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# Brauer Group of $\mathbb{R}(X)$ and Eichler Type Theorem

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**Abstract**

The Brauer group of $\mathbb{R}(X)$, the rational function field over the real field, is isomorphic to the continuous direct sum of $\mathbb{Z}/2\mathbb{Z}$. A central division algebra over $\mathbb{R}(X)$ has strong approximation property for $\mathbb{R}[X]$ if and only if it is trivial at the place not corresponding to a prime ideal of $\mathbb{R}[X]$. This is a generalization of Eichler theorem.

We discuss similar problems for algebraic function fields over $\mathbb{R}$ and obtain partial solutions for some cases.

## 1. Brauer groups of $\mathbb{R}((X))$ and $\mathbb{R}(X)$.

Let $\mathbb{R}((X))$ be the field of formal power series over $\mathbb{R}$. It is a complete valuation field with the residue field $\mathbb{R}$. By J.P.Serre "Corps locaux" Chap 12,we have

$$Br\left(\mathbb{R}((X))\right) \simeq Gal(\mathbb{C}/\mathbb{R}) \times Br(\mathbb{R}) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

($Br(K)$ denotes the Brauer group of $K$). We shall determine it more concretely.

Let $D$ be a central division algebra over $\mathbb{R}((X))$. Since $Br(\mathbb{C}((X)))$ is trivial, $D$ splits over $\mathbb{C}((X))$, so that $D$ contains a maximal subfield isomorphic to $\mathbb{C}((X))$. Thus we have

$$D = K + Ki + Kj + Kij, \qquad K = \mathbb{R}((X)),$$

$$i^2 = -1, j^2 = f \in K^\times, ji = -ij.$$

We shall denote this $D$ by $\{-1, f\}$.

Since $\{-1, f\} \simeq \{-1, f'\} \iff ff'^{-1} \in \mathrm{N}_{K(\sqrt{-1})/K}(K(\sqrt{-1})^\times) = (K^2 + K^2)^\times$, we have $Br(K) \simeq K^\times/(K^2 + K^2)^\times$, whose complete representative system is given by $\{1, -1, X, -X\}$ so that

$$Br(\mathbb{R}((X))) = \{\mathbb{R}((X)), \mathbb{H}((X)), \{-1, X\}, \{-1, -X\}\}.$$

Note that $\mathbb{R}((X)) = \{-1, 1\}$ and $\mathbb{H}((X)) = \{-1, -1\}$ where $\mathbb{H}$ is the usual quaternion algebra over $\mathbb{R}$. $\mathbb{H}((X))$ is unramified over $\mathbb{R}((X))$, while $\{-1, X\}$ and $\{-1, -X\}$ are ramified.

Next, we shall determine the Brauer group of $\mathbb{R}(X)$.

**Theorem 1** (1) Every central division algebra over $\mathbb{R}(X)$ has the index $\leq 2$, hence if it is not trivial, it is a quaternion algebra over $\mathbb{R}(X)$,
(2) $Br(\mathbb{R}(X)) \simeq \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})_0^{\mathbb{R}} \simeq (\mathbb{Z}/2\mathbb{Z})_0^{\mathbb{R} \coprod \{\mathrm{sgn}\}}$, where $(\mathbb{Z}/2\mathbb{Z})_0^{\mathbb{R}}$ denotes the continuous direct sum of $\mathbb{Z}/2\mathbb{Z}$, namely the aggregation of all finite subsets of $\mathbb{R}$ with the group operation: $A \cdot B =$ the symmetric difference of $A$ and $B$.

**Proof** Let $D$ be a central division algebra over $\mathbb{R}(X)$. Then by the same reason as before, $\mathbb{C}(X)$ is a splitting field of $D$. This proves (1), and some maximal subfield of $D$ is isomorphic to $\mathbb{C}(X)$. Thus $D$ is in the form of $D = \{-1, f\}$ for some $f \in K^\times$, $K = \mathbb{R}(X)$, and we have $Br(\mathbb{R}(X)) \simeq K^\times/(K^2 + K^2)^\times$.

If $f = \varphi^2 + \psi^2$, then $f(a) \geq 0$ for $^\forall a \in \mathbb{R}$. Conversely, if $f(a) \geq 0$ for $^\forall a \in \mathbb{R}$, then $f$ is decomposed into the product $f = \prod_i (X - a_i)^2 \prod_j (X - \alpha_j)(X - \overline{\alpha_j})$,
$a_i \in \mathbb{R}, \alpha_j \in \mathbb{C} \setminus \mathbb{R}$. Since $(X - \alpha_j)(X - \overline{\alpha_j}) = \mathrm{N}_{K(\sqrt{-1})/K}(X - \alpha_j)$, we have $f \in K^2 + K^2$. Therefore, as a complete representative system of $K^\times/(K^2 + K^2)^\times$, we get
$\{\pm(X - a_1) \cdots (X - a_n) \mid a_i \in \mathbb{R}, \text{ mutually distinct }\}$.

For $f = \pm(X - a_1)\cdots(X - a_n)$, $D = \{-1, f\}$ is trivial at $a$ such that $f(a) > 0$. It is ramified at $a_i$ and at the non-prime place (which will be denoted by $\infty$) if the degree of $f$ is odd. Since $\{-1, f_1\} \otimes_{\mathbb{R}(X)} \{-1, f_2\} \sim \{-1, f_1 f_2\}$, the multiplication in $Br(\mathbb{R}(X))$ corresponds to the symmetric difference of the sets of ramified places. Thus we have obtained the desired result (2).

**Remark** A discrete valuation is called real (or imaginary) if its residue field is $\mathbb{R}$ (or $\mathbb{C}$). The set of all real places will be denoted by $RP(K)$. For $K = \mathbb{R}(X)$, we have $RP(K) = \mathbb{R} \coprod \{\infty\}$.

Then, we have $Br(\mathbb{R}(X)) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(\mathbb{R}(X))}$. The isomorphism is given as follows. Suppose that a central division algebra $D$ over $\mathbb{R}(X)$ corresponds to a finite subset $A$ of $RP(\mathbb{R}(X)) = \mathbb{R} \coprod \{\infty\}$. $D$ is ramified at every $a \in A \setminus \{\infty\}$, and at $\infty$ if $|A \setminus \{\infty\}|$ is odd. There are two $D$s which are ramified at no place. They are attributed to $\mathbb{Z}/2\mathbb{Z}$ at $\infty$.

**Corollary** $\mathbb{R}(X)$ satisfies Hasse's principle.

$\{-1, -1\}$ is unramified but non-trivial at every place. All other non-trivial $\{-1, f\}$ are ramified at some places.

## 2. Brauer group of $\mathbb{R}(X, Y)$.

Let $K$ be a finite extension of $\mathbb{R}(X)$, namely an algebraic function field of one variable over $\mathbb{R}$. In other words, $K = \mathbb{R}(X, Y)$, $Y$ is algebraic over $\mathbb{R}(X)$.

If $\sqrt{-1} \in K$, then $K$ is an algebraic function field of one variable over $\mathbb{C}$, so that $Br(K)$ is trivial.

Hereafter we shall assume that $\sqrt{-1} \notin K$. Since $Br(K(\sqrt{-1}))$ is trivial, a central division algebra $D$ over $K$, splits over $K(\sqrt{-1})$. This implies that $D$ is a quaternion algebra and $D = \{-1, f\}$ for some $f \in K^\times$. From this we see that $Br(K)$ has the exponent 2, and $Br(K) \simeq K^\times/(K^2 + K^2)^\times$.

A valuation on $K$ which is trivial on $\mathbb{R}^\times$ is called a place. The residue field of a place $v$ is $\mathbb{R}$ or $\mathbb{C}$, according to which $v$ is called real or imaginary. (Note that this terminology differs from the ones used for algebraic number fields).

For an imaginary place $v$, $D_v$ is trivial over $K_v$. For a real place $v$, $D_v$ is one of four algebras over $K_v$. The one is trivial, another one is an unramified quaternion, and the other two are ramified quaternions. See the results in §1.

Let $RP(K)$ be the set of all real places. Since the place of $K(\sqrt{-1}) = \mathbb{C}(X,Y)$ are in one-to-one correspondence with points of a compact Riemann surface $\mathfrak{R}$, and since a real place $v$ of $K$ does not decompose in $K(\sqrt{-1})$, $RP(K)$ is identified with a subset of $\mathfrak{R}$.

For a real place $v$ of $K$, we have $^{\exists}\varphi \in K$, $\mathrm{ord}_v(\varphi) = 1$. Then, $\varphi(z)$ is a local coordinate in a neighbourhood of the corresponding $z_v \in \mathfrak{R}$. Since $z \in RP(K)$ is equivalent to $\varphi(z) \in \mathbb{R}$ in this neighbourhood, $RP(K)$ is a one-dimensional real manifold. Since $\mathfrak{R}$ is compact, $RP(K)$ consists of $\nu$ closed curves, where $\nu$ is the number of connected components of $RP(K)$.

**Theorem 2** We have $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K)}$.
The isomorphism is given as follows: Fix a point $z_i$ $(1 \leq i \leq \nu)$ from each connected component of $RP(K)$. Suppose that $Br(K) \ni D$ corresponds to a finite subset $A$ of $RP(K)$. Then, $D$ is ramified at $A \setminus \{z_1, \cdots, z_\nu\}$ and possibly at $z_i$. The ramification at $z_i$ is determined by the rule that $D$ is ramified at even number of places on each connected component of $RP(K)$.

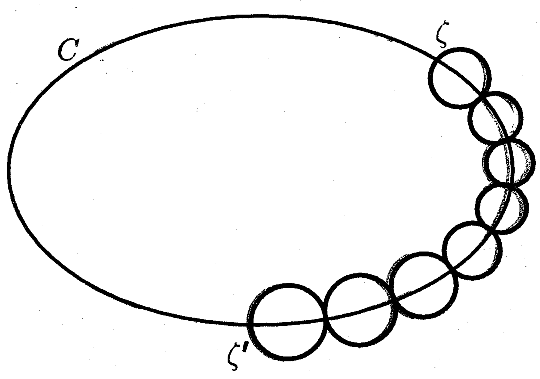
There are $2^\nu$ different division algebras which are ramified at no real place. They are attributed to $(\mathbb{Z}/2\mathbb{Z})^{\{z_1,\cdots,z_\nu\}}$.

**Proof** Let $Br_1(K)$ be the group of all division algebras which are ramified at no real place. Then, $D = \{-1, f\} \in Br_1(K)$ is equivalent to that $\mathrm{ord}_z(f)$ is even for every $z \in RP(K)$, namely that $f(z)$ has definite sign on each connected component of $RP(K)$.

As shown later, Hasse's principle holds for $K = \mathbb{R}(X,Y)$. Therefore, $D = \{-1, f\}$ is trivial if and only if $f$ is non-negative on $RP(K)$, so that we have $|Br_1(K)| \leq 2^\nu$. The equality holds if for any connected component $C$ of $RP(K)$, there exists $f \in K^\times$ such that $f(z) \leq 0$ on $C$ but $f(z) \geq 0$ on $RP(K) \setminus C$. Since $RP(K)$ is mapped homeomorphically into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1 \leq i \leq 4}, T_1(z) = \frac{X(z)}{X(z)^2+1}, T_2(z) = \frac{1}{X(z)^2+1}, T_3(z) = \frac{Y(z)}{Y(z)^2+1}, T_4(z) = \frac{1}{Y(z)^2+1}$, and since the function $F$ defined by $F(z) = -1$ on $C$ and $F(z) = 1$ on $RP(K) \setminus C$ is continuous on $RP(K)$, the polynomial approximation theorem of Weierstrass assures that there exists a polynomial $P(T_i)$ such that $P(T_i(z)) < 0$ on $C$ but $P(T_i(z)) > 0$ on $RP(K) \setminus C$. This completes the proof of $Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{\{z_1,\cdots,z_\nu\}}$.

Take any $f \in K^\times$. If $\operatorname{ord}_{z_0}(f)$ is odd for $z_0 \in RP(K)$, then $f(z)$ changes its sign when $z$ crosses $z_0$. Since a connected component $C$ of $RP(K)$ is a closed curve, $f(z)$ must change its sign even times on $C$, therefore $D = \{-1, f\}$ is ramified at even number of places on $C$.

Now, we shall show that for any two points $\zeta$ and $\zeta'$ on $C$, there exists $f \in K^\times$ such that $D = \{-1, f\}$ is ramified at $\zeta$ and $\zeta'$, but not ramified at other real places. Again we shall map $RP(K)$ into $\mathbb{R}^4$ by $z \mapsto (T_i(z))_{1\le i\le 4}$. Since $C$ is a closed analytic curve, there are $\zeta = \zeta_0,\ \zeta_1,\ \cdots\ \zeta_n = \zeta'$ ($\zeta_i \in C$) and small spheres $S_j : \sum_{i=1}^{4}(T_i - a_{ij})^2 = r_j^2$ such that $S_j \cap RP(K) = \{\zeta_{j-1}, \zeta_j\}$.




Then, $f = \prod_{j=1}^{n}\left\{\sum_{i=1}^{4}(T_i(z) - a_{ij})^2 - r_j^2\right\}$ satisfies $\operatorname{ord}_\zeta(f) = \operatorname{ord}_{\zeta'}(f) = 1$, $\operatorname{ord}_{\zeta_i}(f) = 2$ $(1 \le i \le n-1)$, and $\operatorname{ord}_z(f) = 0$ for $z \in RP(K) \setminus \{\zeta_i\}$. This $f$ is the desired element of $K^\times$.

Thus we have proved $Br(K)/Br_1(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K)\setminus\{z_1,\cdots,z_\nu\}}$, so combining with the result for $Br_1(K)$, we get $Br(K) \simeq (\mathbb{Z}/2\mathbb{Z})_0^{RP(K)}$.

**Remark** $K$ satisfies Hasse's principle as a result of the following lemma.

Let $\square K$ be the set of all sums of squares, $\square K = \{\sum x_i^2 | x_i \in K\}$.

**Lemma** Let $K = \mathbb{R}(X, Y)$ be an algebraic function field over $\mathbb{R}$.

(1) For $f \in K^\times, f \in \square K$ if and only if $f(z) \geq 0$ for $^\forall z \in RP(K)$. Especially, if $RP(K) = \phi$ then $\square K = K$.

(2) Every element of $\square K$ can be written as a sum of two squares.

We shall omit the proof here, and refer to [6], Th.3.2, Chap.3 and Th.2.1, Chap.4.

**Corollary** $K = \mathbb{R}(X, Y)$ satisfies Hasse's priciple.

**Proof** $D = \{-1, f\}$ is locally trivial if and only if $f(z) \geq 0$ for $^\forall z \in RP(K)$, which is equivalent to $f \in K^2 + K^2 = \mathrm{N}_{K(\sqrt{-1})/K}\left(K(\sqrt{-1})^\times\right)$, hence $D = \{-1, f\}$ is trivial.

## 3. Approximation in idele groups.

Let $R$ be a Dedekind domain, and $K$ be its quotient field. Every prime ideal $p$ of $R$ defines the $p$-adic valuation on $K$. This is called a prime valuation. Besides $p$-adic valuations, we often consider some others, which are called non-prime valuations. For instance, all valuations trivial on $k^\times$ when $K$ is an algebraic function field over the constant field $k$.

We define the adele ring $R_{\mathbb{A}}$ of $R$ by $R_{\mathbb{A}} = \prod_p R_p$, where $p$ runs over all prime valuations and $R_p$ denotes the completion of $R$ at the place $p$. Also we define the adele ring $K_{\mathbb{A}}$ of $K$ by $K_{\mathbb{A}} = K \otimes_R R_{\mathbb{A}} \simeq \bigcup_S (\prod_{p \in S} K_p \times \prod_{p \notin S} R_p)$ where $S$ runs over all finite set of prime valuations. The idele group $K_{\mathbb{A}}^\times$ is defined as the group of inversible elements of $K_{\mathbb{A}}$. It is written in the form of $K_{\mathbb{A}}^\times = \bigcup_S (\prod_{p \in S} K_p^\times \times \prod_{p \notin S} R_p^\times)$.

The fundamental system of neighbourhoods of 0 in $K_{\mathbb{A}}$ is given by $\{V(S, n)\}$, where

$$V(S, n) = \prod_{p \in S} p^n R_p \times \prod_{p \notin S} R_p.$$

Similarly, the fundamental system of neighbourhoods of 1 in $K_{\mathbb{A}}^\times$ is given by $\{U(S, n)\}$, where

$$U(S,n)=\prod_{p\in S}(1+p^nR_p)\times\prod_{p\notin S}R_p^\times.$$

Let $D$ be a central division algebra over $K$. A finitely generated $R$-submodule of $D$ is called an $R$-lattice, and if it spans $D$ as a $K$-vector space, it is called a full $R$-lattice. An $R$-lattice is called an $R$-order, if it is a subring including 1 (= the unit element of $D$).

The adele ring $D_{\mathbb{A}}$ of $D$ is defined by $D_{\mathbb{A}} = D\otimes_K K_{\mathbb{A}}$. It is written in the form of $D_{\mathbb{A}}=\bigcup_S(\prod_{p\in S}D_p\times\prod_{p\notin S}\Gamma_p)$, where $\Gamma$ is a full $R$-order of $D$ and $\Gamma_p=\Gamma\otimes_R R_p$. The idele group $D_{\mathbb{A}}^\times$ is defined similarly. The fundamental system of neighbourhoods of 1 in $D_{\mathbb{A}}^\times$ is given by

$$U(S,n)=\prod_{p\in S}(1+p^n\Gamma_p)\times\prod_{p\notin S}\Gamma_p^\times.$$

$D$ is diagonally imbedded into $D_{\mathbb{A}}$, and $D^\times$ is diagonally imbedded into $D_{\mathbb{A}}^\times$. $D$ is dense in $D_{\mathbb{A}}$(by chinese remainder theorem), but $D^\times$ is not dense in $D_{\mathbb{A}}^\times$. But $D^\times$ may be dense in some subgroup of $D_{\mathbb{A}}^\times$.

Let $\mathfrak{n}_{D/K}$ be the reduced norm $D\to K$. $\mathfrak{n}_{D/K}$ maps $D^\times$ homomorphically into $K^\times$. We shall denote its kernel by $D^{(1)}$. $\mathfrak{n}_{D/K}$ is uniquely extended as a $K_{\mathbb{A}}$-valued polynomial function on $D_{\mathbb{A}}$. This extension is denoted by the same symbol $\mathfrak{n}_{D/K}$, and its kernel in $D_{\mathbb{A}}^\times$ is denoted by $D_{\mathbb{A}}^{(1)}$.

Eichler theorem ascertains that for global fields, $D^{(1)}$ is dense in $D_{\mathbb{A}}^{(1)}$ (in the topology of $D_{\mathbb{A}}^\times$) if and only if $D_v$ is not a division algebra for some non-prime $v$.

For global fields, we have also $D^{(1)}=[D^\times,D^\times]$, the commutator group of $D^\times$. But for a general $K$, this relation does not hold (For instance, Platonov [3]).

For a general $K$, in the connection with the cancellation problem of $\Gamma$, it seems natural to consider $[D^\times,D^\times]$ rather than $D^{(1)}$. Thus we define the strong approximation property as follows: A central division algebra $D$ is said to have strong approximation property if $[D^\times,D^\times]$ is dense in $[D_{\mathbb{A}}^\times,D_{\mathbb{A}}^\times]$. To find a necessary and sufficient condition for strong approximation property is a generalization of Eichler theorem to a general case.

In the connection with the cancellation problem of $\Gamma$, we consider a little weaker approximation property. $D$ is said to have $D^\times$-approximation, if the closure of $D^\times$ (in the topology of $D_{\mathbb{A}}^{\times}$) contains $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}]$. $D$ is said to have $R_{\mathbb{A}}^{\times}D^{\times}$-approximation, if the closure of $R_{\mathbb{A}}^{\times}D^{\times}$ contains $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}]$. (Both of $D^{\times}$ and $R_{\mathbb{A}}^{\times}$ are contained in $D_{\mathbb{A}}^{\times}$, so $R_{\mathbb{A}}^{\times}D^{\times} \subset D_{\mathbb{A}}^{\times}$.) The last and weakest approximation property is necessary and sufficient for the cancellation of every full $R$-order $\Gamma$ of $D$ (namely $\Gamma \oplus \Gamma \simeq L \oplus \Gamma$ implies $\Gamma \simeq L$, the isomorphism being as $\Gamma$-lattices).

## 4. Eichler theorem for $\mathbb{R}(X)$.

In §1 we have seen that $D = \{-1, f\}$ is trivial at the non-prime place $\infty$ if and only if $f$ is monic of even degree.

**Theorem 3** If $D_\infty$ is not trivial, then $D^{\times}$ is discrete in $D_{\mathbb{A}}^{\times}$ and $R_{\mathbb{A}}^{\times}D^{\times}$ is closed in $D_{\mathbb{A}}^{\times}$.

**Corollary** If $D_\infty$ is not trivial, then $R_{\mathbb{A}}^{\times}D^{\times}$-approximation property does not hold.

**Proof of Corollary** It suffices to show $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}] \not\subset R_{\mathbb{A}}^{\times}D^{\times}$. For a real place $a$, we shall identify $D_a^{\times}$ with the subgroup $D_a^{\times} \times \prod_{p \neq a}(1)_p$ of $D_{\mathbb{A}}^{\times}$. It is clear that $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}] \cap D_a^{\times} = [D_a^{\times}, D_a^{\times}]$. Since $D_a$ is a quaternion (or a matrix) algebra over $K_a$, we have $[D_a^{\times}, D_a^{\times}] = D_a^{(1)}$, so that $[D_a^{\times}, D_a^{\times}] \not\subset K_a^{\times}$.

On the other hand, if $x = (x_p) \in R_{\mathbb{A}}^{\times}D^{\times} \cap D_a^{\times}$, then we have ${}^{\exists}d \in D^{\times}, {}^{\forall}p$ (prime place), ${}^{\exists}r_p \in R_p^{\times}, x_p = r_p d$. For $p \neq a$, we have $x_p = 1$ so that $d = r_p^{-1} \in R_p^{\times} \subset K_p^{\times}$, so that $d \in D^{\times} \cap K_p^{\times} = K^{\times}$, hence $x_a = r_a d \in R_a^{\times}K^{\times} \subset K_a^{\times}$. This assures $R_{\mathbb{A}}^{\times}D^{\times} \cap D_a^{\times} \subset K_a^{\times}$ so that $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}] \not\subset R_{\mathbb{A}}^{\times}D^{\times}$.

**Proof of Theorem 3** $D = \{-1, f\}$ means that

$$D = K + Ki + Kj + Kij$$

$$i^2 = -1,\ j^2 = f,\ ji = -ij.$$

Then $\Gamma = R + Ri + Rj + Rij$ is a full $R$-order of $D$ ($K = \mathbb{R}(X), R = \mathbb{R}[X]$).

A fundamental neighbourhood of 1 in $D_{\mathbb{A}}^{\times}$ is $U(g) = \prod_p(1 + g\Gamma_p) \cap \prod_p \Gamma_p^{\times}$ for $g \in R$ and we have $U(g) \cap D^{\times} = (1 + g\Gamma) \cap \Gamma^{\times}$, so the first half of Theorem 3 is proved if

$(1+g\Gamma)\cap\Gamma^\times=(1)$.

Suppose that $d=\varphi_1+\varphi_2 i+\varphi_3 j+\varphi_4 ij\in(1+g\Gamma)\cap\Gamma^\times,\varphi_i\in R$. This means that $\varphi_1\equiv 1 \mod g,\varphi_i\equiv 0 \mod g$ for $i\geq 2$, and $\mathfrak{n}_{D/K}(d)=\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)\in R^\times=\mathbb{R}^\times$. If $g\in R\setminus R^\times$, substituting a zero of $g$, we see that $\mathfrak{n}_{D/K}(d)=1$.

Since each $\varphi_i^2$ has, if not zero, a positive coefficient of the highest degree term, such terms of $\varphi_1^2$ and $\varphi_2^2$ (resp. $\varphi_3^2$ and $\varphi_4^2$) do not cancel.

From $\varphi_1^2+\varphi_2^2-1=f(\varphi_3^2+\varphi_4^2)$, if $f$ is of odd degree, both hand sides should be zero. This implies that $\varphi_3=\varphi_4=0$ and $\varphi_1,\varphi_2\in\mathbb{R}$, which implies $\varphi_2=0$ and $\varphi_1=1$ because $\varphi_2$ is a multiple of $g$.

If $f$ is of even degree with a negative coefficient of the highest degree term, then the highest degree terms of $\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)$ do not cancel, so that we have $\forall i,\varphi_i\in\mathbb{R}$. This again implies $\varphi_i=0$ for $i\geq 2$, and so $\varphi_1=1$.

Thus the first half of Theorem 3 has been proved. Similar discussions show that $(R+g\Gamma)^\times=R^\times$, if $g\in R\setminus R^\times$.

$\Gamma_g=R+g\Gamma$ is a full $R$-order of $D$, and $(\Gamma_g)_\mathbb{A}^\times=\prod_p(R_p+g\Gamma_p)^\times$ is an open subgroup of $D_\mathbb{A}^\times$, so $(\Gamma_g)_\mathbb{A}^\times D^\times$ is open and closed, hence $\bigcap_g(\Gamma_g)_\mathbb{A}^\times D^\times$ is a closed subgroup of $D_\mathbb{A}^\times$, containing $R_\mathbb{A}^\times D^\times$.

We shall show the inverse inclusion. Take any $x\in\bigcap_g(\Gamma_g)_\mathbb{A}^\times D^\times$, then $\forall g,\exists\gamma_g\in(\Gamma_g)_\mathbb{A}^\times,\exists d_g\in D^\times,x=\gamma_g d_g$. Since $(\Gamma_g)_\mathbb{A}^\times\cap D^\times=(R+g\Gamma)^\times=R^\times$, $d_g$ is determined modulo $R^\times$, so if $g_1$ is a multiple of $g$, then $d_{g_1}$ differs from $d_g$ only modulo $R^\times$. This implies that we can choose $d_g$ independently of $g$, thus $\exists d\in D^\times,xd^{-1}\in\bigcap_g(\Gamma_g)_\mathbb{A}^\times$.

But we have $R_\mathbb{A}^\times=\bigcap_g(\Gamma_g)_\mathbb{A}^\times$, because $\forall p,\bigcap_g(R_p+g\Gamma_p)^\times=R_p^\times$. Thus the proof of the second half of Theorem 3 is completed.

**Theorem 4** If $D_\infty$ is trivial, hence if $f$ is monic of even degree, then $[D^\times,D^\times]=D^{(1)}$ is dense in $[D_\mathbb{A}^\times,D_\mathbb{A}^\times]$.

This theorem is divided into the following two parts.

**Theorem 4.1** If $f$ is monic of even degree, then for $g,h\in R$ such that $(h,gf)=1$, we have

$$(1+g\Gamma)\cap(i+h\Gamma)\cap\Gamma^{\times}\neq\phi.$$

**Theorem 4.2** The conclusion part of Theorem 4.1 is equivalent to strong approximation property.

**Proof of Theorem 4.1** It suffices to show the existence of $\varphi_i \in R, 1 \leq i \leq 4$ such that

$$\varphi_1 \equiv 1 \pmod{g}, \quad \varphi_i \equiv 0 \pmod{g},\ 2 \leq i \leq 4$$
$$\varphi_2 \equiv 1 \pmod{h}, \quad \varphi_i \equiv 0 \pmod{h},\ i=1,3,4, \quad \text{and}$$

$$\varphi_1^2+\varphi_2^2-f(\varphi_3^2+\varphi_4^2)=1. \tag{1}$$

Put $\varphi_1 = 1+g^2fu_1, \varphi_2 = g^2fu_2, \varphi_i = gu_i (i=3,4)$, then the required congruence modulo $g$ is automatically satisfied. Substituting them into (1) and dividing both sides by $g^2f$, we get

$$2u_1+g^2fu_1^2+g^2fu_2^2-(u_3^2+u_4^2)=0. \tag{2}$$

Since $(h^2, g^2f)=1$, $g^2f$ is inversible in $R/h^2R$, so there exist $\psi, \psi' \in R$ such that

$$g^2f\psi = 1+h^2\psi'.$$

Put $u_1 = -\psi+h^2v_1, u_2 = \psi+h^2v_2, u_i = hv_i (i=3,4)$, then the required congruence modulo $h$ is automatically satisfied. Substituting them into (2), we get

$$-2\psi+2h^2v_1+g^2f\{2\psi^2+2h^2\psi(v_2-v_1)+h^4(v_1^2+v_2^2)\}=h^2(v_3^2+v_4^2).$$

Since $-2\psi+2g^2f\psi^2=-2\psi(1-g^2f\psi)=2h^2\psi\psi'$, we have

$$2\psi\psi'+2(1-g^2f\psi)v_1+2g^2f\psi v_2+g^2fh^2(v_1^2+v_2^2)=v_3^2+v_4^2. \tag{3}$$

Put $v_1=(1-g^2f\psi)w$ and $v_2=g^2f\psi w$, then we get

$$2\psi\psi'+\{(1-g^2f\psi)^2+(g^2f\psi)^2\}(2w+g^2fh^2w^2)=v_3^2+v_4^2. \tag{4}$$

A polynomial $P \in R = \mathbb{R}[X]$ belongs to $R^2 + R^2$, if and only if $P(a) \geq 0$ for ${}^\forall a \in \mathbb{R}$, as shown in the proof of Theorem 1. So it suffices to show that the left hand side of (4) is everywhere non-negative for some $w \in R$.

Put $2\psi\psi' = F$ and $g^2fh^2 = G$, then $(1-g^2f\psi)^2 + (g^2f\psi)^2 = 1 - 2g^2f\psi(1-g^2f\psi) = 1 + 2g^2f\psi h^2\psi' = 1 + FG$, so we have

$$F + (1+FG)(2w + Gw^2) \geq 0. \tag{5}$$

The above calculation also shows $1+FG \geq \frac{1}{2}$, namely $FG \geq -\frac{1}{2}$. Since $f$ is monic of even degree,we have $\lim_{t\to\pm\infty} G(t) = \infty$ so that ${}^\exists M > 0, {}^\forall t \in \mathbb{R}, G(t) \geq -M$. Since $\{t \mid G(t) \leq 0\}$ is compact, $F$ is bounded there, so ${}^\exists N > 0, |F(t)| \leq N$ for $G(t) \leq 0$.

The left hand side of (5) is zero for

$$w = \frac{1}{G}\left\{-1 \pm (1+FG)^{-\frac{1}{2}}\right\}.$$

Since $(1+t)^{-\frac{1}{2}} \leq 1 - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2$ for $t \geq -\frac{1}{2}$, if we set $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G$, then (5) is satisfied for $G \geq 0$. Let $P$ be an everywhere positive polynomial of two variables $s$ and $t$, then $w = -\frac{F}{2} + \frac{3}{\sqrt{2}}F^2G + P(G, FG)$ satisfies (5) for $G \geq 0$.

The condition (5) is satisfied also for $G < 0$, if

$$-1 \leq -\frac{t}{2} + \frac{3}{\sqrt{2}}t^2 + sP(s,t) \leq -1 + (1+t)^{-\frac{1}{2}} \tag{6}$$

on $\Delta = \{(s,t) \mid -M \leq s \leq 0,\ t \geq -\frac{1}{2},\ |t| \leq N|s|\}$. The condition (6) is satisfied if

$$\varepsilon \geq \frac{1}{s}\left\{1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2\right\} + P(s,t) \geq 0$$

on $\Delta$, where $\varepsilon \leq (1+NM)^{-\frac{1}{2}}/M$. Since $\alpha(s,t) = \frac{1}{s}\left\{1 - (1+t)^{-\frac{1}{2}} - \frac{t}{2} + \frac{3}{\sqrt{2}}t^2\right\}$ is non-positive and continuous on $\Delta$ (it is continuous at $(0,0)$ because of $|t| \leq N|s|$), such a polynomial $P(s,t)$ exists by virtue of polynomial approximation theorem of Weierstrass. $P(s,t)$ can be assumed everywhere positive, because we can put $P = Q^2 + \frac{\varepsilon}{2}$, $Q$ being an approximating polynomial of $\sqrt{|\alpha(s,t)|}$. Thus Theorem 4.1 has been proved.

**Proof of Theorem 4.2** Let $H$ be the closure of $[D^\times, D^\times] = D^{(1)}$ in $D^\times_{\mathbb{A}}$. Let $p_0$ be a prime place where $D$ is unramified, and let $i_{p_0} = (1, \cdots, 1, i, 1, \cdots) \in D^\times_{\mathbb{A}}$ be the element of $D^\times_{\mathbb{A}}$ whose $p_0$-coordinate is $i$, while other coordinates are 1.

The proof is completed by the following steps, which are slight modifications of ones given in [1] §51.

**Step 1** The conclusion part of Theorem 4.1 is equivalent to that $^{\forall}p_0$ (where $D_{p_0}$ is unramified), $i_{p_0} \in H$ (note that $i_{p_0} \in D_{p_0}^{(1)} = [D_{p_0}^{\times}, D_{p_0}^{\times}] \subset [D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}]$).

**Step 2** Identify $D_{p_0}^{(1)}$ with a subgroup $D_{p_0}^{(1)} \times \prod_{p \neq p_0} (1)_p$ of $D_{\mathbb{A}}^{\times}$, then $H \cap D_{p_0}^{(1)}$ is a closed normal subgroup of $D_{p_0}^{(1)}$.

**Step 3** If $D$ is unramified at $p_0$, then $i_{p_0} \in H$ implies $D_{p_0}^{(1)} \subset H$.

If $D_{p_0}$ is a matrix algebra, the assertion is a result of simplicity of $PSL(2, K_{p_0})$. If $D_{p_0}$ is an unramified quaternion algebra, since $x = a + bi + cj + dij \in D_{p_0}^{(1)}$ satisfies $x^2 - 2ax + 1 = 0$, the condition $x \in H$ depends only on $a$. (Here we identify $x \in D_{p_0}^{(1)}$ with $x_{p_0} = (1, \cdots, 1, x, 1, \cdots) \in D_{\mathbb{A}}^{\times}$.)

Take any $x = a + bi + cj + dij \in D_{p_0}^{(1)}$. Since $b^2 + c^2 + d^2$ has a root in $K_{p_0}$, we have $^{\exists}e \in K_{p_0}, b^2 + c^2 + d^2 = e^2$. If $i \in H$, then $-ai + ej \in H$, therefore $i(-ai + ej) = a + eij \in H$, hence $x \in H$. This means $D_{p_0}^{(1)} \subset H$.

**Step 4** Assume the conclusion part of Theorem 4.1. For a finite set $S$ of prime places, we have $\prod_{p \in S} D_p^{(1)} \times \prod_{p \notin S} (1)_p \subset H$.

If $D$ is unramified on $S$, the assertion is a consequence of Step 3.

Let $S_0$ be the set of all prime places where $D$ is ramified. The assertion for $S = S_0$ follows from the fact that $D^{(1)}$ is dense in $\prod_{p \in S_0} D_p^{(1)}$ in the product topology of $D_p^{\times}$.

**Step 5** $\bigcup_S \left( \prod_{p \in S} D_p^{(1)} \times \prod_{p \notin S} (1)_p \right)$ is dense in $[D_{\mathbb{A}}^{\times}, D_{\mathbb{A}}^{\times}]$.

Combining the five assertions above, we complete the proof of Theorem 4.2.

## 5. Eichler theorem for $\mathbb{R}(X, Y)$.

For an algebraic function field $K = \mathbb{R}(X, Y)$, we shall fix a set $P$ of valuations (which are trivial on $\mathbb{R}^{\times}$). A valuation $v \in P$ is called a prime place and $v \notin P$ is called a non-prime place. We assume that there exists a non-prime place. Then, $R_P = \{x \in K | {}^\forall v \in P, v(x) \leq 1\}$ is a Dedekind domain and $K$ is its quotient field. A prime ideal of $R_P$ is given by $p_v = \{x \in R_P | v(x) < 1\}$ for $v \in P$.

The adele ring and the idele group are constructed using prime places only. We shall write $R$ instead of $R_P$.

We consider the following property(E):

(E) A central division algebra $D$ over $K$ has strong approximation property, if $D$ is trivial at some non-prime place.

The converse of the property(E) holds always as shown below.

**Theorem 5** If a central division algebra $D$ is non-trivial at every non-prime place, then $D$ does not have $R_{\mathbb{A}}^\times D^\times$-approximation property.

**Remark** Before proving this theorem, we shall mention about the product formula. The formula is expressed as follows using $\mathrm{ord}_v$; $v(x) = \theta^{\mathrm{ord}_v(x)}$ $(0 < \theta < 1)$.

$$ {}^\forall x \in K^\times, \sum_{v:\text{real}} \mathrm{ord}_v(x) + 2 \sum_{v:\text{imag.}} \mathrm{ord}_v(x) = 0, $$

where the sum is taken over all places, prime or not.

**Proof** Similar discussions as the proof of Theorem 3 show that it suffices to prove that

$$ (R + g\Gamma)^\times = R^\times \text{ for } g \in R \setminus R^\times. $$

Let $D = \{-1, f\}, f \in R$. The assumption of Theorem 5 means that all non-prime places are real and that for every non-prime place $v$, $\mathrm{ord}_v(f)$ is odd or $\mathrm{ord}_v(f)$ is even with a negative coefficient of the lowest degree term with respect to the prime element $\pi_v$.

Suppose that $\varphi_1 + \varphi_2 i + \varphi_3 j + \varphi_4 ij \in (R + g\Gamma)^\times$, then $\varphi_1 \in R, \varphi_i \in gR (2 \leq i \leq 4)$, and $\varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2) \in R^\times$. Put $\varphi = \varphi_1^2 + \varphi_2^2 - f(\varphi_3^2 + \varphi_4^2)$, then $\varphi \in R^\times$ implies $\mathrm{ord}_v(\varphi) = 0$ for every prime place $v$. As for a non-prime place $v$, the assumption on $f$ implies that the lowest degree terms do not cancel, so that $\mathrm{ord}_v(\varphi) = \mathrm{Min}(2\mathrm{ord}_v(\varphi_1), 2\mathrm{ord}_v(\varphi_2), \mathrm{ord}_v(f) + 2\mathrm{ord}_v(\varphi_3), \mathrm{ord}_v(f) + 2\mathrm{ord}_v(\varphi_4))$, if $\varphi_i \neq 0$.

Combining this with the product formula, we have

$$ \sum_{\text{non-prime}} \mathrm{ord}_v(\varphi_i) \geq 0 \quad (i = 1, 2), $$

$$\sum_{\text{non-prime}} \mathrm{ord}_v(\varphi_i) \geq \frac{1}{2} \sum_{\text{prime}} \alpha_v \mathrm{ord}_v(f) \quad (i = 3, 4)$$

where $\alpha_v = 1$ for a real $v$ and $\alpha_v = 2$ for an imaginary $v$. Since $\varphi_i \in R$ and $f \in R$, we have $\mathrm{ord}_v(\varphi_i) \geq 0$ and $\mathrm{ord}_v(f) \geq 0$ for a prime place $v$, hence again from the product formula, we must have $\mathrm{ord}_v(\varphi_i) = 0$ for every prime place $v$. This means $\varphi_i \in R^\times$. For $i \geq 2$, this contradicts with $\varphi_i \in gR$, so we must have $\varphi_i = 0$, which in turn implies $\varphi_1 \in R^\times$. This completes the proof of $(R + g\Gamma)^\times = R^\times$.

**Remark** Property(E) depends not only on $K$, but also on $R$, or equivalently on the choice of non-prime places. However:

**Theorem 6** (1) Suppose that property(E) holds whenever $R$ has only one non-prime place, then it holds for any $R$.

(2) For the rational function field $K = \mathbb{R}(X)$, property(E) holds for any $R$.

**Proof of (1)** Let $P(R)$ be the set of all prime places for the Dedekind domain $R$. Then $P(R') \subset P(R)$ implies $R \subset R'$. We shall denote the idele group of $D$ with respect to $R$ by $D^\times_{\mathbb{A}}(R)$. Then $P(R) = P(R') \coprod P(R_1)$ implies that $D^\times_{\mathbb{A}}(R)$ is the product topological group of $D^\times_{\mathbb{A}}(R')$ and $D^\times_{\mathbb{A}}(R_1)$, because of $D^\times_{\mathbb{A}}(R) = \bigcup_S \left( \prod_{v \in S} D_v^\times \times \prod_{v \in P(R) \setminus S} \Gamma_v^\times \right)$ where $S$ runs over all finite subsets of $P(R)$.

$D^\times$ is imbedded diagonally in $D^\times_{\mathbb{A}}$, and strong approximation property means precisely that the image $i_R(D^{(1)})$ is dense in $[D^\times_{\mathbb{A}}(R), D^\times_{\mathbb{A}}(R)]$.

If $P(R') \subset P(R)$, then the projection $D^\times_{\mathbb{A}}(R) \to D^\times_{\mathbb{A}}(R')$ maps $i_R(D^{(1)})$ onto $i_{R'}(D^{(1)})$ and $[D^\times_{\mathbb{A}}(R), D^\times_{\mathbb{A}}(R)]$ onto $[D^\times_{\mathbb{A}}(R'), D^\times_{\mathbb{A}}(R')]$. Therefore, if $i_R(D^{(1)})$ is dense in $[D^\times_{\mathbb{A}}(R), D^\times_{\mathbb{A}}(R)]$, then $i_{R'}(D^{(1)})$ is dense in $[D^\times_{\mathbb{A}}(R'), D^\times_{\mathbb{A}}(R')]$.

Now suppose that $D$ is trivial at some non-prime place $v$ of a given $R$. Let $P_0$ be the set of all places other than $v$, and suppose that property(E) holds for $R_0$ corresponding to $P_0$, then $i_{R_0}(D^{(1)})$ is dense in $[D^\times_{\mathbb{A}}(R_0), D^\times_{\mathbb{A}}(R_0)]$, hence $i_R(D^{(1)})$ is dense in $[D^\times_{\mathbb{A}}(R), D^\times_{\mathbb{A}}(R)]$, so property(E) holds for $R$.

**Remark** The proof of Theorem 4.2 does work for a general algebraic function field $K = \mathbb{R}(X, Y)$ and its Dedekind domain $R$. So, strong approximation property holds for $D = \{-1, f\}$, if $(1 + g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \phi$ for ${}^\forall g, h \in R$ such that $(gf, h) = 1$.

Also the proof of Theorem 4,1 works partially. For $\psi, \psi' \in R$ such that $g^2 f\psi = 1 + h^2\psi'$, put $F = 2\psi\psi'$ and $G = g^2 f h^2$. Then, we have $(1+g\Gamma) \cap (i + h\Gamma) \cap \Gamma^\times \neq \phi$ if $^\exists w \in R,\ F + (1+FG)(2w + Gw^2) \in R^2 + R^2$.

Suppose that $R$ has only one non-prime place $v$, then $f \in R$ means that $f$ does not have a pole other than $v$. If $v$ is real and $D_v$ is trivial, then $\mathrm{ord}_v(f)$ is even and $f(z)$ is positive near $v$. Since $RP(K)$ is compact, this implies that $f$, hence $G$, is bounded from below on $RP(K)$, and that $F$ is bounded on $\{z \in RP(K) | G(z) \leq 0\}$. If $v$ is imaginary, then both $F$ and $G$ are bounded on $RP(K)$. So, similar discussions as the proof of Theorem 4.1 show that $^\exists w \in R,\ F + (1+FG)(2w + Gw^2) \geq 0$ on $RP(K)$.

The proof for general $K$ fails only because the condition "$\varphi \in R$ and $\varphi \geq 0$ on $RP(K)$" does not imply $\varphi \in R^2 + R^2$. Since Hasse's principle is satisfied, $\varphi \in K^2 + K^2$ is assured, but $\varphi \in R^2 + R^2$ is not concluded. We shall give a counter example for an elliptic function field $K = \mathbb{R}(X,Y), Y^2 = (X-a)(X-b)(X-c)$. If $\alpha \in \mathbb{R}$ is smaller than $\mathrm{Min}(a,b,c)$, then we have $X - \alpha > 0$ on $RP(K)$. $X - \alpha$ has a double pole at the non-prime place $v$, while an element of $R^2 + R^2 = \mathrm{N}_{K(\sqrt{-1})/K}(R + \sqrt{-1}R)$ should have $\mathrm{ord}_v \leq -4$.

**Proof of Theorem 6 (2)**

Let $K = \mathbb{R}(X)$ and suppose that $R$ has only one non-prime place $v$.

If $R \neq \mathbb{R}[X]$, then $v$ corresponds to an irreducible polynomial $p$, and $\varphi \in R$ is equivalent to $\varphi = g/p^\nu, g \in \mathbb{R}[X]$ and $\deg g \leq \nu \deg p$. Here we can assume that $\nu$ is even. Then $\varphi \geq 0$ on $RP(K)$ implies $g \geq 0$ on $RP(K)$, so $g$ is of even degree and can be written as $g = g_1^2 + g_2^2,\ g_i \in \mathbb{R}[X],\ \deg g_i \leq \frac{1}{2}\deg g$. Therefore $\varphi = \left(g_1/p^{\nu/2}\right)^2 + \left(g_2/p^{\nu/2}\right)^2$ and $\deg g_i \leq \frac{\nu}{2}\deg p$, so that $\varphi \in R^2 + R^2$.

From the remark above, this completes the proof of Theorem 6 (2).

## References

[1] Curtis-Reiner: "*Methods of representation theory*" vol.1,vol.2. Interscience (1981,1987)

Especially §23 *lattices and orders*, §51 *Jacobinski's cancellation theorem*

[2] H.Hijikata: *On the decomposition of lattices over orders.* to appear

[3] V.P.Platonov: *The Tannaka-Artin problem and reduced $K$-theory.* Math.USSR.Izvestija **10** (1976)211-243

[4] A.Albert: "*Structure of algebras*" Amer.Math.Soc.Colloq.Publ. **24** (1939)

[5] J.P.Serre: "*Corps locaux*" Hermann (1968)

[6] W.Scharlau: "*Quadratic and Hermitian forms*" Springer,GMW **270** (1985)

[7] A.Pfister: *Zur Darstellung definiter Funktionen als Summe von Quadraten.* Invent. Math. 4 (1967)229-237

[8] B.Fein, M.Schacher: *Brauer groups of rational function fields over global fields.* Springer, Lecture Notes in Math. **844** (1981)46-74