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ガロア・タイヒミュラー塔の普遍定義体について
—- 極大退化曲線の変形と織田予想

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§概要。

本稿では、n 個のマークされた点をもつ種数 g の代数曲線のモジュライ空間 $M_{g,n}$ に関する、互いに関連した2つの結果について報告します。

その1つは、$M_{g,n}$ のコンパクト化 $\overline{M}_{g,n}$における "0次元カスプ", 即ち極大退化曲線をパラメトライズする $\overline{M}_{g,n}$ の点の形式近傍に（有限通りの付加構造に対応して）標準的な座標系が導入できること。

もう1つは、曲線の副 $\mathfrak{C}$ 基本群の外部自己同型群における$\pi_1(M_{g,n})$ の普遍モノドロミー表現の「定義体」として現れる $\mathbb{Q}$ の代数拡大体 $\mathbb{Q}_{g,n}^{(\mathfrak{C})}$ が、$2-$ $-2g-n<0$ である限り、(g,n) によらないであろう、という織田孝幸氏の予測にたいする一つの結果です。(ここで $\mathfrak{C}$ は full 即ち 商、部分、拡大で閉じている有限群のクラスとする。) 新しく得られたのは

定理. $\mathbb{Q}_{g,n}^{(\mathfrak{C})} \subset \mathbb{Q}_{0,3}^{(\mathfrak{C})}$.

逆方向の包含関係については、$n>0$, $\mathfrak{C}$: profinite または pro-l の仮定のもとで松本真氏（[Ma]）、中村・高尾尚武氏・上野亮一氏（[NTU]~[N]）等によって、すでにその成立が知られています（参：数理研講究録 884）。

証明は、極大退化曲線の標準変形空間内での形式（管状）近傍の基本群を、ガロアの作用も込めて精密に記述する事によって得られます。また、この記述はさらにいくつかの数論的副産物をもちます。詳しくは RIMS-preprint:

On deformation of maximally degenerate stable marked curves
and Oda's problem

（以下その序文）及びその続編（準備中）を御参照下さい。

§1 Introduction

1.1. In this article, we shall study deformation of a maximally degenerate stable marked curve, from the point of view of Galois representation, i.e., with the aim of comparing Galois actions on the fundamental groups of the original and deformed curves.

We shall start with a construction involving an *explicit parametrization* of a universal deformation of such a degenerate curve X^0 (see §1.2, §2). Then we shall study a certain 1-parameter subfamily of deformation from the Galois theoretic viewpoint. Our first step for this is to construct a "tangential base point" on the total space of deformation, outside X^0 but near each of its singular points. Our explicit parametrization of deformation is crucial in this construction. Paths connecting these base points will then be compared with paths *around* and paths *inside* X^0. (This, of course, involves comparison of Galois actions.) They are all "within" the formal neighborhood of X^0. Since the family is 1-dimensional so that X^0 is a divisor, Grothendieck-Murre theory [GM] can be fully used. This study is presented in §3 (the main result is quoted in §1.3 below).

This will then be applied to the following prediction by Oda ("between lines" in §2 of [O]). The moduli stack $M_{g,n}$ over $\mathbb{Q}$ of n-point marked smooth curves of genus g has a canonical l-adic tower of coverings arising from the monodromy representation of $\pi_1(M_{g,n})$ on the pro-l fundamental group of an n-point punctured smooth curve of genus g. He predicted that the constant field $\mathbb{Q}_{g,n}^{(\text{pro-}l)}$ of this tower is independent of (g,n), as long as (g,n) is "hyperbolic", i.e., $2-2g-n<0$. (The special case of this prediction, the independence of $\mathbb{Q}_{0,n}^{(\text{pro-}l)}$ on $n(\geq 3)$, had previously been communicated to Ihara by Deligne [De].) We shall prove that $\mathbb{Q}_{g,n}^{(\text{pro-}l)} \subset \mathbb{Q}_{0,3}^{(\text{pro-}l)}$. Combined with the already established inclusion [NTU]~[N], [Ma], this will confirm his prediction $\mathbb{Q}_{g,n}^{(\text{pro-}l)} = \mathbb{Q}_{0,3}^{(\text{pro-}l)}$ under the additional assumption $n \geq 1$ (see §1.4 and §4 for details).

1.2. Now let us be more precise on each of the main points. We shall study, in §2, deformation of any "$\mathbf{P}^1_{01\infty}$-diagram" over $\mathbb{Q}$, which is a synonym for "maximally degenerate stable marked curve over $\mathbb{Q}$ with which each irreducible component is smooth". By definition, a $\mathbf{P}^1_{01\infty}$-diagram over $\mathbb{Q}$ is a pair $(X^0, \mathrm{Mark}(X^0))$ of

(i) a geometrically connected reduced proper curve X^0 over $\mathbb{Q}$, with only ordinary double singularities $\mathrm{Sing}(X^0)$,

and

(ii) a finite (possibly empty) set $\mathrm{Mark}(X^0)$ of smooth $\mathbb{Q}$-rational points of X^0 specified ("marked"),

satisfying the condition that each irreducible component X^0_λ of X^0 can be identified with the projective line $\mathbf{P}^1$ over $\mathbb{Q}$ in such a way that

$$(\mathrm{Sing}(X^0) \cup \mathrm{Mark}(X^0)) \cap X^0_\lambda = \{0, 1, \infty\}.$$

It is well-known that each $\mathbf{P}^1_{01\infty}$-diagram over $\mathbb{Q}$ has a universal deformation over the spectrum S of the algebra of formal power series $\mathbb{Q}[[q_1, \cdots, q_m]]$, where $m = |\mathrm{Sing}(X^0)|$. But the general belief seems to have been that there is no canonical choice, *over* $S = \mathrm{Spec}\ \mathbb{Q}[[q_1, \cdots, q_m]]$, of such a universal family. In other words, the algebra $\mathbb{Q}[[q_1, \cdots, q_m]]$ is canonical but the generators $q_1, \cdots, q_m$ are not. We shall show that just by adding a pair of combinatorial structures $(\mathcal{J}, i)$ on $(X^0, \mathrm{Mark}(X^0))$ one can define a "canonical" universal family $(X/S, \mathrm{Mark}(X))$ ($\mathrm{Mark}(X)$ consists of sections $S \to X$ extending $\mathrm{Mark}(X^0)$). Thus, once $(\mathcal{J}, i)$ is given, each of $q_1, \cdots, q_m$ can be regarded as a distinguished formal holomorphic function on the formal neighborhood of the point of the moduli stack $M_{g,n}^{stable}$ that corresponds to $(X^0, \mathrm{Mark}(X^0))$. Here, g is the arithmetic genus of X^0 and $n = |\mathrm{Mark}(X^0)|$.

The additional structure consists of

(a) a tangential structure $\mathcal{J}$ on X^0; giving $\mathcal{J}$ is equivalent to choosing, for each λ and $P \in \mathrm{Sing}(X^0) \cap X^0_\lambda$, one point from the two-point set

$$((\mathrm{Sing}(X^0) - P) \cup \mathrm{Mark}(X^0)) \cap X^0_\lambda,$$

and

(b) an ordering of $\mathrm{Sing}(X^0)$; i.e., a bijection $i : \mathrm{Sing}(X^0) \xrightarrow{\sim} \{1, 2, \cdots, m\}$.

Our construction of $(X/S, \mathrm{Mark}(X))$ is via its formal completion along X^0, which is obtained by pasting "standard" affine formal deformations in a natural way (Theorems 1(§2.3), 1′(§2.4)).

This is a generalization of Tate elliptic curves, and corresponds to a special case of Mumford curves (see §2.4.2). From the point of view of Mumford uniformization theory [Mu], we are only choosing a special Schottky group. But our construction is at least what makes it directly applicable to our main purpose, and also, hopefully, what makes it clear why this special choice of a deformation is canonical.

We shall actually include curves and deformations over $\mathbb{Z}$.

1.3. Now restrict $(X/S, \mathrm{Mark}(X))$ to the diagonal Spec $\mathbb{Q}[[q]]$ of S defined by $q_1 = \cdots = q_m = q$, and let $(C, \mathrm{Mark}(C))$ be the generic member of the restricted family. Then C is a proper *smooth* curve over the quotient field $\mathbb{Q}((q))$ of $\mathbb{Q}[[q]]$, and $\mathrm{Mark}(C)$ consists of finitely many $\mathbb{Q}((q))$-rational points. For any field K (always of characteristic 0 in this article), denote by $\overline{K}$ its algebraic closure, and by $G_K = \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group. Let $\mathcal{C}$ be any class of finite groups which is *almost full*, i.e., closed under the formation of taking subgroups, factor groups and finite direct products. Then the main result of §3 reads as follows.

Theorem 2′(§3.5). *If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on the maximal pro-$\mathcal{C}$ quotient of the fundamental groupoid of $\mathbb{P}^1 \otimes \overline{\mathbb{Q}} - \{0, 1, \infty\}$ w.r.t. the set of Deligne's tangential base points, then σ has an extension $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}((q))}/\mathbb{Q}((q)))$ that acts trivially on the maximal pro-$\mathcal{C}$ quotient of the fundamental groupoid of $C \otimes \overline{\mathbb{Q}((q))} - \mathrm{Mark}(C)$ w.r.t. the set of "tangential base points" $\tilde{\mu}$ ($\mu \in \mathrm{Sing}(X^0)$) defined in §3.3. In particular, the outer action of $\tilde{\sigma}$ on the maximal pro-$\mathcal{C}$ quotient of the fundamental group of $C \otimes \overline{\mathbb{Q}((q))} - \mathrm{Mark}(C)$ is trivial.*

Remark. $(\tilde{\mu})$ and the choice of $\tilde{\sigma}$ are related to each other (see §3 for details).

Actually what we obtain is not just the comparison of the *kernels* of Galois actions, but the comparison of the actions themselves. An explicit description of the Galois action on the fundamental groupoid of $C \otimes \overline{\mathbb{Q}((q))} - \mathrm{Mark}(C)$, in terms of a graph of groups associated with X^0 and the fundamental groupoid of

$\mathsf{P}^1 \otimes \overline{\mathbb{Q}} - \{0,1,\infty\}$, together with applications, will be given in a subsequent article (in preparation).

1.4. Now we come back to Oda's prediction. Let (g,n) be a pair of non-negative integers satisfying $2-2g-n<0$, and $M_{g,n}$ be the moduli stack over $\mathbb{Q}$ of proper smooth curves of genus g with n (ordered) marked points. Then the fibering $M_{g,n+1} \to M_{g,n}$ defined by "forgetting the $(n+1)$-th marked point" gives a universal family of n-point punctured smooth curves of genus g over $M_{g,n}$. If ξ is any geometric point of $M_{g,n}$ and C_ξ is the fiber above ξ, there is a canonical exact sequence of profinite groups (algebraic fundamental groups)

$$1 \to \pi_1(C_\xi, \tilde{\xi}) \to \pi_1(M_{g,n+1}, \tilde{\xi}) \to \pi_1(M_{g,n}, \xi) \to 1, \tag{1}$$

where $\tilde{\xi}$ is any geometric point of C_ξ. This defines an outer action of $\pi_1(M_{g,n},\xi)$ on $\pi_1(C_\xi,\tilde{\xi})$ (by conjugation), and hence also that on the maximal pro-$\mathcal{C}$ quotient $\pi_1^{(\mathcal{C})}(C_\xi,\tilde{\xi})$, for any almost full class $\mathcal{C}$ of finite groups:

$$\varphi : \pi_1(M_{g,n},\xi) \to \mathrm{Out}\, \pi_1^{(\mathcal{C})}(C_\xi,\tilde{\xi}). \tag{2}$$

Now the projection on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathrm{Ker}\,\varphi$ is of the form $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{g,n}^{(\mathcal{C})})$, with a Galois extension $\mathbb{Q}_{g,n}^{(\mathcal{C})}$ over $\mathbb{Q}$ which is independent of the choice of ξ. The basic prediction by Oda is that $\mathbb{Q}_{g,n}^{(\mathcal{C})}$ would not depend on (g,n). But since $\mathbb{Q}_{g,n}^{(\mathcal{C})} \subset \mathbb{Q}_{g,n+1}^{(\mathcal{C})}$ and since, for each $g \geq 0$, there exists a $\mathsf{P}^1_{01\infty}$-diagram of type (g,n) over $\mathbb{Q}$ for sufficiently large n (§2.1), Theorem 2′ cited above (applied to ξ corresponding to $(C \otimes \overline{\mathbb{Q}((q))}, \mathrm{Mark}(C))$) gives immediately the following

Theorem 3A(§4.2). $\mathbb{Q}_{g,n}^{(\mathcal{C})} \subset \mathbb{Q}_{0,3}^{(\mathcal{C})}(+)$.

Here, $\mathbb{Q}_{0,3}^{(\mathcal{C})}(+)$ is the kernel of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action on the maximal pro-$\mathcal{C}$ quotient of the fundamental groupoid of $\mathsf{P}^1 \otimes \overline{\mathbb{Q}} - \{0,1,\infty\}$ w.r.t. the set of Deligne's tangential base points.

When $\mathcal{C}$ consists of all finite l-groups, where l is a fixed prime, then $\mathbb{Q}_{g,n}^{(\mathcal{C})} = \mathbb{Q}_{g,n}^{(\text{pro-}l)}$ and $\mathbb{Q}_{0,3}^{(\text{pro-}l)} = \mathbb{Q}_{0,3}^{(\text{pro-}l)}(+)$ (even if $l=2$). We thus obtain

Theorem 3B. $\mathbb{Q}_{g,n}^{(\text{pro-}l)} \subset \mathbb{Q}_{0,3}^{(\text{pro-}l)}$.

This inclusion has previously been proved in the special case where $2g \equiv 0 \pmod{l-1}$, using appropriate l-covers of $\mathsf{P}^1 - \{0,1,\infty\}$ ([Ma]).

A comparison "at each level" of the weight filtrations of $\mathbb{Q}_{g,n}^{(\text{pro-}l)}$ and $\mathbb{Q}_{0,3}^{(\text{pro-}l)}$ will be discussed in §4.3.

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