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The Tate conjecture and the semisimplicity conjecture for \( t \)-modules*

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§0. Introduction.

Let \( l \) be a prime number. Let \( k \) be an algebraic number field and \( A \) an abelian variety over \( k \) of dimension \( d \). Then the \( l \)-adic Tate module

\[
V_l(A) \overset{\text{def}}{=} (\varprojlim \ker(l^n \cdot \text{id} : A(\overline{k}) \to A(\overline{k}))) \otimes \mathbb{Z}_l \mathbb{Q}_l
\]

is a \( 2d \)-dimensional vector space over \( \mathbb{Q}_l \) on which \( \text{Gal}(\overline{k}/k) \) acts. Thus, fixing a basis of \( V_l(A) \), we obtain an \( l \)-adic Galois representation

\[
\rho_{A,l} : \text{Gal}(\overline{k}/k) \to GL_{2d}(\mathbb{Q}_l).
\]

The following theorem of Faltings is important.

**Theorem (0.1).**

(i) (Tate conjecture.)

\[
\text{Hom}_k(A, A') \otimes \mathbb{Q}_l \simeq \text{Hom}_{\mathbb{Q}_l[\text{Gal}(\overline{k}/k)]}(V_l(A), V_l(A')).
\]

(ii) (Semisimplicity conjecture.) \( V_l(A) \) is a semisimple \( \mathbb{Q}_l[\text{Gal}(\overline{k}/k)] \)-module.

These conjectures can be also formulated for the \( l \)-adic Galois representations attached to more general motives, but they are still widely open.

Another problem is: What \( l \)-adic Galois representations come from abelian varieties (or motives)? We might hope for characterization of such representations in terms of \( p \)-adic theory at the places of \( k \) above \( p = l \). In the case of abelian varieties, the following partial results are known (Serre, Tate, Raynaud, Deligne,...).

**Theorem (0.2).**

(i) For each place \( v \) of \( k \) above \( l \), \( \rho_{A,l}|_{\text{Gal}(\overline{k}_v/k_v)} \) is a Hodge-Tate representation, i.e. has a Hodge-Tate decomposition. (In fact, it seems to be known, moreover, to be a potentially semistable representation.)

(ii) Let \( \rho \) be an \( l \)-adic representation of \( \text{Gal}(\overline{k}/k) \) which is potentially abelian. (Namely, the image of \( \text{Gal}(\overline{k}/k) \) by \( \rho \) admits an abelian open subgroup.) If \( \rho|_{\text{Gal}(\overline{k}_v/k_v)} \) is a Hodge-Tate representation for all place \( v \) of \( k \) above \( l \), then \( \rho \) is 'generated' by

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*This lecture was given in Japanese.
the representations attached to potentially CM abelian varieties and Artin representations.

In the present article, we consider $t$-adic Galois representations instead of $l$-adic Galois representations. A $t$-adic Galois representation is, by definition, a continuous group homomorphism $\text{Gal}(k^{\text{sep}}/k) \to GL_n(\mathbb{F}_q((t)))$, where $k$ is a field of characteristic equal to $\text{char}(\mathbb{F}_q)$. (For the definition, we do not have to restrict the characteristic of the field $k$, but we do not have any interesting theory so far in the case $\text{char}(k) \neq \text{char}(\mathbb{F}_q)$.) Here, the analogues of abelian varieties and motives are Drinfeld modules, Anderson’s abelian $t$-modules, or more general objects, which yield $t$-adic Galois representations by taking their $t$-adic Tate modules.

In this new setting, the Tate conjecture has been proved independently by Taguchi [1][2] and the author [3]. See also [4]. In the present article, we consider mainly the semisimplicity conjecture and the problem of characterizing ‘geometric’ (or ‘motivic’) $t$-adic representations.

§1. Pink’s restricted modules.

Pink introduced the concept of restricted modules (in 1994) in order to approach the semisimplicity conjecture for $t$-modules. (In fact, he also gave a proof of the conjecture, different from ours.) Roughly speaking, the category of restricted $k(t)\{\tau\}$-modules is that of $t$-motives modulo isogeny. Here, $k$ is a field containing $\mathbb{F}_q$, $t$ is an indeterminate, and the (generally noncommutative) ring $k(t)\{\tau\}$ is defined to be the ring whose underlying abelian group is a (left) $k(t)$-vector space with basis $\{\tau^i\}_{i=0,1,\ldots}$ and whose multiplication rule is:

$$
\left(\sum_i f_i \tau^i\right) \left(\sum_j g_j \tau^j\right) = \sum_i \sum_j f_i \sigma^i (g_j) \tau^{i+j},
$$

where $\sigma$ is defined by:

$$
\sigma \left(\sum_i c_i t^i\right) = \sum_i c_i^q t^i.
$$

**Definition (1.1).** Let $M$ be a left $k(t)\{\tau\}$-module.

(i) We say that $M$ is restricted, if $\dim_{k(t)} M < \infty$ and

$$
\tau_{\text{linear}} : M^{(q)} \overset{\text{def}}{=} k(t) \otimes_{\sigma, k(t)} M \to M, \ f \otimes x \mapsto f \tau x
$$

is an isomorphism.

(ii) Assume $M$ to be restricted. Then we say that $M$ is étale (at $t = 0$), if there exists an $O_{k(t)}\{\tau\}$-submodule $\mathcal{M}$ of $M$ which is finitely generated as an $O_{k(t)}\{\tau\}$-module such that $\tau_{\text{linear}}$ induces an isomorphism from $M^{(q)} \overset{\text{def}}{=} O_{k(t)} \otimes_{\sigma, O_{k(t)}} \mathcal{M}$ to $\mathcal{M}$. Here $O_{k(t)} \overset{\text{def}}{=} k(t) \cap k[[t]] = k[t]_{(t)}$.

**Remark (1.2).**

(i) Similarly, we define the concept of restricted and étale restricted $F\{\tau\}$-modules for each subfield $F$ of $k((t))$ containing $k(t)$ with $\sigma(F) \subset F$. Examples of such $F$
are: $k((t))$, $Q \overset{\text{def}}{=} \text{Frac}(k \otimes \mathbb{F}_q((t)))$, $Q^h \overset{\text{def}}{=} \text{Frac}(k \otimes \mathbb{F}_q((t))^h)$, etc., where $\mathbb{F}_q((t))^h$ is the algebraic closure of $\mathbb{F}_q(t)$ in $\mathbb{F}_q((t))$.

(ii) In the definition above, the analogue of $(\mathbb{Q}, l, \mathbb{Q}_l)$ is $(\mathbb{F}_q(t), t, \mathbb{F}_q((t)))$. This is only for simplicity, and we can develop our theory for more general setting like [3].

**Example (1.3).** Let $(G, \phi)$ be a Drinfeld $\mathbb{F}_q[t]$-module or an abelian $t$-module of Anderson's. Then

$$M \overset{\text{def}}{=} k(t) \otimes_{k[t]} \text{Hom}(\mathbb{F}_q\text{-module schemes}/k)(G, G_a)$$

becomes a restricted $k(t)\{\tau\}$-module. It is étale, unless the ideal $(t)$ is the 'characteristic' of $\phi$.

The following proposition gives a relation between restricted modules and $t$-adic Galois representations.

**Proposition (1.4).**

We have the following category equivalence:

$$(\text{étale restricted } k((t))\{\tau\}-\text{modules}) \simeq (t\text{-adic representations of } \text{Gal}(k^\text{sep}/k))$$

$$M \quad \mapsto \quad \hat{V}(M) \overset{\text{def}}{=} (k^\text{sep}((t)) \otimes_{\mathbb{F}_q((t))} M)^\tau$$

$$\hat{D}(V) \overset{\text{def}}{=} (k^\text{sep}((t)) \otimes_{\mathbb{F}_q((t))} V)^{\text{Gal}(k^\text{sep}/k)} \quad \mapsto \quad V.$$ 

Here $\tau$ (resp. $\text{Gal}(k^\text{sep}/k)$) acts diagonally on $k^\text{sep}((t)) \otimes_{k((t))} M$ (resp. $k^\text{sep}((t)) \otimes_{\mathbb{F}_q((t))} V$), and $(\cdot)^\tau$ (resp. $(\cdot)^{\text{Gal}(k^\text{sep}/k)}$) means the $\tau$-invariant (resp. $\text{Gal}(k^\text{sep}/k)$-invariant) part. The action of $\text{Gal}(k^\text{sep}/k)$ on $\hat{V}(M)$ (resp. $\tau$ on $\hat{D}(V)$) is induced by its action on $k^\text{sep}((t))$.

**Definition (1.5).** For an étale restricted $k(t)\{\tau\}$-module $M$, we write $\hat{V}(M)$ instead of $\hat{V}(k((t)) \otimes_{k((t))} M)$, and call it the ($t$-adic) Tate module of $M$. Similar notation is employed for an étale restricted $F\{\tau\}$-module. (cf. Remark (1.2)(i).)

The following example explains why we call $\hat{V}(M)$ Tate module.

**Example (1.6).** In the case of Example (1.3), we have

$$\hat{V}(M) \simeq V_t(G)^* = \text{Hom}_{\mathbb{F}_q((t))}(V_t(G), \mathbb{F}_q((t))),$$

where

$$V_t(G) \overset{\text{def}}{=} (\lim_{\text{inf}} \text{Ker}(\phi_{t^n} : G(k) \to G(k))) \otimes_{\mathbb{F}_q[[t]]} \mathbb{F}_q((t)).$$

§2. Tate conjecture and semisimplicity conjecture.

From now on, we assume that $k$ is a finitely generated field over $\mathbb{F}_q$. 

Theorem (2.1). (Tate conjecture.)
Let $M$ and $M'$ be étale restricted $k(t)\{\tau\}$-modules. Then,
\[
\mathrm{Hom}_{k(t)\{\tau\}}(M, M') \otimes \mathbb{F}_q((t)) \cong \mathrm{Hom}_{\mathbb{F}_q((t))}[\mathrm{Gal}(k^{\text{sep}}/k)](\hat{V}(M), \hat{V}(M')).
\]

Theorem (2.2). (Semisimplicity conjecture.)
Let $M$ be an étale restricted $k(t)\{\tau\}$-module, and assume that $M$ is semisimple as a $k(t)\{\tau\}$-module. Then $\hat{V}(M)$ is a semisimple $\mathbb{F}_q((t))[\mathrm{Gal}(k^\text{sep}/k)]$-module.

Remark (2.3). In the semisimplicity conjecture, the assumption of semisimplicity of the $k(t)\{\tau\}$-module $M$ excludes objects like semia-abelian varieties.

The outline of the proof of these theorems is given in the next section.

§3. ‘Geometric’ $t$-adic Galois representations.

The $t$-adic representations (of $\mathrm{Gal}(k^{\text{sep}}/k)$) attached to étale restricted $k(t)\{\tau\}$-modules or, more generally, those attached to étale restricted $Q^h\{\tau\}$-modules are worth calling geometric representations. (See Remark (1.2)(i) for the definition of $Q^h$ and $Q$.)

Definition (3.1). We say that a $t$-adic representation of $\mathrm{Gal}(k^{\text{sep}}/k)$ is quasi-geometric, if it is isomorphic to the $t$-adic representation attached to an étale restricted $Q\{\tau\}$-module.

Although we have not yet established any good theory of geometric $t$-adic representations, we have a good theory of quasi-geometric $t$-adic representations, as follows.

Remark (3.2). If $k$ is finite, all $t$-adic representations are quasi-geometric, since $Q$ then coincides with $k((t))$.

Now we have the following diagrams of categories and functors:

\[
\begin{array}{ccc}
(\text{étale restricted } k(t)\{\tau\}\text{-modules}) & Q \otimes_{k(t)} \downarrow & (\text{étale restricted } Q\{\tau\}\text{-modules})
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow_{k((t))} & \quad & \downarrow_{Q} \\
(\text{étale restricted } k((t))\{\tau\}\text{-modules}) & \rightarrow & (\text{quasi-geometric } t\text{-adic representations})
\end{array}
\]

\[
\begin{array}{ccc}
(\text{étale restricted } k((t))\{\tau\}\text{-modules}) & \simeq & (t\text{-adic representations}).
\end{array}
\]

Lemma (3.3).
(i) Let $M$ and $M'$ be étale restricted $k(t)\{\tau\}$-modules. Then,
\[
\mathrm{Hom}_{k(t)\{\tau\}}(M, M') \otimes \mathbb{F}_q((t)) \cong \mathrm{Hom}_{Q\{\tau\}}(Q \otimes_{k(t)} M, Q \otimes_{k(t)} M').
\]

(ii) Let $M$ be an étale restricted $k(t)\{\tau\}$-module, and assume that $M$ is semisimple as a $k(t)\{\tau\}$-module. Then $Q \otimes_{k(t)} M$ is a semisimple $Q\{\tau\}$-module.

This lemma, which is rather easy to prove, reduces the Tate conjecture (2.1) and the semisimplicity conjecture (2.2) to the following:
Theorem (3.4).
(i) The functor
\[ \text{étale restricted } Q\{\tau\}-\text{modules} \xrightarrow{k((t)) \otimes \cdot} \text{étale restricted } k((t))\{\tau\}-\text{modules} \]
is fully faithful.
(ii) The subcategory (quasi-geometric $t$-adic representations) is stable under taking subquotients in the category ($t$-adic representations).

Our proof of this theorem borrows a technique in $p$-adic Hodge theory. The main point is to construct a commutative ring $B$, which is a subring of $k^{\mathrm{sep}}((t))$ stable under the actions of $\tau$ and $\text{Gal}(k^{\mathrm{sep}}/k)$, satisfying the following properties:
(i) $B^\tau = \mathbb{F}_q((t))$.
(ii) $B^{\text{Gal}(k^{\mathrm{sep}}/k)} = Q$.
(iii) For each étale restricted $Q\{\tau\}$-module $M$, the canonical isomorphism
\[ k^{\mathrm{sep}}((t)) \otimes \hat{V}(\mathbb{F}_q((t)))M \simeq k^{\mathrm{sep}}((t)) \otimes M \]
comes from a (unique) isomorphism
\[ B \otimes \mathbb{F}_q((t)) \hat{V}(M) \simeq B \otimes Q \hat{V}(M). \]

Remark (3.5). Roughly speaking, the condition (iii) says that $B$ contains the entries of a 'period matrix' of $M$.

Theorem (3.4)(i) follows directly from the properties of $B$. In fact, the inverse map of
\[ \text{Hom}_{Q\{\tau\}}(M, M') \to \text{Hom}_{k((t))\{\tau\}}(k((t)) \otimes M, k((t)) \otimes M') \]
\[ = \text{Hom}_{\mathbb{F}_q((t)) \text{Gal}(k^{\mathrm{sep}}/k)}(\hat{V}(M), \hat{V}(M')) \]
is defined to map $f \in \text{Hom}_{\mathbb{F}_q((t)) \text{Gal}(k^{\mathrm{sep}}/k)}(\hat{V}(M), \hat{V}(M'))$ to the restriction of $\text{id}_B \otimes f : B \otimes \hat{V}(M) \to B \otimes \hat{V}(M')$ to the Gal($k^{\mathrm{sep}}/k$)-invariant parts.

Definition (3.6). For each $t$-adic representation $V$ of $\text{Gal}(k^{\mathrm{sep}}/k)$, we define
\[ D(V) = (B \otimes V)^{\text{Gal}(k^{\mathrm{sep}}/k)}. \]

From the properties of $B$, we can easily deduce the following theorem, which completes the proof of Theorem (3.4)(ii).
Theorem (3.7).
Let $V$ be a $t$-adic representation of $\text{Gal}(k^{\text{sep}}/k)$. Then the following are equivalent:

(i) $V$ is quasi-geometric;
(ii) $\dim_{\mathbb{Q}} D(V) = \dim_{\mathbb{F}_q((t))} V$;
(iii) $k((t)) \otimes_{\mathbb{Q}} D(V) \cong \hat{D}(V)$.

In particular, any subquotients of a quasi-geometric representation are again quasi-geometric.

Finally, we mention the construction of the ring $B$. Fix a proper normal model $X$ of $k$ over $\mathbb{F}_q$, and define $\Sigma$ to be the set of the points of codimension 1 in $X$. Let $X^{\text{sep}}$ be the normalization of $X$ in $k^{\text{sep}}$, and define $\Sigma^{\text{sep}}$ to be the set of the points of codimension 1 in $X^{\text{sep}}$. Denote by $w_{\bar{x}}$ the additive valuation of $k^{\text{sep}}$ associated to $\bar{x} \in \Sigma^{\text{sep}}$ (normalized as $w_{\bar{x}}(k^\times) = \mathbb{Z}$). Define the subring $B^+$ of $k^{\text{sep}}((t))$ by:

$f = \sum a_i t^i \in B^+ \iff$ for all $\bar{x} \in \Sigma^{\text{sep}}$, \{w_{\bar{x}}(a_i)\}_i$ is bounded below and, for almost all $\bar{x} \in \Sigma^{\text{sep}}$, $w_{\bar{x}}(a_i) \geq 0$ for all $i$. Here 'for almost all $\bar{x} \in \Sigma^{\text{sep}}$, ...' means 'there exists a finite subset $\Sigma_0$ of $\Sigma$ and, for all $\bar{x} \in \Sigma^{\text{sep}}$ not above $\Sigma_0$, ...'. Next define the subset $S$ of $k^{\text{sep}}((t))$ by

$$S = \{ f \in k^{\text{sep}}((t))^\times \mid \sigma(f)f^{-1} \in k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)) \},$$

which turns out to be a multiplicative subset of $B^+$. Now the ring $B$ is defined by

$$B = S^{-1}B^+.$$

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