A note on geometric changes of complete solutions of first order differential equations

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0. Introduction

Our purpose is to give a framework for understanding geometric changes of singularities appearing in solutions of completely integrable first order differential equations and then to study the special case of ordinary differential equations to get a well-known result.

This paper is closely related to the study [3] which uses Arnold’s result([1], [2]). We consider first order differential equations in the context of contact geometry([5]). And we employ a method when classifying functions up to diffeomorphisms preserving discriminant sets, which uses explicit coordinate changes arising from vector fields preserving discriminant sets ([4]). We hope that the same method suffices (with suitable modifications) to describe generic changes of singularities of solutions for first order partial differential equations with complete integral.

We would like to thank Professor J. W. Bruce for introducing me to this method.

1. Complete solutions and discriminant sets

First we shall describe the geometric structure connected with first order differential equations following S. Izumiya’s formulation([5]). Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of $n$-variables functions which may be considered as $\mathbb{R}^{2n+1}$ with natural coordinates given by $(x_1, \cdots, x_n, y, p_1, \cdots, p_n)$, where $(x_1, \cdots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We have the natural projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R} ; \pi(x,y,p) = (x,y)$.

A system of first order differential equations (or, briefly, an equation) is defined to be an immersion germ $l : (\mathbb{R}^r,0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$, where $n+1 \leq r \leq 2n$. Let $\theta$ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$. By the philosophy of Lie, we may define the notion of solutions as follows. An (abstract) solution of $l$ is a Legendrian
immersion \(i : L \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) such that \(i(L) \subset l(\mathbb{R}^r)\), where \(L\) is a \(n\)-dimensional manifold and the Legendrian immersion is an immersion \(i : L \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) such that \(i^*\theta = 0\).

Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a smooth function. Then \(j^1f : \mathbb{R}^n \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) is a Legendrian embedding. Hence, in our terminology, the (classical) solution of \(l\) is a smooth function \(f\) such that \(j^1f(\mathbb{R}^n) \subset l(\mathbb{R}^r)\). On the other hand, we can show that an (abstract) solution \(i : L \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) is given by (at least locally) a jet extension \(j^1f\) of a smooth function \(f\) if and only if \(\pi \circ i\) is a non-singular map. Thus the graph of the (abstract) solution \(\pi \circ i(L)\) in \(\mathbb{R}^n \times \mathbb{R}\) may have singularities.

We say that \(l\) is completely integrable (or \(l\) has an (abstract) complete solution) if there exists a submersion germ \(\mu = (\mu_1, \ldots, \mu_{r-n}) : (\mathbb{R}', 0) \rightarrow \mathbb{R}'^{-n}\) such that \(l_t = l \mid \mu^{-1}(t) : \mu^{-1}(t) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) is an abstract solution of \(l\) for any \(t \in \mathbb{R}'^{-n}\). Then \(\mu\) is called a complete integral of \(l\) and the pair \((\mu, l) : (\mathbb{R}', 0) \rightarrow \mathbb{R}'^{-n} \times J^1(\mathbb{R}^n, \mathbb{R})\) is called an equation germ with complete integral.

In order to study generic types of singularities appearing in solutions of completely integrable equations, we now introduce a natural equivalence relation among equations with complete integral([5]). Let \((\mu, l) : (\mathbb{R}', 0) \rightarrow (\mathbb{R}'^{-n} \times J^1(\mathbb{R}^n, \mathbb{R}),(t_0, (x_0, y_0, p_0)))\) and \((\mu', l') : (\mathbb{R}', 0) \rightarrow (\mathbb{R}'^{-n} \times J^1(\mathbb{R}^n, \mathbb{R}),(t_1, (x_1, y_1, p_1)))\) be equation germs with complete integral. We say that \((\mu, l)\) and \((\mu', l')\) are equivalent as equations with complete integral if there exist diffeomorphism germs \(\phi : (\mathbb{R}'^{-n}, t_0) \rightarrow (\mathbb{R}'^{-n}, t_1), \Phi : (\mathbb{R}', 0) \rightarrow (\mathbb{R}', 0), \kappa : (\mathbb{R} \times \mathbb{R}, (x_0, y_0)) \rightarrow (\mathbb{R} \times \mathbb{R}, (x_1, y_1))\) and a contact diffeomorphism germ \(K : (J^1(\mathbb{R}^n, \mathbb{R}),(x_0, y_0, p_0)) \rightarrow (J^1(\mathbb{R}^n, \mathbb{R}),(x_1, y_1, p_1))\) such that the following diagram is commute:

\[
\begin{array}{ccc}
(\mathbb{R}'^{-n}, t_0) & \xleftarrow{\mu} & (\mathbb{R}', 0) \\
\downarrow \phi & & \downarrow \Phi \\
(\mathbb{R}'^{-n}, t_1) & \xleftarrow{\mu'} & (\mathbb{R}', 0)
\end{array}
\quad\begin{array}{cc}
\xrightarrow{\pi} & \xrightarrow{\kappa} \\
\xrightarrow{K} & \xrightarrow{\kappa}
\end{array}
\quad
\begin{array}{ccc}
(\mathbb{R} \times \mathbb{R}^n) & \rightarrow & (\mathbb{R} \times \mathbb{R})
\end{array}
\]

Let \(f : (\mathbb{R}'^{-n} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\) be a function germ such that \(\text{rank}(\partial f / \partial t_i, \partial^2 f / \partial t_i \partial q_j) = r - n\). We call such a function germ a complete family of function germs. We now define a map germ \(L_f : (\mathbb{R}'^{-n} \times \mathbb{R}^n, 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})\) by \(L_f(t, q) = (\partial f / \partial q(t, q), \sum_{i=1}^n \partial f / \partial q_i(t, q) \cdot q_i - f(t, q), q)\).

Then \(L_f\) is an immersion germ if and only if \(f\) is a complete family of function germs. Hence \((\pi_1, L_f)\) is an equation germ with complete integral, where \(\pi_1(\mathbb{R}'^{-n} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}'^{-n}, 0)\) is the canonical projection. Then we have the following proposition.

**Proposition 1.1.** ([5]). Let \((\mu, l) : (\mathbb{R}', 0) \rightarrow (\mathbb{R}'^{-n} \times J^1(\mathbb{R}^n, \mathbb{R}),(t_0, (x_0, y_0, p_0)))\) be an equation germ with complete integral. Then there exists a complete family of function germs \(f : (\mathbb{R}'^{-n} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\) such that \((\mu, l)\) and \((\pi_1, L_f)\) are equivalent as equations with complete integral.
This proposition guarantees that it is enough to study $L_f$ for studying singularities of solutions of equations with complete integral.

Now we show how the graphs of abstract complete solutions of equations relate to discriminant sets of an unfolding of some function (a family of height functions).

Let $f : (R^{r-n} \times R^n, 0) \to (R, 0)$ be a complete family of function germs.

We consider the following set:

$$\Sigma_f' = \{(t, \partial f/\partial q(t, q), \partial f/\partial q(t, q) \cdot q - f(t, q)) \mid t \in R^{r-n}, q \in R^n \} \subset R^{r-n} \times R^n \times R.$$  

For a fixed $t \in R^{r-n}$, $\Sigma_f' = \Sigma_f' \cap \{t\} \times R^n \times R$ is the graph of an abstract solution of the completely integrable equation $L_f$ and is clearly the affine dual of the graph $\Gamma_f' = \{(q, f(t, q)) \mid q \in R^n\}$ of $f$. So we refer to the assembled family of duals $\Sigma_f'$ as the big dual.

The big dual can be studied by considering the following $(r-n)$ parameter family of height functions([2],[3]),

$$H_f : R^n \times (R^{r-n} \times S^n \times R) \to R,$$

where $H_f(q, t, u, z) = (q, f(t, q)).u - z, S^n$ is the unit vectors in $R^{n+1}$ and . denotes the usual inner product in $R^{n+1}$.

Since we are only interested in these graphs near $(0, f(0, 0))$, we consider the germ $F : R^n \times (R^{r-n} \times R^n \times R), (0, 0, 0, 0) \to R, 0$ defined by

$$F(q, t, \lambda, z) = (q, f(t, q)).(\lambda_1 - \partial f/\partial q_1(0, 0), \lambda_2 - \partial f/\partial q_2(0, 0), \ldots, \lambda_n - \partial f/\partial q_n(0, 0), 1) - z.$$  

We can naturally regard $F$ as a $(r+1)$-parameter unfolding of $F_0(q) = F(q, 0, 0, 0) : (R^n, 0) \to R, 0$. Then the discriminant set of $F$ is (by definition) the set germ $D_F, 0 = \{(t, \lambda, z) \in R^{r-n} \times R^n \times R \mid F(q, t, \lambda, z) = \partial F/\partial q(q, t, \lambda, z) = 0 \text{ for some } q\}, 0$.

Geometrically the discriminant set can be thought of as the big dual, that is, the sections $t= \text{constant of } D_F$ are locally diffeomorphic to the duals $\Sigma_f'$ of the graphs $\Gamma_f'$.

Therefore in order to see geometrically how the graphs of abstract complete solution of equations change, we need to consider the natural projection germ of the discriminant set $D_F$ to the $t$-parameter, i.e. $p_1 : (R^{r+1}, D_F), 0 \to R^{r-n}, 0 : p_1(t, \lambda, z) = t$.

### 2. Functions on discriminant sets

In this section we study the special case $n = 1$ and $r = 2$, i.e. the case of ordinary differential equations. Then we need to consider the following 3-parameter unfolding.

$$F : R \times R^3, (0, 0) \to R, 0 \text{ given by } F(q, t, \lambda, z) = (q, f(t, q)).(\lambda - \partial f/\partial q(0, 0), 1) - z,$$

where $f : (R \times R, 0) \to (R, 0)$ is a complete family of function germs, i.e. $\text{rank}(\partial f/\partial t, \partial^2 f/\partial t \partial q) |_{0} = 1$.

First we consider the discriminant set $D_F$ of $F$ for "generic" complete family of function germs $f$. We shall define the genericity of complete family of function germs ([5]). Let $U \times V$ be an open subset of $R \times R$ and $CF(U \times V, R) = \{f \in C^\infty(U \times V, R) \mid \text{rank}(f_t, f_{tq}) = 1 \text{ at any } (t, q) \in U \times V\}$. By Proposition 1.1 we may consider that $CF(U \times V, R)$ is the space
of equations with complete integral. A subset of $CF(U \times V, \mathbb{R})$ is called generic if it is open and dense in $CF(U \times V, \mathbb{R})$.

Let $P$ be a property of complete family of function germs $f: \mathbb{R} \times \mathbb{R}, 0 \rightarrow \mathbb{R}$. The property $P$ is said to be generic if for some neighbourhood $U \times V$ of 0 in $\mathbb{R} \times \mathbb{R}$ the set $P(U \times V) = \{ f \in CF(U \times V, \mathbb{R}) \mid$ the germ $f: (U \times V, (t, q)) \rightarrow \mathbb{R}$ has the property $P$ for any point $(t, q) \in U \times V \}$ is generic in $CF(U \times V, \mathbb{R})$.

Now we obtain the following:

**Proposition 2.1.** For a generic complete family of function germs $f$, $F$ contains only $A_1$, $A_2$ and $A_3$ singularities and all these singularities are versally unfolded by $F$.

**Proof.** $F_0 = F(\cdot, 0)$ has an $A_k$ ($k \geq 1$) singularity at $q = 0$ if $f^{(2)}(0, 0) = \cdots = f^{(k)}(0, 0) = 0$ and $f^{(k+1)}(0, 0) \neq 0$.

For $\frac{\partial F}{\partial z}(q, 0) = -1$, $\frac{\partial F}{\partial \lambda}(q, 0) = q$ and $\frac{\partial F}{\partial t}(q, 0) = \frac{\partial f}{\partial t}(0, q)$, so $A_1$ and $A_2$ singularities are always versally unfolded and an $A_3$ is versally unfolded if and only if $\frac{\partial^2 f}{\partial q^2 \partial t}(0, 0) \neq 0$.

We now define subsets of $J^4(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as follows: $S_1 = \{ \frac{\partial^2 f}{\partial q^2}(t, q) = \frac{\partial^3 f}{\partial q^3}(t, q) = \frac{\partial^4 f}{\partial q^4}(t, q) = 0 \}$, $S_2 = \{ \frac{\partial^2 f}{\partial q^2}(t, q) = \frac{\partial^3 f}{\partial q^3}(t, q) = \frac{\partial^3 f}{\partial q^3 \partial t}(t, q) = 0 \}$. Then consider $j^4 f: \mathbb{R} \times \mathbb{R}, 0 \rightarrow J^4(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. By the transversality theorem we get the result.

Then for generic complete family of function germs $f$, the discriminant set germ $D_F$ at 0 of $F$ is diffeomorphic to a plane, cuspidal edge or swallowtail in 3-space. To see how the duals change, we need to consider the natural projection of these discriminant sets $D_F$ to the $t$ parameter, i.e., $p_1: (\mathbb{R}^3, D_F), 0 \rightarrow \mathbb{R}, 0$, where $p_1(t, \lambda, z) = t$.

Let $G$ be the standard versal unfolding $G(q, b) = \pm q^{k+1} + b_1 q^{k-1} + \cdots + b_{k-1} q + b_k$, where $b \in \mathbb{R}^3$ and $1 \leq k \leq 3$. Using the fact that $F$ is a versal unfolding of an $A_k$ singularity $(k \leq 3)$ we can find smooth germs $\psi: \mathbb{R} \times \mathbb{R}^3, 0 \rightarrow \mathbb{R}, 0, \psi: \mathbb{R}^3 \rightarrow \mathbb{R}^3, 0$ with $\psi(-, 0): \mathbb{R}, 0 \rightarrow \mathbb{R}, 0$ a diffeomorphism, $\psi$ a diffeomorphism and $F(\psi(q, b), \psi(b)) = G(q, b)$. The discriminant set of $G$ is mapped by $\psi$ to the discriminant set of $F$ ($\psi$ being a discriminant preserving diffeomorphism), and $\psi_1$, the first component of $\psi$, is the function on $D_G$ corresponding to the natural projection $p_1$ of $D_F$ to the $t$-axis, as in the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{R} \times \mathbb{R}^3 & \stackrel{(F, id)}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^3 \\
\uparrow (\phi, \psi) & \quad \quad & \uparrow id \times \psi \\
\mathbb{R} \times \mathbb{R}^3 & \stackrel{(G, id)}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^3 \\
\uparrow id \times \psi & \quad \quad & \uparrow \psi & \nearrow \psi_1 \\
\mathbb{R} & \rightarrow & \mathbb{R}^3 & \rightarrow & \mathbb{R} \\
\end{array}
$$

where $p: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the natural projection.

We use the discriminant preserving diffeomorphism $\psi$ to study the function $\psi_1$ instead of the natural projection $p_1$ (see [2], the proof of Theorem 1.2).
Proposition 2.2. Let $F(q, t, \lambda, z)$ and $\psi_i(b)$ be as above.

(1) If $F$ is a versal unfolding of $A_1$-singularity and $G(q, b) = q^2 + b_1$, then we have 
$$\partial \psi_1 / \partial b_2(0) \neq 0$$ or 
$$\partial \psi_1 / \partial b_3(0) \neq 0.$$

(2) If $F$ is a versal unfolding of $A_2$-singularity and $G(q, b) = q^3 + b_1q + b_2$, then we have 
$$\partial \psi_1 / \partial b_3(0) \neq 0.$$

(3) If $F$ is a versal unfolding of $A_3$-singularity and $G(q, b) = q^4 + b_1q^2 + b_2q + b_3$, then we have 
$$\partial \psi_1 / \partial b_3(0) \neq 0.$$

Proof. (1) From the chain rule we find that 
$$\frac{\partial F}{\partial a}(\phi(q, 0), 0) \frac{\partial \phi}{\partial b}(q, 0) + \frac{\partial F}{\partial \lambda}(\phi(q, 0), 0) \frac{\partial \psi_i}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \frac{\partial \psi_i}{\partial b}(0) \equiv \delta_i \mod \langle q \rangle,$$

where $\delta$ is the usual Kronecker symbol and $i = 1 \sim 3$.

Since $F$ is an unfolding of an $A_1$-singularity, $\partial F / \partial q(\phi(q, 0), 0)$ has a Taylor series starting with terms of degree at least 1. So we get 
$$\frac{\partial \psi_1}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \equiv q^{2-1} \mod \langle q^2 \rangle,$$

$$\frac{\partial \psi_2}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \equiv a_{i=1} q^{3-1} \mod \langle q^3 \rangle (i = 1 \sim 3).$$

Therefore $\partial \psi_2 / \partial b_3(0)$ is regular. Hence we have $\partial \psi_1 / \partial b_3(0) \neq 0$.

(2) In the same way as in (1) we get 
$$\frac{\partial \psi_1}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \equiv q^{2-1} \mod \langle q^2 \rangle,$$

$$\frac{\partial \psi_2}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \equiv a_{i=1} q^{3-1} \mod \langle q^3 \rangle (i = 1 \sim 3).$$

Therefore $\partial \psi_2 / \partial b_3(0) = \partial \psi_3 / \partial b_3(0) = 0$, which is a contradiction. Hence we have $\partial \psi_1 / \partial b_3(0) \neq 0$.

(3) In the same way as in (1) we get 
$$\frac{\partial \psi_i}{\partial b}(0) + \frac{\partial F}{\partial z}(\phi(q, 0), 0) \equiv a_{i=1} q^{3-1} \mod \langle q^3 \rangle (i = 1 \sim 3).$$

Therefore $\partial \psi_2 / \partial b_3(0) = \partial \psi_3 / \partial b_3(0) = 0$, which is a contradiction. Hence we have $\partial \psi_1 / \partial b_3(0) \neq 0$.
0 (or $\partial \psi_3 / \partial b_1(0) \neq 0$) and \[
\begin{pmatrix}
a_{22} & a_{32} \\
a_{23} & a_{33}
\end{pmatrix}
\begin{pmatrix}
\partial \psi_2 / \partial b_1(0) \\
\partial \psi_2 / \partial b_3(0)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
Therefore $\partial \psi_2 / \partial b_1(0) = \partial \psi_3 / \partial b_1(0) = 0$, which is a contradiction. Hence we have $\partial \psi_1 / \partial b_1(0) \neq 0$. This completes the proof.

Using the conditions on $\psi_1$ of Proposition 2.2 we classify function germs $\psi_1 : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ up to local diffeomorphisms of $\mathbb{R}^3$ preserving the standard discriminant set $D_G$ of $G$. Then we get the following.

**Proposition 2.3.** Let $\psi_1(b)$ be as above.

1. If $G(q, b) = q^2 + b_1$ and $\partial \psi_1 / \partial b_2(0) \neq 0$ (or $\partial \psi_1 / \partial b_3(0) \neq 0$), then $\psi_1$ is equivalent, via a discriminant preserving diffeomorphism, to the trivial projection onto $b_2$-coordinate (or $b_3$-coordinate) of a product discriminant set (i.e. a plane).

2. If $G(q, b) = q^3 + b_1q + b_2$ and $\partial \psi_1 / \partial b_3(0) \neq 0$, then $\psi_1$ is equivalent, via a discriminant preserving diffeomorphism, to the trivial projection onto $b_3$-coordinate of a product discriminant set (i.e. a cuspidal edge).

3. If $G(q, b) = q^4 + b_1q^2 + b_2q + b_3$ and $\partial \psi_1 / \partial b_1(0) \neq 0$, then $\psi_1$ is equivalent, via a discriminant preserving diffeomorphism, to the projection of the standard discriminant set (i.e. the swallowtail) onto $b_1$-coordinate. We call it the standard swallowtail projection.

**Proof.** The standard method of obtaining diffeomorphisms is to integrate smooth vector fields. If the diffeomorphism is to preserve the discriminant, then the vector fields must be tangent to the discriminant (in the sense of being tangent to the smooth strata in the natural stratification of the discriminant).

1. The discriminant of $G(q, b)$ is the set $D_G = \{(0, b_2, b_3)\}$. Then we can obtain a free basis for the $\mathcal{E}(3, 1)$-module of vector fields tangent to the set $D_G$ as follows [4]:

$$\Omega = \mathcal{E}(3, 1) \{ b_i \partial / \partial b_1, \partial / \partial b_2, \partial / \partial b_3 \},$$

where $\mathcal{E}(3, 1)$ is the ring of $C^\infty$ map-germs $f : (\mathbb{R}^3, 0) \to \mathbb{R}^1$. Then the integration yields diffeomorphisms $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ preserving $D_G$, whose 1-jet $j^1 \phi(0)$ are

$$\begin{cases} 
(b_1, b_2, b_3) \to (b_1, k_1 b_1 + l_{11} b_2 + l_{12} b_3, k_2 b_1 + l_{21} b_2 + l_{22} b_3), & \text{where } \det(l_{ij})_{i,j=1,2} \neq 0, \\
(b_1, b_2, b_3) \to (k b_1, b_2, b_3), & \text{where } k \neq 0.
\end{cases}$$

Let $j^1 \psi_1(0) = c_1 b_1 + c_2 b_2 + c_3 b_3$, where $c_2 \neq 0$ or $c_3 \neq 0$. Hence changing coordinates

$$(b_1, b_2, b_3) \to (b_1, c_1 b_1 + c_2 b_2 + c_3 b_3, b_2) \quad \text{or} \quad (b_1, b_2, b_3) \to (b_1, c_1 b_1 + c_2 b_2 + c_3 b_3, b_2)$$

turns $j^1 \psi_1(0)$ into $f : (b_1, b_2, b_3) \to b_2$, which satisfy the following determinacy condition $\Omega_0 \cdot f \supset M_3$, where $M_3$ is the maximal ideal of $\mathcal{E}(3, 1)$ and $\Omega_0 = \{ \xi \in \Omega : \xi|_0 = 0 \}$. (We can get the following as in the similar way to the ordinary determinacy theorem. That is, if $\Omega_0 \cdot f \supset M_3^k$, then $f$ is $k$-determined with respect to $D_G$-preserving diffeomorphisms.) Therefore $f$ is 1-determined with respect to the $D_G$-preserving diffeomorphisms and hence $\psi_1$ and $f$ are equivalent, via a $D_G$-preserving diffeomorphism.
(2) The discriminant of $G(q,b)$ is the set $D_G = \{(b_1, b_2, b_3) \mid 4b_1^3 + 27b_2^2 = 0 \}$. We can obtain a free basis for the $\mathcal{E}(3,1)$-module of vector fields tangent to the set $D_G$ as follows([4]):

$$\Omega = \mathcal{E}(3,1)\{9b_2\partial/\partial b_1 - 2b_1^2\partial/\partial b_2, 2b_1\partial/\partial b_1 + 3b_2\partial/\partial b_2, \partial/\partial b_3\}.$$ 

Then the integration yields diffeomorphisms $\phi: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ preserving $D_G$, whose

1-jet $j^1\phi(0)$ are

$$\begin{cases}
(b_1, b_2, b_3) \to (b_1, b_2, b_3 + nb_2), & \text{where } n \neq 0,
(b_1, b_2, b_3) \to (b_1 + kb_2, b_2, b_3),
(b_1, b_2, b_3) \to (kb_1, b_2, b_3), & \text{where } k^3 = l^2 (k, l > 0).
\end{cases}$$

Let $j^1\psi_1(0) = c_1b_1 + c_2b_2 + c_3b_3$, where $c_3 \neq 0$. Hence changing coordinates $(b_1, b_2, b_3) \to (b_1, b_2, c_1b_1 + c_2b_2 + c_3b_3)$ turns $j^1\psi_1(0)$ into $f : (b_1, b_2, b_3) \to b_3$, which satisfy the following determinacy condition $\Omega_0.f \supset \mathcal{M}_3$. Therefore $f$ is 1-determined with respect to the $D_G$-preserving diffeomorphisms and hence $\psi_1$ and $f$ are equivalent, via a $D_G$-preserving diffeomorphism.

(3) The discriminant of $G(q,b)$ is the standard swallowtail set. We can obtain a free basis for the $\mathcal{E}(3,1)$-module of vector fields tangent to the set $D_G$ as follows([4]):

$$\Omega = \mathcal{E}(3,1)\{2b_1\partial/\partial b_1 + 3b_2\partial/\partial b_2 + 4b_3\partial/\partial b_3, 6b_2\partial/\partial b_1 + (8b_3 - 2b_1^2)\partial/\partial b_2 - b_1b_2\partial/\partial b_3,
(16b_3 - 4b_1^2)\partial/\partial b_1 - 8b_1b_2\partial/\partial b_2 - 3b_2^2\partial/\partial b_3\}.$$ 

Then the integration yields diffeomorphisms $\phi: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ preserving $D_G$, whose

1-jet $j^1\phi(0)$ are

$$\begin{cases}
(b_1, b_2, b_3) \to (b_1 + 3kb_2 + 6k^2b_3, b_2 + 4kb_3), & \text{(i)}
(b_1, b_2, b_3) \to (b_1 + kb_2 + b_3), & \text{(ii)}
(b_1, b_2, b_3) \to (kb_1, kb_2, b_3), & \text{(iii)}
\end{cases}$$

Let $j^1\psi_1(0) = c_1b_1 + c_2b_2 + c_3b_3$, where $c_1 \neq 0$. By (iii) $k = |c_1|$, $j^1\psi_1(0)$ is equivalent to $\pm c_1 b_1 + c_2 b_2 + c_3 b_3$. By (ii) $t = \pm c_3^2$ we get $\pm c_1 + c_2 t b_2$. Then we get $\pm (b_1 - \frac{3}{2}c_2^2 b_2)$ by (i) $k = \pm \frac{1}{3}c_2^2$. Finally by (ii) $t = -\frac{3}{2}c_2^2$ we get $f : (b_1, b_2, b_3) \to \pm b_1$, which satisfy the following determinacy condition $\Omega_0.f \supset \mathcal{M}_3$. Therefore $f$ is 1-determined with respect to the $D_G$-preserving diffeomorphisms and hence $\psi_1$ and $f$ are equivalent, via a $D_G$-preserving diffeomorphism. This completes the proof.

From Proposition 1.1 ~ 2.3, for almost all first order ordinary differential equations with complete integral the local models for the changes in the graphs of solutions are the followings.

(1) the graphs of solutions near $q_0$ are all diffeomorphic to lines.
(2) the graphs of solutions near $q_0$ are all diffeomorphic to cusps.
(3) the family of graphs of solutions near $q_0$ are obtained as sections of the standard swallowtail projection.
References


