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A WEAK SET THEORY WITH GLOBALIZATION

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1. Introduction

Here we deal with a complete lattice $\mathcal{L}$ with unary operations $\neg$ and $\Box$, which are called weak complement (Definition 2.1) and globalization (Definition 2.2), respectively. The pseudo-complement on a complete Heyting algebra, and orthogonal complement on a quantum logic are both examples of weak complement. The operation $\Box$ defined by

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases}$$

is a model of globalization, and the set $\{\Box a \mid a \in \mathcal{L}\}$ forms a complete Boolean algebra.

We will formulate a set theory on $V^{\mathcal{L}}$, and call it a lattice valued set theory (**LZF**). The set theory **LZF** has double structure. One is of the set theory on $V^{\mathcal{L}}$ and the other is of its external set theory.

2. Complete lattice with a weak complement and a globalization

For a subset $\{a_\alpha\}_\alpha$ of a complete lattice $\mathcal{L}$, the least upper bound of $\{a_\alpha\}_\alpha$ is denoted by $\bigvee_\alpha a_\alpha$, and the greatest lower bound of $\{a_\alpha\}_\alpha$ is denoted by $\bigwedge_\alpha a_\alpha$. The smallest element of $\mathcal{L}$ is denoted by 0, and the largest element is denoted by 1.

Definition 2.1. We say a unary operation $\neg$ on a complete lattice $\mathcal{L}$ is a *weak copmlement*, if the following conditions are satisfied for all elements a, b of $\mathcal{L}$.

N1: $\neg 0 = 1, \quad \neg 1 = 0$
N2: $a \wedge \neg a = 0$
N3: $a \leqslant \neg\neg a$
N4: $\neg(a \vee b) = \neg a \wedge \neg b$

Definition 2.2. Let $\mathcal{L}$ be a complete lattice with a weak complement $\neg$. $\Box$ is called a *globalization* on $\mathcal{L}$, if the following conditions are satisfied for all elements a, b, a_k, b_k, c_k $(k \in K)$ of $\mathcal{L}$.

G1: $\Box a \leqslant a$
G2: $\neg \Box a = \Box \neg \Box a$
G3: $\bigwedge_k \Box a_k \leqslant \Box \bigwedge_k a_k$

G4: If $\Box a \leqslant b$, then $\Box a \leqslant \Box b$
G5: $\Box a \wedge \bigvee_k b_k = \bigvee_k(\Box a \wedge b_k)$; $\quad a \wedge \bigvee_k \Box b_k = \bigvee_k(a \wedge \Box b_k)$
G6: $\Box a \vee \neg \Box a = 1$
G7: If $a \wedge \Box c \leqslant b$, then $\neg b \wedge \Box c \leqslant \neg a$

In what follows, $\mathcal{L}$ denotes a complete lattice with a weak complement $\neg$ and a globalization $\Box$.

Definition 2.3. We define the implication $\rightarrow$ on $\mathcal{L}$ by

$$(a \rightarrow b) = \bigvee\{c \in \mathcal{L} \mid (c = \Box c) \wedge (a \wedge c \leqslant b)\}.$$

Then we have

Lemma 2.1. (1) $(a \rightarrow b) = 1$ iff $a \leqslant b$.
(2) $\Box(a \rightarrow b) = (a \rightarrow b)$.
(3) $a \wedge (a \rightarrow b) \leqslant b$
(4) $(a \rightarrow b) \leqslant (\neg b \rightarrow \neg a)$

It is easy to see :

Lemma 2.2. *Let* $a, b \in \mathcal{L}$ *and* $\{a_k\}_{k\in K}, \{b_k\}_{k\in K} \subset \mathcal{L}$. *Then*
(1) *If* $a \leq b$ *then* $\Box a \leq \Box b$
(2) $\Box(\bigwedge_k a_k) = \bigwedge_k \Box a_k$
(3) $\Box a = \Box \Box a$
(4) $\bigwedge_k \Box a_k = \Box \bigwedge_k \Box a_k$
(5) $\bigvee_k \Box a_k = \Box \bigvee_k \Box a_k$

We denote $\neg \Box \neg$ by $\Diamond$. Then we have

Lemma 2.3. *Let* $a, b \in \mathcal{L}$ *and* $\{a_k\}_{k\in K} \subset \mathcal{L}$.
(1) $a \leqslant \Diamond a$
(2) *If* $a \leqslant \Box b$ *then* $\Diamond a \leqslant \Box b$
(3) $\Diamond \bigvee_k a_k = \bigvee_k \Diamond a_k$
(4) $\Diamond(\Box a \wedge b) \leqslant \Box a \wedge \Diamond b$

Complete Boolean algebra (cBa), complete Heyting algebra (cHa), and quantum-logic are all complete lattice with a weak complement $\neg$ and a globalization $\Box$, if $\Box$ is defined by

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

3. $\mathcal{L}$-VALUED UNIVERSE $V^{\mathcal{L}}$

Let $\mathcal{L}$ be a complete lattice with weak complement $\neg$ and globalization $\Box$. $\mathcal{L}$-valued universe $V^{\mathcal{L}}$ is constructed by induction, in the same way as Boolean valued universe V^B.

$$\begin{aligned} V_\alpha^{\mathcal{L}} &= \{u \mid \exists\beta<\alpha\, \exists\mathcal{D}u \subset V_\beta^{\mathcal{L}}(u : \mathcal{D}u \to \mathcal{L})\} \\ V^{\mathcal{L}} &= \bigcup_{\alpha\in\mathrm{On}} V_\alpha^{\mathcal{L}} \end{aligned}$$

The least α such that $u \in V_\alpha^{\mathcal{L}}$ is called the *rank* of u. For $u, v \in V^{\mathcal{L}}$, $[\![u=v]\!]$ and $[\![u \in v]\!]$ are defined by induction on the rank of u, v.

$$\begin{aligned} [\![u=v]\!] &= \bigwedge_{x\in\mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \to [\![x \in u]\!]) \\ [\![u \in v]\!] &= \bigvee_{x\in\mathcal{D}v} [\![u=x]\!] \wedge v(x). \end{aligned}$$

We say an element p of $\mathcal{L}$ is $\Box$-*closed* if $p = \Box p$. Since formulas of the form $p \to q$ is $\Box$-closed, $[\![u=v]\!]$ is $\Box$-closed. We denote $\Box(a \in b)$ by $a \overset{\Box}{\in} b$.

Lemma 3.1. *For $x, y \in V^{\mathcal{L}}$ and $\{b_k\}_k \subset \mathcal{L}$, $[\![x=y]\!] \wedge \bigvee_k b_k = \bigvee_k [\![x=y]\!] \wedge b_k$.*

Proof. By G5. $\Box$

Lemma 3.2. *For $p, q, r \in \mathcal{L}$, if $\Box p \wedge q \leq r$, then $\Box p \leq (q \to r)$.*

Proof. Immediate from the definition of $\to$. $\Box$

Lemma 3.3. *Let $u, v \in V^{\mathcal{L}}$. Then*

(1) $[\![u=v]\!] = [\![v=u]\!]$
(2) $[\![u=u]\!] = 1$
(3) *If $x \in \mathcal{D}u$ then $u(x) \leq [\![x \in u]\!]$.*

Proof. (1) is obvious. (2) and (3) are proved by induction on the rank of u. Let $x \in \mathcal{D}u$. Since $[\![x=x]\!] = 1$ by induction hypothesis,

$$u(x) \leq \bigvee_{x'\in\mathcal{D}u} [\![x=x']\!] \wedge u(x') \leqslant [\![x \in u]\!]. \text{ Hence } [\![u=u]\!] = 1. \quad \Box$$

Theorem 3.4. *For $u, v, w \in V^{\mathcal{L}}$,*

(1) $[\![u=v \wedge v=w]\!] \leq [\![u=w]\!]$
(2) $[\![u=v \wedge v \in w]\!] \leq [\![u \in w]\!]$
(3) $[\![u=v \wedge w \in v]\!] \leq [\![w \in u]\!]$

Proof. (1) We proceed by induction. Assume that $u, v, w \in V^{\mathcal{L}}_{\alpha}$. By Lemma 2.1,

$$[\![u=v]\!] \wedge u(x) \leqslant (u(x) \to [\![x \in v]\!]) \wedge u(x) \leqslant [\![x \in v]\!]$$

for $x \in \mathcal{D}u$. Hence, by using Lemma 3.1,

$$\begin{aligned}
[\![u=v \wedge v=w]\!] \wedge u(x) &\leqslant [\![v=w]\!] \wedge \bigvee_{y \in \mathcal{D}v} [\![x=y]\!] \wedge v(y) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} ([\![x=y]\!] \wedge [\![v=w]\!] \wedge v(y)) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} ([\![x=y]\!] \wedge \bigvee_{z \in \mathcal{D}w} [\![y=z]\!] \wedge w(z)) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} \bigvee_{z \in \mathcal{D}w} [\![x=y \wedge y=z]\!] \wedge w(z)
\end{aligned}$$

by using induction hypothesis,

$$\begin{aligned}
&\leqslant \bigvee_{z \in \mathcal{D}w} [\![x=z]\!] \wedge w(z) \\
&\leqslant [\![x \in w]\!].
\end{aligned}$$

Since $[\![u=v \wedge v=w]\!]$ is $\Box$-closed,

$$[\![u=v \wedge v=w]\!] \leqslant \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in w]\!]).$$

Similarly, we have

$$[\![u=v \wedge v=w]\!] \leqslant \bigwedge_{z \in \mathcal{D}} (w(z) \to [\![z \in x]\!]).$$

Hence, $[\![u=v \wedge v=w]\!] \leqslant [\![u=w]\!]$.

(2) and (3) follows from (1) and Lemma 3.1. $\Box$

By lattice valued set theory (**LZF**) we mean a set theory on $V^{\mathcal{L}}$ whose atomic formulas are of the form $u = v$ or $u \in v$; and logical operations are $\wedge$, $\vee$, $\neg$, $\to$, $\forall x$, $\exists x$ and $\Box$. Now we extend the definition of $[\![\varphi]\!]$ by the following rules.

$[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$
$[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!]$
$[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$
$[\![\varphi_1 \to \varphi_2]\!] = [\![\varphi_1]\!] \to [\![\varphi_2]\!]$
$[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]$
$[\![\exists x \varphi(x)]\!] = \bigvee_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]$
$[\![\Box\varphi]\!] = \Box[\![\varphi]\!]$

The equality axioms are valid on $V^{\mathcal{L}}$. That is,
Theorem 3.5. *For any formula $\varphi(a)$ and $u, v \in V^{\mathcal{L}}$,*

$$[\![u = v \wedge \varphi(u)]\!] \leqslant [\![\varphi(v)]\!].$$

Proof. If $\varphi(a)$ is an atomic formula, then it is immediate from Theorem 3.3 and 3.4. Now we assume $[\![u = v \wedge \varphi_i(u)]\!] \leqslant [\![\varphi_i(v)]\!]$ for $i = 1, 2$. Then $[\![u = v \wedge \varphi_1(u) \wedge \varphi_2(u)]\!] \leqslant [\![\varphi_1(v) \wedge \varphi_2(v)]\!]$ is obvious. Since $[\![u = v]\!]$ is $\Box$-closed,

$$\begin{aligned}
[\![u = v \wedge (\varphi_1(u) \vee \varphi_2(u))]\!] &= [\![u = v \wedge \varphi_1(u)]\!] \vee [\![u = v \wedge \varphi_2(u)]\!] \\
&\leqslant [\![\varphi_1(v) \vee \varphi_2(v)]\!], \\
[\![u = v \wedge \neg\varphi_1(u)]\!] &\leqslant [\![\neg\varphi_1(v)]\!] \text{ by Lemma 2.1.(4)} \\
[\![u = v \wedge \Box\varphi_1(u)]\!] &\leqslant \Box[\![\varphi_1(v)]\!] \text{ by G4.} \\
[\![u = v \wedge \exists x\varphi_1(u, x)]\!] &= \bigvee_x [\![u = v \wedge \varphi_1(u, x)]\!] \quad \text{by G5,} \\
&\leqslant [\![\exists x\varphi_1(v, x)]\!] \\
[\![u = v \wedge \forall x\varphi_1(u, x)]\!] &= \bigwedge_x [\![u = v \wedge \varphi_1(u, x)]\!] \\
&\leqslant [\![\forall x\varphi_1(v, x)]\!]. \quad \Box
\end{aligned}$$

Theorem 3.6. *For any formula $\varphi(a)$ and $u \in V^{\mathcal{L}}$,*

(1) $[\![\forall x(x \in u \to \varphi(x))]\!] = \bigwedge_{x \in \mathcal{D}u} [\![x \in u \to \varphi(x)]\!]$

(2) $[\![\exists x(x \in u \wedge \varphi(x))]\!] = \bigvee_{x \in \mathcal{D}u} [\![x \in u \wedge \varphi(x)]\!]$

Proof. (1): $[\![\forall x(x \in u \to \varphi(x))]\!] \leqslant \bigwedge_{x \in \mathcal{D}u} [\![x \in u \to \varphi(x)]\!]$ is obvious. By using the fact that $[\![x \in u]\!] \leqslant \bigvee_{x' \in \mathcal{D}u} [\![x = x']\!]$, and Lemma 3.1 and Theorem 3.4, we have

$$\begin{aligned}
\Big(\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!]\Big) \wedge [\![x \in u]\!] &= \Big(\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!]\Big) \wedge [\![x \in u]\!] \wedge \bigvee_{x'' \in \mathcal{D}u} [\![x = x'']\!] \\
&= \bigvee_{x'' \in \mathcal{D}u} \Big(\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!] \wedge [\![x \in u]\!] \wedge [\![x = x'']\!]\Big) \\
&\leqslant [\![\varphi(x)]\!]
\end{aligned}$$

Since $\bigwedge_{x \in \mathcal{D}u}(u(x) \to [\![\varphi(x)]\!]$ is $\Box$-closed, we have

$$\bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![\varphi(x)]\!]) \leqslant [\![\forall x(x \in u \to \varphi(x))]\!].$$

(2): By using $[\![x \in u]\!] \leqslant \bigvee_{x \in \mathcal{D}u} [\![x = x']\!]$ again,

$$\begin{aligned}
[\![\exists x(x \in u \wedge \varphi(x))]\!] &\leqslant \bigvee_{x \in V^{\mathcal{L}}} \bigvee_{x' \in \mathcal{D}u} ([\![x = x']\!] \wedge [\![x \in u \wedge \varphi(x)]\!]) \\
&\leqslant \bigvee_{x' \in \mathcal{D}u} [\![x' \in u \wedge \varphi(x')]\!]. \quad \Box
\end{aligned}$$

Definition 3.1. *Restriction* $u\restriction p$ of $u\in V^{\mathcal{L}}$ by $p\in\mathcal{L}$ is defined by

$$\begin{cases} \mathcal{D}(u\restriction p) = \{x\restriction p \mid x\in\mathcal{D}u\} \\ (u\restriction p)(x\restriction p) = \bigvee\{u(x')\wedge p \mid x'\in\mathcal{D}u,\ x\restriction p = x'\restriction p\} \quad \text{for} \quad x\in\mathcal{D}u. \end{cases}$$

If the rank of u is $\leqslant \alpha$ (i.e. $u \in V_{\alpha}^{\mathcal{L}}$), so is $u\restriction p$, and we have

Theorem 3.7. *If* $u, x \in V^{\mathcal{L}}$, $p, q \in \mathcal{L}$, *and* p *is* $\Box$*-closed(i.e,* $p = \Box p$*), then*

(1) $p \leqslant [\![u = u\restriction p]\!]$
(2) $[\![x\in u\restriction p]\!] = [\![x\in u]\!]\wedge p$
(3) $[\![(u\restriction p)\restriction q = u\restriction(p\wedge q)]\!] = 1.$

Proof. We proceed by induction on the rank of u,

(1) : For $x\in\mathcal{D}u$,
$p\wedge u(x) \leqslant (u\restriction p)(x\restriction p)\wedge[\![x=x\restriction p]\!] \leqslant [\![x\in u\restriction p]\!]$
$(u\restriction p)(x\restriction p) = \bigvee_{x'\in\mathcal{D}u,\, x\restriction p = x'\restriction p} u(x')\wedge p\wedge[\![x=x'=x\restriction p]\!] \leqslant [\![x\restriction p\in u]\!]$.
Therefore, $p \leq [\![u=u\restriction p]\!]$.

(2) : $[\![x\in u\restriction p]\!] = \bigvee_{x'\in\mathcal{D}u}[\![x=x'\restriction p]\!]\wedge\bigvee_{x''\in\mathcal{D}u,\, x''\restriction p=x'\restriction p} u(x'')\wedge p$
$\leqslant [\![x\in u]\!]\wedge p$,
since $x''\restriction p = x'\restriction p$ implies $p \leqslant [\![x''=x']\!]$.
$[\![x\in u]\!]\wedge p \leqslant \bigvee_{x'\in\mathcal{D}u}[\![x=x']\!]\wedge u(x')\wedge p$
$\leqslant \bigvee_{x'\in\mathcal{D}u}[\![x=x'\restriction p]\!]\wedge(u\restriction p)(x'\restriction p)$
$\leqslant [\![x\in u\restriction p]\!]$.
(3) follows from (2).
$\Box$

Now we state axioms of set theory which are valid on the universe $V^{\mathcal{L}}$.

Axiom of extensionality: $\forall x(x\in u \leftrightarrow x\in v) \rightarrow u=v$.
We have $[\![\forall x(x \in u \leftrightarrow x \in v)]\!] = [\![u = v]\!]$ by the definition of $[\![u = v]\!]$. Hence, $[\![\forall x(x\in u \leftrightarrow x\in v) \rightarrow u=v]\!] = 1$.

Axiom of pair: $\forall u, v\exists z\ \forall x(x\in z \leftrightarrow x=u \vee x=v)$.
For $u, v \in V^{\mathcal{L}}$ define z by

$$\begin{cases} \mathcal{D}z = \{u, v\} \\ z(t) = 1 \quad \text{for} \quad t\in\mathcal{D}z \end{cases}$$

Then $[\![x\in z]\!] = \bigvee_{t\in\mathcal{D}z}[\![x=t]\!]\wedge z(t) = [\![x=u]\!]\vee[\![x=v]\!]$.
Therefore, $[\![\forall x(x\in z \leftrightarrow x=u\vee x=v)]\!] = 1$.

Axiom of union: $\forall u \exists v \forall x(x \in v \leftrightarrow \exists y(y \in u \wedge x \in y))$.
For $u \in V^{\mathcal{L}}$ defined v by

$$\begin{cases} \mathcal{D}v = \bigcup_{y \in \mathcal{D}u} \mathcal{D}y \\ v(x) = [\![\exists y(y \in u \wedge x \in y)]\!]. \end{cases}$$

Then, by Theorem 3.6,

$$\begin{aligned} [\![\exists y(y \in u \wedge x \in y)]\!] &= \bigvee_{y \in \mathcal{D}u} [\![y \in u]\!] \wedge [\![x \in y]\!] \\ &= \bigvee_{y \in \mathcal{D}u} [\![y \in u]\!] \wedge [\![x \in y]\!] \wedge \bigvee_{x' \in \mathcal{D}y} [\![x = x']\!] \\ &= \bigvee_{y \in \mathcal{D}u,\ x' \in \mathcal{D}y} [\![x = x']\!] \wedge [\![x' \in y \wedge y \in u]\!] \\ &= [\![x \in v]\!] \end{aligned}$$

Definition 3.2. For each set x we define $\check{x} \in V^{\mathcal{L}}$ by

$$\begin{cases} \mathcal{D}\check{x} = \{\check{t} \mid t \in x\} \\ \check{x}(\check{t}) = 1 \end{cases}$$

$\check{x}$ is called the *check set associated with* x. For check sets $\check{x}, \check{y}$, we have

$$[\![\check{x} = \check{y}]\!] = \begin{cases} 1 & \text{if } \ x = y \\ 0 & \text{if } \ x \neq y \end{cases} \ ; \quad [\![\check{x} \in \check{y}]\!] = \begin{cases} 1 & \text{if } \ x \in y \\ 0 & \text{if } \ x \notin y. \end{cases}$$

Definition 3.3. Let

$$\mathrm{ck}(x) \overset{\text{def}}{\Longleftrightarrow} \forall y(y \in x \to y \overset{\Box}{\in} x \wedge \mathrm{ck}(y)).$$

Then $[\![\mathrm{ck}(\check{x})]\!] = 1$ for all x.

Axiom of infinity: $\exists u[\exists x(x \overset{\Box}{\in} u) \wedge \forall x(x \overset{\Box}{\in} u \to \exists y \overset{\Box}{\in} u(x \overset{\Box}{\in} y))]$. $\check{\omega}$ associated with the set ω of all natural numbers satisfies

$$[\![\exists x(x \overset{\Box}{\in} \check{\omega}) \wedge \forall x(x \overset{\Box}{\in} \check{\omega} \to \exists y \overset{\Box}{\in} \check{\omega}(x \overset{\Box}{\in} y))]\!] = 1.$$

Axiom of power set: $\forall u \exists v \forall x(x \in v \leftrightarrow x \subset u)$, where $x \subset u \overset{\text{def}}{\Longleftrightarrow} \forall t(t \in x \to t \in u)$.
Let $u \in V_\alpha^{\mathcal{L}}$. For every $x \in V^{\mathcal{L}}$, define x^* by

$$\begin{cases} \mathcal{D}x^* = V_\alpha^{\mathcal{L}} \\ x^*(t) = [\![x \subset u \wedge t \in x]\!]. \end{cases}$$

Then

$$[\![x\subset u\wedge t\in x]\!]\leqslant [\![t\in u]\!]\le \bigvee_{t'\in V_\alpha^{\mathcal{L}}}[\![t=t']\!]$$

Hence,

$$\begin{aligned}[\![x\subset u\wedge t\in x]\!] &\leqslant \bigvee_{t'\in V_\alpha^{\mathcal{L}}}[\![t=t'\wedge x\subset u\wedge t'\in x]\!]\\ &\leqslant [\![t\in x^*]\!].\end{aligned}$$

It follows that for every $x\in V^{\mathcal{L}}$ there exists $x^*\in V_{\alpha+1}^{\mathcal{L}}$ such that $[\![x\subset u]\!]\le[\![x=x^*]\!]$. Now we define v by

$$\begin{cases}\mathcal{D}v=V_{\alpha+1}^{\mathcal{L}}\\ v(x)=[\![x\subset u]\!].\end{cases}$$

Then

$$[\![\forall x(x\in v\leftrightarrow x\subset u)]\!]=1.$$

Axiom of separation: $\forall u\exists v[\forall x(x\in v\leftrightarrow x\in u\wedge\varphi(x))]$.
For a given $u\in V^{\mathcal{L}}$ define v by

$$\begin{cases}\mathcal{D}v=\mathcal{D}u\\ v(x)=[\![x\in u\wedge\varphi(x)]\!]\end{cases}$$

Then

$$[\![\forall x(x\in v\leftrightarrow x\in u\wedge\varphi(x))]\!]=1.$$

Axiom of collection: $\forall u\exists v[\forall x(x\in u\to\exists y\varphi(x,y))\to\forall x(x\in u\to\exists y\overset{\square}{\in}v\varphi(x,y))]$.
Let

$$p=[\![\forall x(x\in u\to\exists y\varphi(x,y))]\!]=\bigwedge_{x\in\mathcal{D}u}([\![x\in u]\!]\to\bigvee_y[\![\varphi(x,y)]\!]).$$

It suffices to show that there exists v such that

$$p\leqslant[\![\forall x(x\in u\to\exists y\overset{\square}{\in}v\varphi(x,y))]\!].$$

Since $\mathcal{L}$ is a set, for each $x\in\mathcal{D}u$ there exists an ordinal $\alpha(x)$ such that

$$p\wedge[\![x\in u]\!]\leqslant\bigvee_{y\in V_{\alpha(x)}^{\mathcal{L}}}[\![\varphi(x,y)]\!].$$

Hence, by using the axiom of collection externally, there exists an ordinal α such that

$$p\wedge[\![x\in u]\!]\leqslant\bigvee_{y\in V_\alpha^{\mathcal{L}}}[\![\varphi(x,y)]\!]\quad\text{for all } x\in\mathcal{D}u.$$

Now we defined v by

$$\begin{cases} \mathcal{D}v = V_\alpha^{\mathcal{L}} \\ v(y) = 1 \end{cases}$$

Then

$$p \wedge [\![x \in u]\!] \leqslant \bigvee_{y \in \mathcal{D}v} [\![y \overset{\square}{\in} v \wedge \varphi(x,y)]\!] = [\![\exists y \overset{\square}{\in} v \varphi(x,y)]\!] \quad \text{for all } x \in \mathcal{D}u.$$

Since $p = \square p$, we have

$$p \leqslant [\![\forall x(x \in u \to \exists y \overset{\square}{\in} v \varphi(x,y))]\!].$$

Axiom of $\in$-induction: $\forall x[\forall y(y \in x \to \varphi(y)) \to \varphi(x)] \to \forall x \varphi(x)$.
Let $p = [\![\forall x(\forall y(y \in x \to \varphi(y)) \to \varphi(x)]\!]$. We prove $p \leqslant [\![\forall x \varphi(x)]\!] = \bigwedge_{x \in V^{\mathcal{L}}} [\![\varphi(x)]\!]$ by induction on the rank of x. Let $x \in V_\alpha^{\mathcal{L}}$. Since $p \leq [\![\varphi(y)]\!]$ for all $y \in \mathcal{D}x \subset V_{<\alpha}^{\mathcal{L}}$ by induction hypothesis,

$$p \wedge [\![y \in x]\!] \leqslant [\![\varphi(y)]\!] \quad \text{for all } y \in \mathcal{D}x.$$

Hence, by using $p = \square p$, we have

$$p \leqslant [\![\forall y(y \in x \to \varphi(y))]\!].$$

It follows that $p \leqslant [\![\forall x \varphi(x)]\!]$.

Zorn's Lemma : $\mathrm{GL}(u) \wedge \forall v[\mathrm{Chain}(v,u) \to \bigcup v \in u] \to \exists z \, \mathrm{Max}(z,u)$, where

$$\begin{aligned} \mathrm{GL}(u) &\overset{\text{def}}{\iff} \forall x(x \in u \to x \overset{\square}{\in} u), \\ \mathrm{Chain}(v,u) &\overset{\text{def}}{\iff} v \subset u \wedge \forall x,y(x,y \in v \to x \subset y \vee y \subset x), \\ \mathrm{Max}(z,u) &\overset{\text{def}}{\iff} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x). \end{aligned}$$

For $u \in V_\alpha^{\mathcal{L}}$, let

$$p = [\![\mathrm{GL}(u) \wedge \forall v(\mathrm{Chain}(v,u) \to \bigcup v \in u)]\!],$$

and let U be a maximal subset of $V_\alpha^{\mathcal{L}}$ such that

$$\forall x,y \in U([\![x \in u \wedge \exists t(t \in x) \wedge y \in u \wedge \exists t(t \in y)]\!] \wedge p \leqslant [\![x \subset y \vee y \subset x]\!]).$$

U is not empty. Define v by

$$\begin{cases} \mathcal{D}v = U \\ v(x) = p \wedge [\![x \in u \wedge \exists t(t \in x)]\!]. \end{cases}$$

Now it suffices to show that $p \leqslant [\![\mathrm{Max}(\bigcup v, u)]\!]$. Since $p = \Box p$ and $p \wedge v(x) \leqslant [\![x \in u]\!]$ for all $x \in \mathcal{D}v$, we have $p \leqslant [\![v \subset u]\!]$. Hence, by the definition of v, $p \leqslant [\![\mathrm{Chain}(v, u)]\!]$. Therefore, $p \leq [\![\bigcup v \in u]\!]$. Now it suffices to show that

$$p \wedge [\![x \in u \wedge \bigcup v \subset x]\!] \leqslant [\![x \subset \bigcup v]\!] \quad \text{for} \quad x \in \mathcal{D}u.$$

Let $x \in \mathcal{D}u$ and $r = p \wedge [\![x \in u \wedge \bigcup v \subset x]\!]$. Then r is $\Box$-closed, and we have $r \leqslant [\![x = x \restriction r]\!]$ by Theorem 3.7. Hence $x \restriction r \in U$. In fact, for each $y \in U$, we have

$$\begin{aligned}
[\![y \in u \wedge \exists t(t \in y) \wedge (x \restriction r) \in u \wedge \exists t(t \in x \restriction r)]\!] \wedge p &\leqslant [\![y \in v]\!] \wedge r \\
&\leqslant [\![y \subset \bigcup v \subset x]\!] \wedge [\![x = x \restriction r]\!] \\
&\leqslant [\![y \subset x \restriction r]\!] \\
&\leqslant [\![y \subset x \restriction r \vee x \restriction r \subset y]\!].
\end{aligned}$$

It follows that

$$\begin{aligned}
r \wedge x(t) &\leqslant [\![x = x \restriction r \wedge x \in u \wedge t \in x]\!] \wedge p \\
&\leqslant [\![x = x \restriction r \wedge x \restriction r \in u \wedge \exists t(t \in x \restriction r)]\!] \wedge p \\
&\leqslant [\![x = x \restriction r]\!] \wedge v(x \restriction r) \\
&\leqslant [\![x \in v]\!] \leq [\![x \subset \bigcup v]\!] \\
\text{Therefore,}\quad r &\leqslant [\![x \subset \bigcup v]\!].
\end{aligned}$$

Definition 3.4. $\Diamond$ is the logical operation defined by $\Diamond\varphi \overset{\text{def}}{\Longleftrightarrow} \neg\Box\neg\varphi$.

Axiom of $\Diamond$: $\forall u \exists v \forall x(x \in v \leftrightarrow \Diamond(x \in u))$. For a given $u \in V^{\mathcal{L}}$, defined v by

$$\begin{cases} \mathcal{D}v = \mathcal{D}u \\ v(x) = [\![\Diamond(x \in u)]\!]. \end{cases}$$

By using Lemma 2.3,

$$\begin{aligned}
[\![\Diamond(x \in u)]\!] &= \Diamond \bigvee_{x' \in \mathcal{D}u} [\![x = x']\!] \wedge u(x') \\
&\leqslant \bigvee_{x' \in \mathcal{D}u} [\![x = x']\!] \wedge [\![\Diamond(x' \in u)]\!] = [\![x \in v]\!].
\end{aligned}$$

Hence

$$[\![\forall x(x \in v \leftrightarrow \Diamond(x \in u))]\!] = 1.$$

Now we posturate the above axioms, that is the following GA1–GA11, as the nonlogical axioms of our lattice valued set theory **LZF**.

GA1. Equality: $\forall u \forall v[u = v \wedge \varphi(u) \rightarrow \varphi(v)]$.

GA2. Extensionality: $\forall u, v[\forall x(x \in u \leftrightarrow x \in v) \rightarrow u = v]$.
GA3. Pairing: $\forall u, v \exists z[\forall x(x \in z \leftrightarrow (x = u \vee x = v))]$.
The set z satisfying $\forall x(x \in z \leftrightarrow (x = u \vee x = v))$ is denoted by $\{u, v\}$.
GA4. Union: $\forall u \exists z[\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))]$.
The set z satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ is denoted by $\bigcup u$.
GA5. Power set: $\forall u \exists z[\forall x(x \in z \leftrightarrow x \subset u)]$, where

$$x \subset u \overset{\text{def}}{\iff} \forall y(y \in x \rightarrow y \in u).$$

The set z satisfying $\forall x(x \in z \leftrightarrow x \subset u)$ is denoted by $\mathcal{P}(u)$.
GA6. Infinity: $\exists u[\exists x(x \overset{\square}{\in} u) \wedge \forall x(x \overset{\square}{\in} u \rightarrow \exists y \overset{\square}{\in} u(x \overset{\square}{\in} y))]$.
GA7. Separation: $\forall u \exists v[\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))]$.
The set v satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ is denoted by $\{x \in u \mid \varphi(x)\}$.
GA8. Collection:

$$\forall u \exists v[\forall x(x \in u \rightarrow \exists y \varphi(x, y)) \rightarrow \forall x(x \in u \rightarrow \exists y \overset{\square}{\in} v \varphi(x, y))].$$

GA9. $\in$-induction: $\forall x[\forall y(y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x)$.
GA10. Zorn: $GL(u) \wedge \forall v[\text{Chain}(v, u) \rightarrow \bigcup v \in u] \rightarrow \exists z \, \text{Max}(z, u)$, where

$$\begin{aligned} \text{GL}(u) &\overset{\text{def}}{\iff} \forall x(x \in u \rightarrow x \overset{\square}{\in} u), \\ \text{Chain}(v, u) &\overset{\text{def}}{\iff} v \subset u \wedge \forall x, y(x, y \in v \rightarrow x \subset y \vee y \subset x), \\ \text{Max}(z, u) &\overset{\text{def}}{\iff} z \in u \wedge \forall x(x \in u \wedge z \subset x \rightarrow z = x). \end{aligned}$$

GA11. Axiom of $\Diamond$: $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u))$.
The set z satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ is denoted by $\Diamond u$.

Theorem 3.8. *If $P(x_1, \cdots, x_n)$ is a bounded formula with n free variables $x_1, \cdots, x_n$, then*

$$[\![P(\check{u}_1, \cdots, \check{u}_n)]\!] = \begin{cases} 1, & \text{if } P(u_1, \cdots, u_n) \\ 0, & \text{if } \neg P(u_1, \cdots, u_n). \end{cases}$$

Proof. By induction on the complexity of P. $\square$

If $[\![\varphi]\!] = 1$ then we say φ holds in $V^{\mathcal{L}}$.

Corollary 3.9. *If a formula $P(x, x_1, \cdots, x_n)$ is a bounded formula with free variables $x, x_1, \cdots, x_n$ and defines a unique set $u \in U$ such that $P(u, u_1, \cdots, u_n)$, i.e.*

$$u \in U \wedge P(u, u_1, \cdots, u_n) \wedge \forall x[x \in U \wedge P(x, u_1, \cdots, u_n) \rightarrow x = u],$$

then, in $V^{\mathcal{L}}$, $P(x, x_1, \cdots, x_n)$ defines a unique check set $\check{u} \in \check{U}$ such that $P(\check{u}, \check{u}_1, \cdots, \check{u}_n)$, i.e.

$$\check{u} \in \check{U} \wedge P(\check{u}, \check{u}_1, \cdots, \check{u}_n) \wedge \forall x(x \in \check{U} \wedge P(x, \check{u}_1, \cdots, \check{u}_n) \rightarrow x = \check{u})$$

holds in $V^{\mathcal{L}}$.

Definition 3.5. A relation $\sim$ is an *equivalence relation* on a set G if

(1) $a \sim b \rightarrow a \in G \wedge b \in G$
(2) $a \in G \rightarrow a \sim a$
(3) $a \sim b \rightarrow b \sim a$
(4) $a \sim b \wedge b \sim c \rightarrow a \sim c$.

If $\sim$ is an equivalence relation on G, we use the following usual notations.

$$\begin{aligned} [a] &= \{b \in G \mid a \sim b\} \text{ for } a \in G, \\ G/\!\sim &= \{[a] \mid a \in G\}. \end{aligned}$$

Corollary 3.10. *Let* $P(x_1, x_2)$ *be a bounded formula with free variables* x_1, x_2 *which defines an equivalence relation* $\sim$ *on* G, *i.e.*

$$\forall a, b \in G(a \sim b \longleftrightarrow P(a, b)).$$

Then the corresponding relation $\sim$ *on* $V^{\mathcal{L}}$ *defined by*

$$u \sim v \overset{\text{def}}{\Longleftrightarrow} u \in \check{G} \wedge v \in \check{G} \wedge P(u, v),$$

satisfies :

(1) $[\![\sim$ *is an equivalence relation on* $\check{G}]\!] = 1$,
(2) $[\![[\check{a}] = [a]\check{}\,]\!] = 1$ *for* $a \in G$,
(3) $[\![\check{G}/\!\sim\, = (G/\!\sim)\check{}\,]\!] = 1$,

where

$$[\check{a}] = \{b \in \check{G} \mid b \sim \check{a}\} \ , \ \check{G}/\!\sim\, = \{[a] \mid a \in \check{G}\}.$$

Proof.

(1) is obvious by Corollary 3.9

(2) $$\begin{aligned} [\![x \in [\check{a}]]\!] &= [\![x \in \check{G} \wedge P(\check{a}, x)]\!] \\ &= \bigvee_{b \in G} [\![x = \check{b} \wedge P(\check{a}, \check{b})]\!] \\ &= \bigvee_{b \in G} [\![x = \check{b} \wedge \check{b} \in [a]\check{}\,]\!] \\ &= [\![x \in [a]\check{}\,]\!] \end{aligned}$$

(3) $$\begin{aligned} [\![x \in \check{G}/\!\sim]\!] &= [\![\exists u \in \check{G}(x = [u])]\!] \\ &= \bigvee_{b \in G} [\![x = [\check{b}] = [b]\check{}\,]\!] \\ &= [\![x \in (G/\!\sim)\check{}\,]\!] \quad \square \end{aligned}$$

Definition 3.6. For elements u, v of a set G, the pair $\langle u, v\rangle$ of u, v is defined by

$$\langle u, v\rangle \overset{\text{def}}{=} \{\{u\}, \{u, v\}\},$$

and the set of all pairs $\langle u, v\rangle$ with $u, v \in G$ is denoted by $G \times G$.

$$G \times G \overset{\text{def}}{=} \{\langle u, v\rangle \mid u \in G \wedge v \in G\}.$$

Since $\{\check{x_1}, \cdots, \check{x_n}\} = \{x_1, \cdots, x_n\}\check{}$ holds on $V^{\mathcal{L}}$, we have

$$[\![\langle \check{u}, \check{v}\rangle = \langle u, v\rangle\check{}\,]\!] = 1 \text{ and}$$
$$[\![\check{G} \times \check{G} = (G \times G)\check{}\,]\!] = 1.$$

4. NUMBERS

The set ω of all natural numbers is constructed from 0 by the successor function $x \mapsto x{+}1$, where 0 is the empty set and $x{+}1 = x \cup \{x\}$. The integers are constructed as equivalence classes of pairs of natural numbers, the rational numbers are constructed as equivalence classes of pairs of integers, and finally, the real numbers are constructed by Dedekind's cuts of rational numbers. We denote the set of all integers by $\mathbb{Z}$, the set of all rational numbers by $\mathbb{Q}$, the set of all real numbers by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$.

4.1. Natural numbers in $V^{\mathcal{L}}$.

Now we define the set of natural numbers in $V^{\mathcal{L}}$. It will be equal to $\check{\omega}$.

$\check{0}$ is the empty set in $V^{\mathcal{L}}$, i.e.

$$[\![\forall x \neg (x \in \check{0})]\!] = 1.$$

Let $\mathrm{Suc}(x)$ be the formula, which means "x is a successor", defined by

$$\mathrm{Suc}(x) \overset{\text{def}}{\Longleftrightarrow} x = \check{0} \vee \exists y(x = y + 1)$$

where $y{+}1 = y \cup \{y\}$, and let $\mathrm{HSuc}(x)$ be the formula, which means "x is a hereditary successor", defined by

$$\mathrm{HSuc}(x) \overset{\text{def}}{\Longleftrightarrow} \mathrm{Suc}(x) \wedge \forall y(y \in x \to \mathrm{Suc}(y)).$$

Lemma 4.1. $\forall x[\mathrm{HSuc}(x) \to (\mathrm{ck}(x) \wedge \mathrm{Tr}(x) \wedge \forall y(y \in x \to \mathrm{HSuc}(y))]$

Proof. (Grayson) Using $\in$-induction, let $p = [\![\mathrm{HSuc}(x)]\!]$. Then $p = \Box p$ and $p \leqslant [\![x = \check{0} \vee \exists y(x = y + 1)]\!]$. And

$$\begin{aligned} p \wedge [\![x = \check{0}]\!] &\leqslant [\![\mathrm{ck}(x) \wedge \mathrm{Tr}(x) \wedge \forall y(y \in x \to \mathrm{HSuc}(x))]\!] \\ p \wedge [\![\exists y(x = y + 1)]\!] &\leqslant \bigvee_y p \wedge [\![x = y + 1 \ \wedge \ \mathrm{HSuc}(y)]\!] \\ &\leqslant [\![\mathrm{ck}(x) \wedge \mathrm{Tr}(x) \wedge \forall y(y \in x \to \mathrm{HSuc}(x))]\!] \end{aligned}$$

$\Box$

x is called a *natural number* if HSuc(x). Since $\{\check{x}\} = \{x\}^{\check{}}$ and $\check{x} \cup \check{y} = (x \cup y)^{\check{}}$ hold in $V^{\mathcal{L}}$ for any sets x, y, we can see, by $\in$-induction (GA9), that $\check{\omega}$ is the set of all natural numbers in $V^{\mathcal{L}}$. That is,

$$[\![\forall x(x \in \check{\omega} \leftrightarrow \mathrm{HSuc}(x))]\!] = 1.$$

Moreover, the check sets $\check{+}$, and $\check{\cdot}$ associated with the operations $+$, $\cdot$ on ω coincide with addition $+$ and multiplication $\cdot$ on $\check{\omega}$ in $V^{\mathcal{L}}$. That is, let

$$\begin{cases} \mathcal{D}(+) = \{\langle \check{m}, \check{n}, (m+n)^{\check{}} \rangle \mid m, n \in \omega\} \\ +(\check{m}, \check{n}, (m+n)^{\check{}}) = 1 \end{cases}$$

$$\begin{cases} \mathcal{D}(\cdot) = \{\langle \check{m}, \check{n}, (m \cdot n)^{\check{}} \rangle \mid m, n \in \omega\} \\ \cdot\, (\check{m}, \check{n}, (m \cdot n)^{\check{}}) = 1. \end{cases}$$

We denote $(x, y, z) \in +$, and $(x, y, z) \in \cdot$ by $x + y = z$, and $x \cdot y = z$, respectively. Then $+$, $\cdot$ are operations on $\check{\omega}$ in $V^{\mathcal{L}}$, and for $m, n \in \omega$,

$$[\![\check{m} + \check{n} = (m+n)^{\check{}} \wedge \check{m} \cdot \check{n} = (m \cdot n)^{\check{}}]\!] = 1.$$

Similarly, the relation associated with the relation $\leqslant$ on ω is also denoted by $\leqslant$ in $V^{\mathcal{L}}$. $\leqslant$ is the relation on $\check{\omega}$. That is, let

$$\begin{cases} \mathcal{D}(\leqslant) = \{\langle \check{m}, \check{n} \rangle \mid m, n \in \omega,\ m \leqslant n\} \\ \leqslant (\check{m}, \check{n}) = 1. \end{cases}$$

We denote $\leqslant (x, y)$ by $x \leqslant y$. Then, $m \leqslant n$ iff $[\![\check{m} \leqslant \check{n}]\!] = 1$ for all $m, n \in \omega$, and

$$\forall m, n(m, n \in \check{\omega} \to (m \leqslant n \leftrightarrow \exists l(l \in \check{\omega} \wedge m + l = n))$$

holds in $V^{\mathcal{L}}$.

It follows that if $\varphi(x_1, \cdots, x_n)$ is a bounded formula constructed in terms of the relations $\in$, $=$, $\leq$ and functions $+$, $\cdot$, then for all $x_1, \cdots, x_n \in \omega$

$$\varphi(x_1, \cdots, x_n) \Longleftrightarrow [\![\varphi(\check{x}_1, \cdots, \check{x}_n)]\!] = 1.$$

4.2. Integers.

Integers are defined to be equivalence classes of $\omega \times \omega$, where the equivalence relation $\sim$ is defined by

$$u \sim v \overset{\text{def}}{\Longleftrightarrow} \exists m, n, p, q \in \omega(u = \langle m, n \rangle \wedge v = \langle p, q \rangle \wedge m + q = n + p).$$

That is, $\mathbb{Z} = \omega \times \omega / \sim$.

On $V^{\mathcal{L}}$, the corresponding equivalence relation $\sim$ on $\check{\omega} \times \check{\omega}$ is defined by Corollary 3.10. i.e.

$$u \sim v \overset{\text{def}}{\Longleftrightarrow} \exists m, n, p, g \in \check{\omega}(u = \langle m, n \rangle \wedge v = \langle p, g \rangle \wedge m + q = n + p),$$

and

$$[\![\check{\omega} \times \check{\omega}/\sim = (\omega \times \omega/\sim)^{\vee} = \check{\mathbb{Z}}]\!] = 1.$$

Namely, $\check{\mathbb{Z}}$ is the set of all integers in $V^{\mathcal{L}}$, and operations $+$ and $\cdot$ on $\check{\mathbb{Z}}$ are defined so that

$$[\![\check{a} + \check{b} = (a+b)^{\vee} \wedge \check{a} \cdot \check{b} = (a \cdot b)^{\vee}]\!] = 1,$$

$$[\![\check{a} \leqslant \check{b}]\!] = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{if } a > b, \end{cases} \qquad [\![\check{a} < \check{b}]\!] = \neg[\![\check{a} \geqslant b]\!].$$

4.3. Rational numbers.

In order to define the set $\mathbb{Q}$ of rational numbers, we define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ by

$$u \sim v \overset{\text{def}}{\Longleftrightarrow} \exists a, b, a', b' \in \mathbb{Z}(u = \langle a, b\rangle \wedge v = \langle a', b'\rangle \wedge ab' = a'b).$$

Then the set $\mathbb{Q}$ is defined to be $(\mathbb{Z} \times \mathbb{Z})/\sim$. On $V^{\mathcal{L}}$, the corresponding equivalence relation $\sim$ on $(\check{\mathbb{Z}} \times \check{\mathbb{Z}})$ is defined, by Corollary 3.10, i.e.

$$u \sim v \overset{\text{def}}{\Longleftrightarrow} \exists a, b, a', b' \in \check{\mathbb{Z}}(u = \langle a, b\rangle \wedge v = \langle a', b'\rangle \wedge ab' = a'b).$$

and

$$[\![(\check{\mathbb{Z}} \times \check{\mathbb{Z}})/\sim = \check{\mathbb{Q}}]\!] = 1.$$

Namely, $\check{\mathbb{Q}}$ is the set of all rational numbers in $V^{\mathcal{L}}$. Moreover, operations $+, \cdot$ and relations $\leqslant, <$ on $\check{\mathbb{Q}}$ are defined so that

$$[\![\check{a} + \check{b} = (a+b)^{\vee} \wedge \check{a} \cdot \check{b} = (a \cdot b)^{\vee}]\!] = 1$$

$$[\![\check{a} \leqslant \check{b}]\!] = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{if } a > b\,, \end{cases} \qquad [\![\check{a} < \check{b}]\!] = \neg[\![\check{a} \geqslant \check{b}]\!].$$

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