{p \in S} (1 + p^n R_p) \times \prod_{p \notin S} R_p^x.
\]

Let $D$ be a central division algebra over $K$. A finitely generated $R$-submodule of $D$ is called an $R$-lattice, and if it spans $D$ as a $K$-vector space, it is called a full $R$-lattice. An $R$-lattice is called an $R$-order, if it is a subring including 1 (= the unit element of $D$).

The adele ring $D_A$ of $D$ is defined by
$D_A = D \otimes_K K_A$. It is written in the form of
$D_A = \bigcup_S \left( \prod_{p \in S} D_p \times \prod_{p \notin S} \Gamma_p \right)$, where $\Gamma$ is a full $R$-order of $D$ and $\Gamma_p = \Gamma \otimes_R R_p$. The idele group $D_A^\times$ is defined similarly. The fundamental system of neighbourhoods of 1 in $D_A^\times$ is given by
\[
U(S, n) = \prod_{p \in S} (1 + p^n \Gamma_p) \times \prod_{p \notin S} \Gamma_p^x.
\]

$D$ is diagonally imbedded into $D_A$, and $D_A^\times$ is diagonally imbedded into $D_A^\times$. $D$ is dense in $D_A$(by chinese remainder theorem), but $D_A^\times$ is not dense in $D_A^\times$. But $D_A^\times$ may be dense in some subgroup of $D_A^\times$.

Let $\Pi_{D/K}$ be the reduced norm $D \to K$. $\Pi_{D/K}$ maps $D^\times$ homomorphically into $K^\times$. We shall denote its kernel by $D^{(1)}$. $\Pi_{D/K}$ is uniquely extended as a $K_A$-valued polynomial function on $D_A$. This extension is denoted by the same symbol $\Pi_{D/K}$, and its kernel in $D_A^\times$ is denoted by $D_A^{(1)}$.

Eichler theorem ascertainment that for global fields, $D^{(1)}$ is dense in $D_A^{(1)}$ (in the topology of $D_A^\times$) if and only if $D_v$ is not a division algebra for some non-prime $v$.

For global fields, we have also $D^{(1)} = [D^\times, D^\times]$, the commutator group of $D^\times$. But for a general $K$, this relation does not hold (For instance, Platonov [3]).

For a general $K$, in the connection with the cancellation problem of $\Gamma$, it seems natural to consider $[D^\times, D^\times]$ rather than $D^{(1)}$. Thus we define the strong approximation property as follows: A central division algebra $D$ is said to have strong approximation property if $[D^\times, D^\times]$ is dense in $[D_A^\times, D_A^\times]$. To find a necessary and sufficient condition for strong approximation property is a generalization of Eichler theorem to a general case.

In the connection with the cancellation problem of $\Gamma$, we consider a little weaker approximation property. $D$ is said to have $D^\times$-approximation, if the closure of $D^\times$ (in the
topology of $D_{\mathbb{A}}^{\times}$ contains $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}]$. $D$ is said to have $R_{\mathbb{A}}^{\times}D^{\times}$-approximation, if the closure of $R_{\mathbb{A}}^{\times}D^{\times}$ contains $[D^{\times}, D_{\mathbb{A}}^{\times}]$. (Both of $D^{\times}$ and $R_{\mathbb{A}}^{\times}$ are contained in $D_{\mathbb{A}}^{\times}$, so $R_{\mathbb{A}}^{\times}D^{\times} \subset D_{\mathbb{A}}^{\times}$.)

The last and weakest approximation property is necessary and sufficient for the cancellation of every full $R$-order $\Gamma$ of $D$ (namely $\Gamma \oplus \Gamma \simeq L \oplus \Gamma$ implies $\Gamma \simeq L$, the isomorphism being as $\Gamma$-lattices).

4. **Eichler theorem for $\mathbb{R}(X)$**.

In §1 we have seen that $D = \{-1, f\}$ is trivial at the non-prime place $\infty$ if and only if $f$ is monic of even degree.

**Theorem 3** If $D_{\infty}$ is not trivial, then $D^{\times}$ is discrete in $D_{\mathbb{A}}^{\times}$ and $R_{\mathbb{A}}^{\times}D^{\times}$ is closed in $D_{\mathbb{A}}^{\times}$.

**Corollary** If $D_{\infty}$ is not trivial, then $R_{\mathbb{A}}^{\times}D^{\times}$-approximation property does not hold.

**Proof of Corollary** It suffices to show $[D^{\times}_{\mathbb{A}}, D^{\times}_{\mathbb{A}}] \not\subset R^{\times}_{\mathbb{A}}D^{\times}$. For a real place $a$, we shall identify $D^{\times}_{a}$ with the subgroup $D^{\times}_{a} \times \prod_{p \neq a}(1)_{p}$ of $D^{\times}_{\mathbb{A}}$. It is clear that $[D^{\times}_{\mathbb{A}}, D^{\times}_{\mathbb{A}}] \cap D^{\times}_{a} = [D^{\times}_{a}, D^{\times}_{a}]$. Since $D_{a}$ is a quaternion (or a matrix) algebra over $K_{a}$, we have $[D^{\times}_{a}, D^{\times}_{a}] = D^{(1)}_{a}$, so that $[D^{\times}_{a}, D^{\times}_{a}] \not\subset K^{\times}_{a}$.

On the other hand, if $x = (x_{p}) \in R^{\times}_{\mathbb{A}}D^{\times} \cap D^{\times}_{a}$, then we have $\exists d \in D^{\times}, \forall p$ (prime place), $\exists r_{p} \in R^{\times}_{p}, x_{p} = r_{p}d$. For $p \neq a$, we have $x_{p} = 1$ so that $d = r_{p}^{-1} \in R^{\times}_{p} \subset K^{\times}_{p}$, so that $d \in D^{\times} \cap K^{\times}_{p} = K^{\times}_{a}$, hence $x_{a} = r_{a}d \in R^{\times}_{a}K^{\times} \subset K^{\times}_{a}$. This assures $R^{\times}_{\mathbb{A}}D^{\times} \cap D^{\times}_{a} \subset K^{\times}_{a}$ so that $[D^{\times}_{\mathbb{A}}, D^{\times}_{\mathbb{A}}] \not\subset R^{\times}_{\mathbb{A}}D^{\times}$.

**Proof of Theorem 3** $D = \{-1, f\}$ means that

$$D = K + Ki + Kj + Kij$$

$$i^{2} = -1, j^{2} = f, ji = -ij.$$  

Then $\Gamma = R + Ri + Rj + Rij$ is a full $R$-order of $D$ ($K = \mathbb{R}(X), R = \mathbb{R}[X]$).

A fundamental neighbourhood of 1 in $D^{\times}_{\mathbb{A}}$ is $U(g) = \prod_{p}(1 + g\Gamma_{p}) \cap \prod_{p}\Gamma^{\times}_{p}$ for $g \in R$ and we have $U(g) \cap D^{\times} = (1 + g\Gamma) \cap \Gamma^{\times}$, so the first half of Theorem 3 is proved if
Suppose that \( d = \varphi_1 + \varphi_2 i + \varphi_3 j + \varphi_4 ij \in (1 + g\Gamma) \cap \Gamma^\times \), \( \varphi_i \in R \). This means that \( \varphi_1 \equiv 1 \pmod{g} \), \( \varphi_i \equiv 0 \pmod{g} \) for \( i \geq 2 \), and \( \Pi_{D/K}(d) = \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \in R^\times = \mathbb{R}^\times \).

If \( g \in R \setminus R^\times \), substituting a zero of \( g \), we see that \( \Pi_{D/K}(d) = 1 \).

Since each \( \varphi_i^2 \) has, if not zero, a positive coefficient of the highest degree term, such terms of \( \varphi_1^2 \) and \( \varphi_2^2 \) (resp. \( \varphi_3^2 \) and \( \varphi_4^2 \)) do not cancel.

From \( \varphi_1^2 + \varphi_2^2 - 1 = f(\varphi_3^2 + \varphi_4^2) \), if \( f \) is of odd degree, both hand sides should be zero. This implies that \( \varphi_3 = \varphi_4 = 0 \) and \( \varphi_1, \varphi_2 \in R \), which implies \( \varphi_2 = 0 \) and \( \varphi_1 = 1 \) because \( \varphi_2 \) is a multiple of \( g \).

If \( f \) is of even degree with a negative coefficient of the highest degree term, then the highest degree terms of \( \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \) do not cancel, so that we have \( \forall i, \varphi_i \in R \).

This again implies \( \varphi_i = 0 \) for \( i \geq 2 \), and so \( \varphi_1 = 1 \).

Thus the first half of Theorem 3 has been proved. Similar discussions show that

\[
(R + g\Gamma)^\times = R^\times, \quad \text{if} \quad g \in R \setminus R^\times.
\]

\[
\Gamma_{g} = R + g\Gamma \quad \text{is a full } R\text{-order of } D, \quad \text{and } (\Gamma_{g})_{A}^{\times} = \prod_{p}(R_{p} + g\Gamma_{p})^{\times} \quad \text{is an open subgroup}
\]

of \( D_{A}^{\times} \), so \((\Gamma_{g})_{A}^{\times} D^{\times} \) is open and closed, hence \( \bigcap_{g}(\Gamma_{g})_{A}^{\times} D^{\times} \) is a closed subgroup of \( D_{A}^{\times} \), containing \( R_{A}^{\times} D^{\times} \).

We shall show the inverse inclusion. Take any \( x \in \bigcap_{g}(\Gamma_{g})_{A}^{\times} D^{\times} \), then \( \forall g, \exists \gamma_{g} \in (\Gamma_{g})_{A}^{\times}, 3d_{g} \in D^{\times}, x = \gamma_{g} d_{g} \). Since \((\Gamma_{g})_{A} \cap D^{\times} = (R + g\Gamma)^\times = R^\times \), \( d_{g} \) is determined modulo \( R^\times \), so if \( g_{1} \) is a multiple of \( g \), then \( d_{g_{1}} \) differs from \( d_{g} \) only modulo \( R^\times \). This implies that we can choose \( d_{g} \) independently of \( g \), thus \( 3d \in D^{\times}, xd^{-1} \in \bigcap_{g}(\Gamma_{g})_{A}^{\times} \).

But we have \( R_{A}^{\times} = \bigcap_{g}(\Gamma_{g})_{A}^{\times} \), because \( \forall p, \bigcap_{g}(R_{p} + g\Gamma_{p})^{\times} = R_{p}^{\times} \). Thus the proof of the second half of Theorem 3 is completed.

**Theorem 4** If \( D_{\infty} \) is trivial, hence if \( f \) is monic of even degree, then \([D^{\times}, D^{\times}] = D^{(1)}\) is dense in \([D_{A}^{\times}, D_{A}^{\times}]\).

This theorem is divided into the following two parts.

**Theorem 4.1** If \( f \) is monic of even degree, then for \( g, h \in R \) such that \( (h, gf) = 1 \), we
have

$$(1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^X \neq \phi.$$ 

**Theorem 4.2** The conclusion part of Theorem 4.1 is equivalent to strong approximation property.

**Proof of Theorem 4.1** It suffices to show the existence of $\varphi_i \in R, 1 \leq i \leq 4$ such that

\[
\begin{align*}
\varphi_1 &\equiv 1 \pmod{g}, \quad \varphi_i \equiv 0 \pmod{g}, \quad 2 \leq i \leq 4 \\
\varphi_2 &\equiv 1 \pmod{h}, \quad \varphi_i \equiv 0 \pmod{h}, \quad i = 1, 3, 4, \text{ and}
\end{align*}
\]

\[\varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) = 1.\] 

(1)

Put $\varphi_1 = 1 + g^2fu_1, \varphi_2 = g^2fu_2, \varphi_i = gu_i(i = 3, 4)$, then the required congruence modulo $g$ is automatically satisfied. Substituting them into (1) and dividing both sides by $g^2f$, we get

\[2u_1 + g^2fu_1^2 + g^2fu_2^2 - (u_3^2 + u_4^2) = 0.\] 

(2)

Since $(h^2, g^2f) = 1$, $g^2f$ is invertible in $R/h^2R$, so there exist $\psi, \psi' \in R$ such that

\[g^2f\psi = 1 + h^2\psi'.\]

Put $u_1 = -\psi + h^2v_1, u_2 = \psi + h^2v_2, u_i = hv_i(i = 3, 4)$, then the required congruence modulo $h$ is automatically satisfied. Substituting them into (2), we get

\[-2\psi + 2h^2v_1 + g^2f\{2\psi^2 + 2h^2\psi(v_2 - v_1) + h^4(v_1^2 + v_2^2)\} = h^2(v_3^2 + v_4^2).\]

Since $-2\psi + 2g^2f\psi^2 = -2\psi(1 - g^2f\psi) = 2h^2\psi\psi'$, we have

\[2\psi\psi' + 2(1 - g^2f\psi)v_1 + 2g^2f\psi v_2 + g^2fh^2(v_1^2 + v_2^2) = v_3^2 + v_4^2.\] 

(3)

Put $v_1 = (1 - g^2f\psi)w$ and $v_2 = g^2f\psi w$, then we get

\[2\psi\psi' + \{(1 - g^2f\psi)^2 + (g^2f\psi)^2\}(2w + g^2fh^2w^2) = v_3^2 + v_4^2.\] 

(4)
A polynomial $P \in R = \mathbb{R}[X]$ belongs to $R^2 + R^2$, if and only if $P(a) \geq 0$ for $\forall a \in R$, as shown in the proof of Theorem 1. So it suffices to show that the left hand side of (4) is everywhere non-negative for some $w \in R$.

Put $2\psi\psi' = F$ and $g^2fh^2 = G$, then $(1-g^2f\psi)^2 + (g^2f\psi)^2 = 1 - 2g^2f\psi(1-g^2f\psi) = 1 + 2g^2f\psi h^2\psi' = 1 + FG$, so we have

$$F + (1 + FG)(2w + Gw^2) \geq 0. \quad (5)$$

The above calculation also shows $1 + FG \geq \frac{1}{2}$, namely $FG \geq -\frac{1}{2}$. Since $f$ is monic of even degree, we have $\lim_{t \to \pm \infty} G(t) = \infty$ so that $3M > 0$, $\forall t \in R, G(t) \geq -M$. Since $\{t \mid G(t) \leq 0\}$ is compact, $F$ is bounded there, so $3N > 0, |F(t)| \leq N$ for $G(t) \leq 0$.

The left hand side of (5) is zero for

$$w = \frac{1}{G} \left\{ -1 \pm (1 + FG)^{-\frac{1}{2}} \right\}. \quad (6)$$

Since $(1+t)^{-\frac{1}{2}} \leq 1 - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2$ for $t \geq -\frac{1}{2}$, if we set $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G$, then (5) is satisfied for $G \geq 0$. Let $P$ be an everywhere positive polynomial of two variables $s$ and $t$, then $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G + P(G, FG)$ satisfies (5) for $G \geq 0$.

The condition (5) is satisfied also for $G < 0$, if

$$-1 \leq -\frac{t}{2} + \frac{3}{\sqrt{2}}t^2 + sP(s, t) \leq -1 + (1+t)^{-\frac{1}{2}} \quad (6)$$

on $\Delta = \{(s, t) \mid -M \leq s \leq 0, t \geq -\frac{1}{3}, |t| \leq N|s|\}$. The condition (6) is satisfied if

$$\epsilon \geq \frac{1}{s} \left\{ 1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2 \right\} + P(s, t) \geq 0$$

on $\Delta$, where $\epsilon \leq (1 + NM)^{-\frac{1}{2}}/M$. Since $\alpha(s, t) = \frac{1}{s} \left\{ 1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2 \right\}$ is non-positive and continuous on $\Delta$ (it is continuous at $(0, 0)$ because of $|t| \leq N|s|$), such a polynomial $P(s, t)$ exists by virtue of polynomial approximation theorem of Weierstrass. $P(s, t)$ can be assumed everywhere positive, because we can put $P = Q^2 + \frac{\epsilon}{2}$, $Q$ being an approximating polynomial of $\sqrt{|\alpha(s, t)|}$. Thus Theorem 4.1 has been proved.

**Proof of Theorem 4.2** Let $H$ be the closure of $[D^\times, D^\times] = D^{(1)}$ in $D'^\times_A$. Let $p_0$ be a prime place where $D$ is unramified, and let $i_{p_0} = (1, \cdots, 1, i, 1, \cdots) \in D'^\times_A$ be the element of $D'^\times_A$ whose $p_0$-coordinate is $i$, while other coordinates are 1.
The proof is completed by the following steps, which are slight modifications of ones given in [1] §51.

**Step 1** The conclusion part of Theorem 4.1 is equivalent to that $\forall_{p_{0}}$ (where $D_{p_{0}}$ is unramified), $i_{p_{0}} \in H$ (note that $i_{p_{0}} \in D_{p_{0}}^{(1)} = [D_{p_{0}}^{x}, D_{p_{0}}^{z}] \subset [D_{A}^{x}, D_{A}^{z}]$).

**Step 2** Identify $D_{p_{0}}^{(1)}$ with a subgroup $D_{p_{0}}^{(1)} \times \prod_{p \neq p_{0}} (1)_{p}$ of $D_{A}^{x}$, then $H \cap D_{p_{0}}^{(1)}$ is a closed normal subgroup of $D_{p_{0}}^{(1)}$.

**Step 3** If $D$ is unramified at $p_{0}$, then $i_{p_{0}} \in H$ implies $D_{p_{0}}^{(1)} \subset H$.

If $D_{p_{0}}$ is a matrix algebra, the assertion is a result of simplicity of $PSL(2, K_{p_{0}})$. If $D_{p_{0}}$ is an unramified quaternion algebra, since $x = a + bi + cj + dij \in D_{p_{0}}^{(1)}$ satisfies $x^{2} - 2ax + 1 = 0$, the condition $x \in H$ depends only on $a$. (Here we identify $x \in D_{p_{0}}^{(1)}$ with $x_{p_{0}} = (1, \ldots, 1, x, 1, \ldots) \in D_{A}^{x}$.)

Take any $x = a + bi + cj + dij \in D_{p_{0}}^{(1)}$. Since $b^{2} + c^{2} + d^{2}$ has a root in $K_{p_{0}}$, we have $\exists e \in K_{p_{0}}, b^{2} + c^{2} + d^{2} = e^{2}$. If $i \in H$, then $-ai + ej \in H$, therefore $i(-ai + ej) = a + eij \in H$, hence $x \in H$. This means $D_{p_{0}}^{(1)} \subset H$.

**Step 4** Assume the conclusion part of Theorem 4.1. For a finite set $S$ of prime places, we have $\prod_{p \in S} D_{p}^{(1)} \times \prod_{p \notin S} (1)_{p} \subset H$.

If $D$ is unramified on $S$, the assertion is a consequence of Step 3.

Let $S_{0}$ be the set of all prime places where $D$ is ramified. The assertion for $S = S_{0}$ follows from the fact that $D^{(1)}$ is dense in $\prod_{p \in S_{0}} D_{p}^{(1)}$ in the product topology of $D_{p}^{x}$.

**Step 5** $\bigcup_{S} \left( \prod_{p \in S} D_{p}^{(1)} \times \prod_{p \notin S} (1)_{p} \right)$ is dense in $[D_{A}^{x}, D_{A}^{z}]$.

Combining the five assertions above, we complete the proof of Theorem 4.2.

5. Eichler theorem for $R(X, Y)$.

For an algebraic function field $K = R(X, Y)$, we shall fix a set $P$ of valuations (which are trivial on $R^{x}$). A valuation $v \in P$ is called a prime place and $v \notin P$ is called a non-
prime place. We assume that there exists a non-prime place. Then, \( R_P = \{ x \in K \mid v(x) \leq 1 \} \) is a Dedekind domain and \( K \) is its quotient field. A prime ideal of \( R_P \) is given by \( p_v = \{ x \in R_P \mid v(x) < 1 \} \) for \( v \in P \).

The adele ring and the idele group are constructed using prime places only. We shall write \( R \) instead of \( R_P \).

We consider the following property (E):

(E) A central division algebra \( D \) over \( K \) has strong approximation property, if \( D \) is trivial at some non-prime place.

The converse of the property (E) holds always as shown below.

**Theorem 5** If a central division algebra \( D \) is non-trivial at every non-prime place, then \( D \) does not have \( R^\times_A D^\times \)-approximation property.

**Remark** Before proving this theorem, we shall mention about the product formula. The formula is expressed as follows using \( \text{ord}_v \):

\[ v(x) = \theta^{\text{ord}_v(x)} (0 < \theta < 1) \]

\[ \forall x \in K^\times, \sum_{v: \text{real}} \text{ord}_v(x) + 2 \sum_{v: \text{imag}} \text{ord}_v(x) = 0, \]

where the sum is taken over all places, prime or not.

**Proof** Similar discussions as the proof of Theorem 3 show that it suffices to prove that

\[ (R + gGamma)^\times = R^\times \text{ for } g \in R \setminus R^\times. \]

Let \( D = \{-1, f\}, f \in R \). The assumption of Theorem 5 means that all non-prime places are real and that for every non-prime place \( v \), \( \text{ord}_v(f) \) is odd or \( \text{ord}_v(f) \) is even with a negative coefficient of the lowest degree term with respect to the prime element \( \pi_v \).

Suppose that \( \varphi_1 + \varphi_2 i + \varphi_3 j + \varphi_4 ij \in (R + gGamma)^\times \), then \( \varphi_1 \in R, \varphi_i \in gR (2 \leq i \leq 4) \), and \( \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \in R^\times \). Put \( \varphi = \varphi_1 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \), then \( \varphi \in R^\times \) implies \( \text{ord}_v(\varphi) = 0 \) for every prime place \( v \). As for a non-prime place \( v \), the assumption on \( f \) implies that the lowest degree terms do not cancel, so that \( \text{ord}_v(\varphi) = \text{Min}(2\text{ord}_v(\varphi_1), 2\text{ord}_v(\varphi_2), \text{ord}_v(f) + 2\text{ord}_v(\varphi_3), \text{ord}_v(f) + 2\text{ord}_v(\varphi_4)) \), if \( \varphi_i \neq 0 \).

Combining this with the product formula, we have

\[ \sum_{\text{non-prime}} \text{ord}_v(\varphi_i) \geq 0 \ (i = 1, 2), \]
\[ \sum_{\text{non-prime}} \text{ord}_v(\varphi_i) \geq \frac{1}{2} \sum_{\text{prime}} \alpha_v \text{ord}_v(f) \quad (i = 3, 4) \]

where \( \alpha_v = 1 \) for a real \( v \) and \( \alpha_v = 2 \) for an imaginary \( v \). Since \( \varphi_i \in R \) and \( f \in R \), we have \( \text{ord}_v(\varphi_i) \geq 0 \) and \( \text{ord}_v(f) \geq 0 \) for a prime place \( v \), hence again from the product formula, we must have \( \text{ord}_v(\varphi_i) = 0 \) for every prime place \( v \). This means \( \varphi_i \in R^\times \). For \( i \geq 2 \), this contradicts with \( \varphi_i \in gR \), so we must have \( \varphi_i = 0 \), which in turn implies \( \varphi_1 \in R^\times \). This completes the proof of \( (R + g\Gamma)^\times = R^\times \).

**Remark** Property(E) depends not only on \( K \), but also on \( R \), or equivalently on the choice of non-prime places. However:

**Theorem 6** (1) Suppose that property(E) holds whenever \( R \) has only one non-prime place, then it holds for any \( R \).

(2) For the rational function field \( K = \mathbb{R}(X) \), property(E) holds for any \( R \).

**Proof of (1)** Let \( P(R) \) be the set of all prime places for the Dedekind domain \( R \). Then \( P(R') \subset P(R) \) implies \( R \subset R' \). We shall denote the idele group of \( D \) with respect to \( R \) by \( D^\times_A(R) \). Then \( P(R) = P(R') \coprod P(R_1) \) implies that \( D^\times_A(R) \) is the product topological group of \( D^\times_A(R) \) and \( D^\times_A(R_1) \), because of \( D^\times_A(R) = \bigcup_S \left( \prod_{v \in S} D^\times_v \times \prod_{v \in P(R) \setminus S} \Gamma^\times_v \right) \) where \( S \) runs over all finite subsets of \( P(R) \).

\( D^\times \) is imbedded diagonally in \( D^\times_A \), and strong approximation property means precisely that the image \( i_R(D^{(1)}) \) is dense in \([D^\times_A(R), D^\times_A(R)]\).

If \( P(R') \subset P(R) \), then the projection \( D^\times_A(R) \to D^\times_A(R') \) maps \( i_R(D^{(1)}) \) onto \( i_R(D^{(1)}) \) and \([D^\times_A(R), D^\times_A(R)]\) onto \([D^\times_A(R'), D^\times_A(R')]\). Therefore, if \( i_R(D^{(1)}) \) is dense in \([D^\times_A(R), D^\times_A(R)]\), then \( i_R(D^{(1)}) \) is dense in \([D^\times_A(R'), D^\times_A(R')]\).

Now suppose that \( D \) is trivial at some non-prime place \( v \) of a given \( R \). Let \( P_0 \) be the set of all places other than \( v \), and suppose that property(E) holds for \( R_0 \) corresponding to \( P_0 \), then \( i_{R_0}(D^{(1)}) \) is dense in \([D^\times_A(R_0), D^\times_A(R_0)]\), hence \( i_R(D^{(1)}) \) is dense in \([D^\times_A(R), D^\times_A(R)]\), so property(E) holds for \( R \).

**Remark** The proof of Theorem 4.2 does work for a general algebraic function field \( K = \mathbb{R}(X,Y) \) and its Dedekind domain \( R \). So, strong approximation property holds for \( D = \{-1, f\} \), if \((1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \phi \) for \( g, h \in R \) such that \((gf, h) = 1 \).
Also the proof of Theorem 4.1 works partially. For \( \psi, \psi' \in R \) such that \( g^2fh^2 = 1 + h^2\psi' \), put \( F = 2\psi\psi' \) and \( G = g^2fh^2 \). Then, we have \( (1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \phi \) if \( \exists w \in R, F + (1 + FG)(2w + Gw^2) \in R^2 + R^2 \).

Suppose that \( R \) has only one non-prime place \( v \), then \( f \in R \) means that \( f \) does not have a pole other than \( v \). If \( v \) is real and \( D_v \) is trivial, then \( \text{ord}_v(f) \) is even and \( f(z) \) is positive near \( v \). Since \( RP(K) \) is compact, this implies that \( f \), hence \( G \), is bounded from below on \( RP(K) \), and that \( F \) is bounded on \( \{ z \in RP(K) | G(z) \leq 0 \} \). If \( v \) is imaginary, then both \( F \) and \( G \) are bounded on \( RP(K) \). So, similar discussions as the proof of Theorem 4.1 show that \( \exists w \in R, F + (1 + FG)(2w + Gw^2) \geq 0 \) on \( RP(K) \).

The proof for general \( K \) fails only because the condition "\( \varphi \in R \) and \( \varphi \geq 0 \) on \( RP(K) \)" does not imply \( \varphi \in R^2 + R^2 \). Since Hasse’s principle is satisfied, \( \varphi \in K^2 + K^2 \) is assured, but \( \varphi \in R^2 + R^2 \) is not concluded. We shall give a counter example for an elliptic function field \( K = \mathbb{R}(X,Y), Y^2 = (X-a)(X-b)(X-c) \). If \( \alpha \in \mathbb{R} \) is smaller than \( \text{Min}(a,b,c) \), then we have \( X - \alpha > 0 \) on \( RP(K) \). \( X - \alpha \) has a double pole at the non-prime place \( v \), while an element of \( R^2 + R^2 = N_{K(\sqrt{-1})/K}(R + \sqrt{-1}R) \) should have \( \text{ord}_v \leq -4 \).

**Proof of Theorem 6 (2)**

Let \( K = \mathbb{R}(X) \) and suppose that \( R \) has only one non-prime place \( v \).

If \( R \neq \mathbb{R}[X] \), then \( v \) corresponds to an irreducible polynomial \( p \), and \( \varphi \in R \) is equivalent to \( \varphi = g/p^\nu, g \in \mathbb{R}[X] \) and \( \deg g \leq \nu \deg p \). Here we can assume that \( \nu \) is even. Then \( \varphi \geq 0 \) on \( RP(K) \) implies \( g \geq 0 \) on \( RP(K) \), so \( g \) is of even degree and can be written as \( g = g_1^2 + g_2^2 \), \( g_i \in \mathbb{R}[X] \), \( \deg g_i \leq \frac{\nu}{2} \deg g \). Therefore \( \varphi = (g_1/p^{\nu/2})^2 + (g_2/p^{\nu/2})^2 \) and \( \deg g_i \leq \frac{\nu}{2} \deg p \), so that \( \varphi \in R^2 + R^2 \).

From the remark above, this completes the proof of Theorem 6 (2).

**References**


Especially \S 23 lattices and orders, \S 51 Jacobinski’s cancellation theorem


