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A Graph-Theoretic Characterization Theorem for Multiplicative Fragment of Non-Commutative Linear Logic (Preliminary Report)*†

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Abstract

It is well-known that every proof net of MNCLL(Multiplicative fragment of Non-Commutative Linear Logic), can be drawn as a plane Danos-Regnier graph (drawing) satisfying the switching condition of Danos-Regnier ([3]). In this paper, we show the reverse direction that every plane Danos-Regnier graph (drawing) with one terminal edge satisfying the switching condition represents a unique MNCLL proof net (unique up to the dual mirror images). In the course of proving this, we also give the characterization of the MNCLL proof nets by means of the notion of strong planity of a Danos-Regnier graph, as well as the notion of a certain long-trip condition, called the stack-condition, of a Danos-Regnier graph, the latter of which is related to Abruci's balanced long-trip condition ([2]). In our full-paper version, we shall also apply our results to Intuitionistic Linear Logic, and obtain a characterization theorem for Multiplicative Intuitionistic Non-Commutative Linear Logic, in terms of signed Danos-Regnier graphs.

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1 Introduction.

It is well-known that the proof nets of Multiplicative (Commutative) Linear Logic (MLL) are characterized by a simple and elegant graph-theoretic condition, saying that any Danos-Regnier graph is a proof net of MLL if and only if it is acyclic and connected under any choice of par-link switching (cf. Danos-Regnier [3]). This condition is sometimes called as the (Danos-Regnier) switching condition. This characterization is a simplified version of a famous result of Girard ([4]), which is called the long-trip condition. It has been well-known that any proof net of Multiplicative Non-Commutative Linear Logic can be drawn as a plane graph. Hence, a proof net of Multiplicative Non-Commutative Linear Logic is a Danos-Regnier graph which not only satisfies the Danos-Regnier condition but also is planar. It has been a long-time open question if or not the reverse direction is true. The purpose of this paper is to answer to this question affirmatively; we show that any plane Danos-Regnier graph drawing with one terminal edge satisfying the switching condition represents a unique proof net of Multiplicative Non-Commutative Linear Logic modulo the mirror images (namely, it is interpretable to exactly two different non-commutative proof nets, each of which is the mirror image of the other). We also give a relationship of our purely graph-theoretic characterization of the non-commutative proof nets and Abruci's characterization ([2]) which uses the notion of a balanced long-trip condition.

In the course of our characterization proof, we introduce new notions of strong planity of a graph, and of the stack condition of a long-trip; roughly speaking, a marked Danos-Regnier graph is strongly planar if it is not only planar but also has a plane drawing extension to a par-link closure in which the ports of all links are rotated in a same direction (either clock-wisely or anti-clock-wisely), where a marked Danos-Regnier graph is a usual Danos-Regnier graph in which each ports of a link has a port name L (Left) or R (Right) or C (Conclusion). (See Section 2 for the formal definition). The stack condition is a modification of Abruci's balanced long trip condition ([2]); Instead of putting a mark at each conclusion node during a long-trip in Abruci's long-trip condition ([2]), our stack condition uses a stack for recording a certain information of a long-trip. (See Section 3 for the definition.)

In the next Section (Section 2) we show that any non-commutative proof net (i.e., a proof net of Multiplicative Non-Commutative Linear Logic) is a strongly planar marked Danos-Regnier graph satisfying the switching condition. In Section 3, we show that any non-commutative proof net is a strongly planar marked Danos-Regnier graph satisfying the switching condition. In Section 4, the equivalence between the stack condition and Abruci's long-trip condition is established. In Section 5, we show that if a marked Danos-Regnier graph satisfies the stack condition it is interpretable as a non-commutative proof net uniquely. In Section 6, we prove that any strongly planar marked Danos-Regnier graph satisfying the switching condition also satisfies the stack condition, which establishes the equivalence between the non-commutative proof nets and the three characterizations above. Since any plane marked Danos-Regnier graph drawing has a unique way to make a strongly planar marked Danos-Regnier graph (unique up to the isomorphic mirror images) as a corollary of the above, we establish the main characterization theorem that any plane Danos-Regnier graph drawing with one terminal edge, satisfying the switching condition, represents a unique non-commutative proof net (modulo the isomorphic mirror images), and vice versa. This characterization theorem gives the relationship between the notion of non-commutativity in logic and the notion of planity in graph theory.

The structure of the Sections is as follows.

$$
\begin{tikzcd}
& \text{Non–commutative Proof Net} \arrow[dr, "\text{S. 3}"] & \\
\text{Stack Condition} \arrow[ur, "\text{S. 5}"] \arrow[d, shift right, "\text{S. 4}"'] & & \text{Strong Planity of marked D–R graphs} \arrow[ll, "\text{S. 6}"'] \arrow[d, shift right, "\text{S. 6}"'] \\
\text{Long Trip Condition} \arrow[u, shift right] & & \text{Planity of D–R graph drawings} \arrow[u, shift right]
\end{tikzcd}
$$

2 Classical System MNCLL.

We denote a sequence of formulas by a capital Greek letter, such as $\Delta, \Gamma, \Sigma, \cdots$. We give the one-sided version of Multiplicative Non-Commutative Linear Logic.

Definition 2.1 *We define the nagation of a formula as follows: For each formula A and B, $(A \otimes B)^\perp = B^\perp \wp A^\perp$, $(A \wp B)^\perp = B^\perp \otimes A^\perp$, and $(A^\perp)^\perp = A$.*

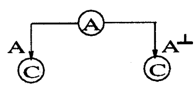
Definition 2.2 *We define the system* MNCLL (Multiplicative fragment of Non-Commutative Linear Logic) (Yetter [10]).
Axioms:
$\vdash A,\ A^\perp$, *where A is a formula.*

$$\frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut), \qquad \frac{\vdash A_1, \cdots A_{n-1}, A_n}{\vdash A_n, A_1, \cdots, A_{n-1}}\ (shift).$$

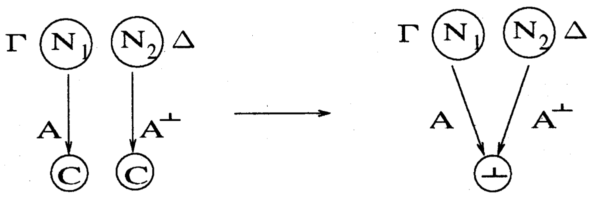
$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ (tensor), \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \wp B, \Delta}\ (par).$$

We define a *non-commutative proof net*, as a graph induced from a derivation in MNCLL as follows. We call an edge a *terminal edge*, if it is connected to a conclusion node.

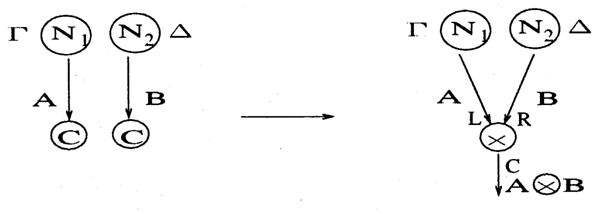
Definition 2.3 *We define a non-commutative proof net by induction on the derivation in MNCLL.*
(Axiom.) *We draw an axiom-link corresponding to* $\vdash A,\ A^{\perp}$ *as follows, so that we obtain a non-commutative proof net with the terminal edges of* $A, A^{\perp}$.



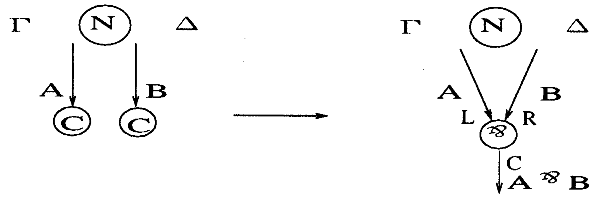
(Cut.) *Assume that sequences* Γ, A *and* $A^{\perp}, \Delta$ *of formulas are the terminal edges of non-commutative proof nets* N_1 *and* N_2 *respectively. Now we draw a cut-link as follows, so that we obtain a new non-commutative proof net with the terminal edges of* Γ, Δ.



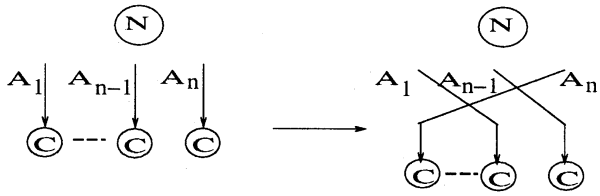
(Tensor.) *Assume that sequences* Γ, A *and* B, Δ *of formulas are the terminal edges of non-commutative proof nets* N_1 *and* N_2 *respectively. Now we draw a tensor-link as follows, so that we obtain a new non-commutative proof net with the terminal edges of* $\Gamma, A \otimes B, \Delta$.



(Par.) *Assume that sequences* Γ, A, B, Δ *of formulas are the terminal nodes of a non-commutative proof net* N. *Now we draw a par-link as follows, so that we obtain a new non-commutative proof net with the terminal edges of* $\Gamma, A \wp B, \Delta$.



(Shift.) *Assume that sequences* $A_1, \cdots, A_{n-1}, A_n$ *of formulas are the terminal nodes of a non-commutative proof net* N. *Now we extend edges* $A_1, \cdots, A_{n-1}$, *and cross them with* A_n, *so that we obtain a new non-commutative proof net with terminal edges* $A_n, A_1, \cdots, A_{n-1}$.



Clearly a non-commutative proof net has the same inductive structure as a proof net of MLL does. Thus we have the proposition:

Proposition 2.4 *A non-commutative proof net is a proof net of MLL.*

Proof. By induction on the number of nodes. □
Now we introduce a notion of D-R graphs:

Definition 2.5 *A directed Danos-Regnier graph (or D-R graph) is a directed graph, which consists of axiom-links, cut-links, tensor-links, par-links and conclusion nodes: An axiom-link has two out-edges; a cut-link has two in-edges; each of a tensor-link and a par-link has two in-edges and one out-edge.*

Definition 2.6 *An edge in a D-R graph connected to a conclusion node is called a terminal edge.*

We will follow Danos and Regnier's convention to denote a formula by an edge and a logical connective by a link in a D-R graph. The following characterization theorem for proof nets of MLL is due to Danos and Regnier.

Theorem 2.7 *(Danos and Regnier [3]) A D-R graph is a proof net of MLL, if and only if it is always acyclic and connected under any choice of par-switchings (see [3] for the notion of par-switchings).*

We call the condition that a D-R graph is always acyclic and connected under any choice of par-switchings, as *the switching condition.*

As we noted earlier, a non-commutative proof net is a proof net of MLL, and so it can be drawn as a D-R graph.

3 Non-Commutative Proof Net Implies Strong Planity

In this section, we introduce a notion of marked D-R graphs. Then we give a notion of strong planity, which is later shown to characterize non-commutative proof nets in terms of marked D-R graphs. Our main theorem in this section is that any non-commutative proof net is strong planar. Finally we explain the relationship between the strong planity of the marked D-R graphs and the planity of the D-R graphs.

Definition 3.1 *A marked D-R graph is a D-R graph, where each of a tensor-link and a par-link has two in-edges labeled L (left) and R (right), respectively, and one out-edge labeled C (conclusion).*

Now we give a few geometric notions of a marked D-R graph necessary to define the strong planity.

Definition 3.2 *A marked D-R graph drawing is said to be uniformly directed if the L-edge, R-edge and C-edge for a link is drawn in a fixed cyclic order uniformly for all tensor-links and par-links, or the links of degree 3.*

Definition 3.3 *Let G be a marked D-R graph. A marked D-R graph $\bar{G}$ with single terminal edge is a closure of G, if it is obtained from G by removing the conclusion nodes from G, and connecting free edges by par-links, and by adding a conclusion node to the single free edge left at the end.*

Definition 3.4 *A marked D-R graph G is said to be strongly planar, if there exists a closure $\bar{G}$ of the graph G, which has a plane and uniformly directed drawing.*

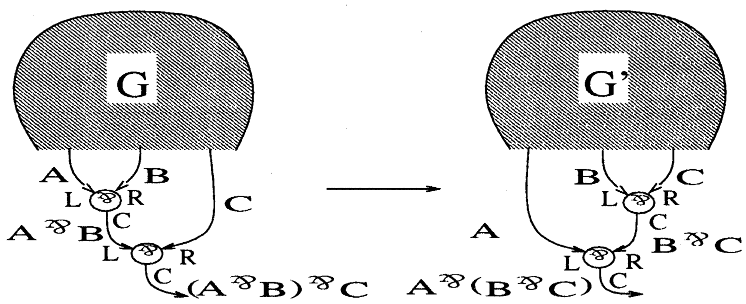
By a *mirror image* of a marked D-R graph drawing, we mean the reflection of the marked D-R graph drawing in the mirror: Thus any clockwisely directed marked D-R graph drawing has the mirror image, which is counter-clockwisely directed.

As a matter of simplicity, for a strongly planar graph G we always consider its uniformly directed plane graph drawing. Moreover we may assume that the links in the graph drawing are clockwisely directed, by taking its mirror image if necessary.

Let the in-edges L (left) and R (right) of a tensor-link (or a par-link) be labeled with formulas A and B, respectively. Then the out-edge C (conclusion) is labeled with the formula $A \otimes B$ (or $A \wp B$, respectively).

Proposition 3.5 *Let $(A \wp B) \wp C$ be an edge in a strongly planar marked D-R graph. Then we can obtain a new strongly planar graph with an edge $A \wp (B \wp C)$.*

Proof. By removing the pars from the graph drawing G with edge $(A\wp B)\wp C$, we obtain 3 free edges A, B and C. Then by connecting B and C first, and then A, we obtain a new plane clockwisely directed marked D-R graph drawing G' with edge $A\wp(B\wp C)$.



□

Definition 3.6 *We call a formula* $(A\wp B)\wp C$ *or* $A\wp(B\wp C)$ *as an associative par instance of* A, B, C.

We naturally extend the notion of the associative par instance for $A_1, \cdots, A_n$.

Definition 3.7 *We define a marked D-R graph with a sequence* Σ *of edges as follows:*
(1) A graph G *is a marked D-R graph with* A, *if and only if it is a marked D-R graph with terminal edge labeled* A.
(2) A G' *is a marked D-R graph with* Γ, A, B, Δ, *if and only if it is obtained from a marked D-R graph* G *with* $\Gamma, A\wp B, \Delta$, *by removing the par-link connecting* A *and* B *by the L-edge and the R-edge, respectively.*

Proposition 3.8 *Assume that a strong planar graph* G *satisfies the switching condition. Let* $\bar{G}$ *be a closure of* G, *which is a strongly planar graph with single terminal edge. Then the following are equivalent:*
(1) The graph G *is a strongly planar graph with* $A_1, \cdots A_n$,
(2) the graph $\bar{G}$ *is a strongly planar graph with single terminal edge being an associative par instance of* $A_1, \cdots, A_n$,
(3) for any associative par instance A *of* $A_1, \cdots, A_n$, *there exists a strongly planar graph with* A *as the single terminal edge.*

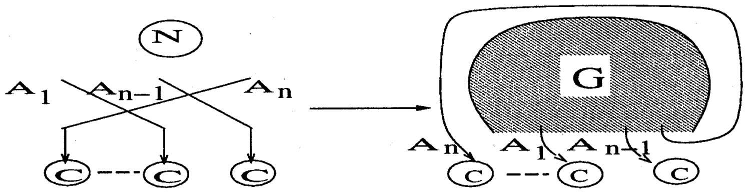
Proof. The equivalence between (1) and (2) follows from Definition 3.7. The equivalence between (2) and (3) follows from Proposition 3.5. □

Theorem 3.9 *If a marked D-R graph satisfying the switching condition is a non-commutative proof net, then it is strongly planar.*

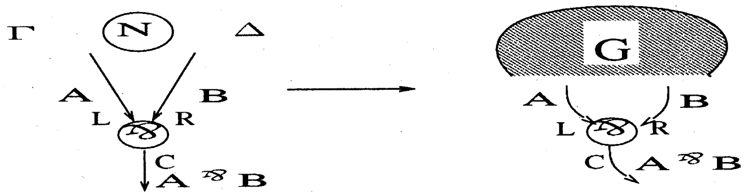
Proof. Let the non-commutative proof net have terminal nodes Σ. We construct by induction on the structure of the non-commutative proof net, a plane clockwisely directed marked D-R graph drawing G with Σ: Our construction preserves planity even when the Shift rule is applied.

Axiom. If the non-commutative proof net only consists of an axiom-link, then the claim trivially holds.

Shift. Assume that the last inference applied to the non-commutative proof net is Shift. Then the terminal nodes are $\Sigma \equiv A_n, A_1, \cdots, A_{n-1}$. By induction hypothesis, a non-commutative proof net with terminal nodes $\Sigma' \equiv A_1, \cdots, A_{n-1}, A_n$ is strongly planar; so it has a plane clockwisely directed marked D-R graph drawing G with $A_1, \cdots, A_{n-1}, A_n$. Then we bring the edge A_n over the drawing without intersecting the drawing itself: This is possible since the drawing is a finite figure. Thus we obtained a new plane clockwisely directed marked D-R graph drawing with $A_n, A_1, \cdots, A_{n-1}$.

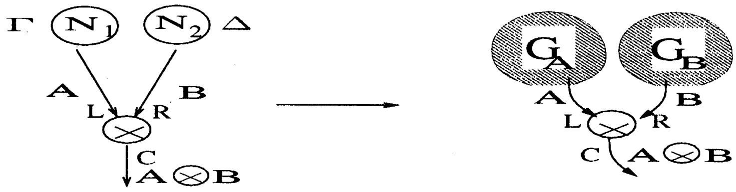


Par. Assume that the last inference applied to the non-commutative proof net is a par-link $A\wp B$: Let $\Sigma \equiv \Gamma, A\wp B, \Delta$. By removing the par-link, we obtain a new non-commutative proof net N with terminal nodes Γ, A, B, Δ. By induction hypothesis, the non-commutative proof net N is strongly planar, and it has a plane clockwisely directed marked D-R graph drawing with Γ, A, B, Δ. By simply connecting the edges A and B by a par-link with the L-edge and R-edge, we obtain a new plane clockwisely directed marked D-R graph drawing with $\Gamma, A\wp B, \Delta$.

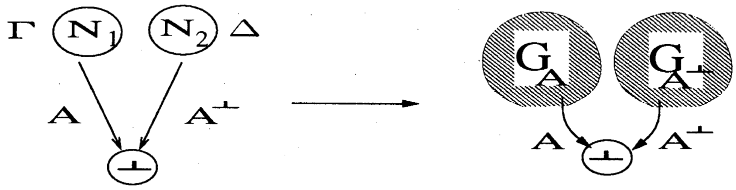


Tensor. Assume that the last link added to the proof net is a tensor-link $A\otimes B$: Let $\Sigma \equiv \Gamma, A\otimes B, \Delta$. By removing the tensor-link, we obtain new non-commutative proof nets N_1 and N_2 with terminal nodes Γ, A and B, Δ, respectively. By induction hypothesis, N_1 with Γ, A and N_2 with B, Δ are strongly planar. Let their marked D-R graphs be G_A and G_B, respectively, whose drawings are plane clockwisely directed. We draw G_A and G_B

apart enough so that there is no crossing between them. By connecting the edges A and B by a tensor-link with the L-edge and the R-edge respectively, we obtain a new plane clockwisely directed marked D-R graph drawing for the D-R graph G with $\Gamma, A\otimes B, \Delta$.



Cut. We can argue similarly to the case of tensor.



□

4 Equivalence between Stack Condition and Abruci's Long Trip Condition.

In this section, we give the notions of the long trip condition and the stack condition, and show the equivalence between the two. The long trip condition was originally given by Abruci, in order to characterize a multiplicative non-commutative Linear Logic MNLL ([2]). The system MNLL is not equivalent to MNCLL; because sequent $\vdash A, A^{\perp}$ is not a theorem, while sequent $\vdash A^{\perp}, A$ is, in MNLL due to the lack of the Shift rule.

The long trip condition is defined by a *special trip*, which is a long trip with restrictions. Because system MNCLL is defined with the Shift rule, the long trip condition for MNCLL will become much simpler than that for MNLL.

The notion of a *stack condition* is obtained from an attempt to analyze the relationship between the strong planity and the long trip condition; we show at the end of this section, the precise correspondence between the long trip condition in MNCLL and the stack condition.

Let us note that marked D-R graph G satisfying the switching condition is by Theorem 2.7, a proof net of MLL. Now we show how to adapt the long trip condition to MNCLL; and we also call the adaptation simply as the long trip condition in what follows. The following list of definitions and theorems are due to Abruci ([2]), unless noted otherwise.

Definition 4.1 *(Abruci [2]). For a given marked D-R graph G with an edge A,*
(1) T is a point of G, iff T is $A\downarrow$ or $A\uparrow$,
(2) we call a sequence $T_1,\cdots,T_n$ of points of G a one-way special trip from $A\uparrow$ (or $A\downarrow$) in G, iff the sequence is portion of the long trip in G from $T_1 = A\uparrow$ to $T_n = A\downarrow$ (or $T_1 = A\downarrow$ to $T_n = A\uparrow$, respectively), with the following switching:
(2.1) every $\otimes$-link is switched on "R" ("right"),
(2.2) every $\wp$-link is switched on "L" ("left").

Let G be a marked D-R graph satisfying the switching condition. By Theorem 2.7, graph G is a proof net of MLL. We say an edge is a *critical node* (a critical vertex of Abruci [2]) , if it is a terminal edge or a R-edge of a par-link.

As mentioned above, in system MNLL of Abruci [2], sequent $\vdash A^\perp, A$ is a theorem, while sequent $\vdash A, A^\perp$ is not. Due to such an asymmetry, Abruci's original long trip condition makes a distinction between traversals $A\uparrow, A^\perp\downarrow$ and $A^\perp\uparrow, A\downarrow$ of an axiom-link by means of the labels $x^C + a$; where C is a critical node of a marked D-R graph satisfying the switching condition, and a is an integer. By contrast, in our system MNCLL, we can start with the following simplified definition.

Definition 4.2 *(Modification of Definition 3.0 (iii) of Abruci [2]) $S(G) = \{x^C; C$ is a critical node of $G\}$.*

Let A be a terminal edge of G. An *assignment for G from A* is defined by a special trip $T_1,\cdots,T_n$ starting from $T_1 = A\uparrow$.

Definition 4.3 *We define an assignment for G from A by induction on i.*
(1) $\mathcal{L}(T_1) = x^A$.
Assume we defined $\mathcal{L}(T_i)$ for $i < n$;
(2) if $T_i = B\downarrow$, and B is a critical node of G (and so $T_{i+1} = B\uparrow$), then $\mathcal{L}(T_{i+1}) = x^B$;
(3) if $T_i = B\downarrow$ and B is the first premise of a par-link with C as second premise, then $\mathcal{L}(T_{i+1}) = \mathcal{L}(C\downarrow)$, if $C\downarrow = T_j$ with $j < i$ and $\mathcal{L}(T_i) = x^C$, or undefined, otherwise;
(4) $\mathcal{L}(T_{i+1}) = \mathcal{L}(T_i)$, in all the other cases.

We say an assignment $\mathcal{L}$ for G is total, iff $\mathcal{L}$ is a total function.

Proposition 4.4 *(Proposition 3.2 of Abruci [2]) Let G be a marked D-R graph satisfying the switching condition. Let $\mathcal{L}$ be a total assignment for G. If $\mathcal{L}'$ is an assignment for G, then $\mathcal{L}'$ is total, and $\mathcal{L} = \mathcal{L}'$.*

Definition 4.5 *(Definition 3.3 of Abruci [2]) Let G be a marked D-R graph satisfying the switching condition. (1) G is good, iff every assignment for G is total: By the previous proposition, if G is good, then all the assignments for G are equal.*
(2) If G is good, the labeled special trip in G is obtained from a special trip by replacing each point $A \downarrow$ *by* $A\mathcal{L}(A \downarrow)$ *and each point* $A \uparrow$ *by* $\mathcal{L}(A \uparrow)A$*, where* $\mathcal{L}$ *is the unique assignment for G.*

Definition 4.6 *(Modification of Definition 3.7 of Abruci [2]) (1) Let* $\mathcal{L}$ *be the unique total assignment for G. We define the binary relation* $\prec$ *(precedes) on the terminal edges of G:*

$$A \prec B \textit{ iff } \mathcal{L}(A \downarrow) = \mathcal{L}(\uparrow B) = x^B.$$

(2) G induces the linear order of the conclusions, iff $\prec$ *is a chain, and every conclusion occurs exactly once in the chain.*

Definition 4.7 *(Definition 3.8 of Abruci [2] [1]) Let G be a marked D-R graph satisfying the switching condition, and let* Σ *be a sequence of the edges in G. Then G with* Σ *satisfies the long trip condition, iff (1) the conclusions of G are exactly the formulas in* Σ*, (2) G is good, and (3) G induces the linear order of the conclusions.*

Lemma 4.8 *(Lemma 3.9 of Abruci [2]) Let G be a non-commutative proof net with conclusions* $\Sigma \equiv A_1, \cdots, A_n$*. Then the labeled special trip in G looks as:*
$(x^{A_1})A_1, \cdots, A_k(x^{A_1}), (x^{A_k})A_k, \cdots, A_2(x^{A_3}), (x^{A_2})A_2, \cdots, A_1(x^{A_2}), \cdots,$ *and no conclusion occurs for every* $1 \le i < k$ *in the portion between* $(x^{A_i})A_i, \cdots, A_i(x^{A_{i+1}})$.

Proof. By the property of a special trip in a proof net of MLL. □
In the long trip condition, the well-defined special trip gives the labels to edges as it visits, but the labels do not necessarily order all the terminal edges in the graph. Thus one needs Definition 4.7 (1) to get a correct notion. In the *stack condition*, on the other hand, any well-defined special trip always order all the terminal edges in the graph.

Definition 4.9 *(1) We define a stack* $\Sigma \equiv A_1, \cdots, A_n$ *as a sequence of formulas. Pop(S) =* A_1*. Let A be a formula. Then Push(A, S)* $\equiv A, S$*.*
(2) Let G be a marked D-R graph satisfying the switching condition, i.e. a proof net of MLL. Let $T_1, \cdots, T_n$ *be a special trip on G. We define a stack state* $S_G(T_i)$ *at a point* T_i *by induction* $i \le n$ *as follows:*

[1] Definition 3.8 in [2] defines Abruci's non-commutative proof net for MNLL: However we call as a non-commutative proof net the inductive structure defined in Section 2 in this paper. Instead, we call Abruci's non-commutative proof net as a marked D-R graph satisfying the switching condition and the long trip condition.

(2.1) $S_G(T_1) \equiv \phi$.
Assume we defined $S_G(T_i)$ *for* $i < n$:
(2.2) Let $T_{i+1} = B \uparrow$ *and* $T_i = B \downarrow$. *Then* $S_G(T_{i+1}) \equiv Push(B, S_G(T_i))$.
(2.3) Let $T_{i+1} = B \wp C \downarrow$ *and* $T_i = B \downarrow$. *Assume* $S_G(T_i) \equiv A_1, \cdots, A_n$. *Then* $S_G(T_{i+1}) \equiv A_2, \cdots, A_n$, *if* $Pop(S_G(T_i)) = A_1 = C$, *and undefined, otherwise. (2.4)* $S_G(T_{i+1}) \equiv S_G(T_i)$ *in all the other cases.*
(2.5) If $S_G(T_i)$ *is undefined, then* $S_G(T_{i+1})$ *is undefined, as well.*

Definition 4.10 *Let* G *be a marked D-R graph satisfying the switching condition, and* $T_1, \cdots, T_n$ *with* $T_1 = C \downarrow$ *be a special trip on* G, *and* C *is a terminal edge in* G. *We say that graph* G *with* Σ *satisfies the stack condition, if* $S_G(T_n) \equiv \Sigma$.

Finally we show the correspondence between the long trip condition and the stack condition.

Lemma 4.11 *Let* G *be a marked D-R graph being good and satisfying the switching condition. Let* T *be a point in* G, *and let* $T_1, \cdots, T_n$ *be a special trip on* G *with* $T_1 = C \downarrow$, *where* C *is a critical node. For any* $1 < i \leq n$, *if* $B = Pop(S_G(T_i))$, *then* $\mathcal{L}(T_i) = x^B$.

Proof. We prove it by induction on the length of the special trip in G from $C \downarrow$. Assume that the claim holds for $i < n$. Since it suffices to show the claim when the stack changes, we have 2 crucial cases: (1) Let $T_{i+1} = B \uparrow$ and $T_i = B \downarrow$. Then $\mathcal{L}(T_{i+1}) = x^B$, and the claim trivially holds. (2) Let $T_{i+1} = B \wp C \downarrow$ and $T_i = B \downarrow$. Assume $S_G(T_{i+1}) \equiv D, \Gamma$ for some sequence Γ of critical nodes in marked D-R graph G. Because $S_G(T_{i+1})$ is well-defined, $S_G(T_i) \equiv C, D, \Gamma$. Hence there is a C, such that $S_G(C \downarrow) \equiv D, \Gamma$ and $T_j = C \downarrow$ for some $j < i$. $\mathcal{L}(C \downarrow) = x^D$ and $\mathcal{L}(B \downarrow) = x^C$. Therefore $\mathcal{L}(B \wp C \downarrow) = x^D$. □

Lemma 4.12 *Let* G *be a marked D-R graph with* Σ *satisfying both the switching condition and the long trip condition. Let* $B \wp C \downarrow$ *be a point in* G. *If* $\mathcal{L}(B \wp C \downarrow) = x^A$, *then the last visited* $C \downarrow$ *satisfies* $\mathcal{L}(C \downarrow) = x^A$.

Proof. Since G is a proof net of MLL and $\mathcal{L}(B \wp C \downarrow) = x^A$, by removing enough par-links from G, we obtain a new marked D-R graph G' with terminal edges $\Gamma, B \wp C, A, \Delta$. The argument in Theorem 4.1 (i) in [2] shows that the long trip condition is preserved under the removal of par-links: Hence the graph G' with $\Gamma, B \wp C, A, \Delta$ satisfies the long trip condition. Again by removing the par-link between B and C, we obtain a new marked D-R graph with Γ, B, C, A, Δ satisfies the long trip condition. By Lemma 4.8, $\mathcal{L}(C \downarrow) = x^A$.
□

Lemma 4.13 *Let G be a marked D-R graph with Σ satisfying both the switching condition and the long trip condition. Let T be a point in G, and let $T_1, \cdots, T_n$ be a special trip on G with $T_1 = C \downarrow$, where C is a critical node. For any $1 < i \leq n$, if $\mathcal{L}(T) = x^B$, then $Pop(S_G(T)) = B$.*

We show by induction on the length of the special trip in G. Assume that the claim holds for $i < n$. We have the following two crucial cases: (1) Let $T_{i+1} = B \uparrow$, and B is a critical node. Then $T_i = B \downarrow$, and so $Pop(S_G(T_{i+1})) = B$. (2) Let $T_{i+1} = B \wp C \downarrow$, and $\mathcal{L}(T_{i+1}) = x^A$. Then $T_i = B \downarrow$ and $\mathcal{L}(B \downarrow) = C$, since $\mathcal{L}(B \wp C \downarrow)$ is defined. By Lemma 4.12, the last visited $C \downarrow$ satisfies $\mathcal{L}(C \downarrow) = x^A$. Since induction hypothesis implies $Pop(S_G(C \downarrow)) = A$, $S_G(C \downarrow) \equiv A, \Gamma$ for some sequence Γ of critical nodes in marked D-R graph G. Because this C is visited last, $S_G(B \downarrow) \equiv C, A, \Gamma$. Then point $B \wp C \downarrow$ follows, hence $Pop(S_G(B \downarrow)) = C$ and $Pop(S_G(T_{i+1})) = A$. □

Theorem 4.14 *Let G be a marked D-R graph with terminal edges Σ satisfying the switching condition. The marked D-R graph G with Σ satisfies the long trip condition, iff G with Σ satisfies the stack condition.*

Proof. Assume G with Σ satisfies the stack condition. Clearly G is good, and has terminal edges exactly equal to formulas in Σ. Let $\Sigma \equiv A_1, \cdots, A_n$. Because of the stack condition of G, A_i is a critical node, $S_G(A_i \downarrow) \equiv A_{i+1}, \cdots, A_n$ and $S_G(A_i \uparrow) \equiv A_i, \cdots, A_n$. Therefore $\mathcal{L}(A_i \uparrow) = x^{A_i}$ and $\mathcal{L}(A_i \downarrow) = x^{A_{i+1}}$ follow from Lemma 4.11. The reverse direction is clear from Lemmas 4.8 and 4.13. □

5 Stack Condition Implies Non-Commutative Proof Net.

In this section, we show that any marked D-R graph satisfying the stack condition is a non-commutative proof net.

Definition 5.1 *A sequence $A_{i+1}, \cdots, A_n, A_1, \cdots, A_i$ $(i \leq n)$ is called as a shift of $A_1, \cdots, A_n$.*

Proposition 5.2 *Let G be a marked D-R graph satisfying the switching condition, Assume for some terminal edge C in G, the special trip $T_1, \cdots, T_n$ with $T_1 = C \downarrow$ satisfies $S_G(T_n) = \Sigma$. Then for any terminal edge D in G, the special trip $T'_1, \cdots, T'_{n'}$ with $T'_1 = D \downarrow$ satisfies $S_G(T'_{n'})$ equal to a shift of Σ, in which D is the rightmost formula.*

Proof. By the property of the special trips. □

Proposition 5.3 *Let G be a marked D-R graph with Σ satisfying both the switching condition and the stack condition. Then for any shift Σ' of Σ, G with Σ' satisfies the stack condition.*

Proof. By Proposition 5.2. □

Definition 5.4 *An edge A is said to be connected to an edge B. if there is a path connecting the edges A and B.*

Theorem 5.5 *Let G be a marked D-R graph with Σ satisfying both the switching condition and the stack condition. Then it is a non-commutative proof net with terminal edges Σ.*

Proof. Because the marked D-R graph G satisfies the switching condition, by Theorem 2.7, G is a proof net of MLL. Thus we may assume its inductive structure on G.
(Axiom.) Clear.
Assume the marked D-R graph G with Σ satisfies the stack condition.
(Par.) Let Σ be $\Gamma, A\wp B, \Delta$. Let G' be a marked D-R graph obtained by removing the par-link between A and B. We show the stack condition on G' with Γ, A, B, Δ follows. Let C be the rightmost formula in Δ. By the stack condition of G, a special trip $T_1, \cdots, T_n$ on G starting $T_1 = C\downarrow$ satisfies $S_G(T_n) = \Gamma, A\wp B, \Delta$. We construct a special trip $T'_1, \cdots, T'_m$ on G', such that $S_G(T'_m) = \Gamma, A, B, \Delta$. We follow the same trip up to $A\downarrow$; let $T_i = A\downarrow$. We define $T'_j = T_j$ $(j \le i)$: Because $T_{i+1} = A\wp B\downarrow$ and $T_{i+2} = A\wp B\uparrow$, we define the rest of the trip as $T'_j = T_{j+2}$ $(i+1 \le j \le m)$.

Now we show that $S_G(T'_m) = \Gamma, A, B, \Delta$. Because $S_G(T_j) = S_G(T'_j)$ for $j \le i$, and $S_G(T_{i+1}) = S_G(A\wp B\downarrow)$ is well-defined, $S_G(T'_i) = S_G(A\downarrow) = B, \Delta$. Hence $S_G(T'_{i+1}) = S_G(A\uparrow) = A, B, \Delta$. Since $T'_j = T'_{j+2}$ $(i+1 \le j \le m)$, the claim holds. The rest of the proof follows from the induction hypothesis applied to G'.
(Tensor.) We may assume there is no par-link in Σ, whose C-edge is a terminal one. By Splitting Lemma ([4]), we moreover may assume the tensor-link $A\otimes B$ is added last. By the stack condition of G, and Proposition 5.3, we assume a special trip $T_1, \cdots, T_n$ on G starting $T_1 = A\otimes B\downarrow$ satisfies $S_G(T_n) \equiv \Sigma$, where $A\otimes B$ is the rightmost formula. Let G_A and G_B be marked D-R graphs obtained from marked D-R graph G by removing the tensor-link between A and B, whose edges are connected to edge A, and are connected to edge B, respectively: Hence G_A and G_B are only connected at $A\otimes B$ in G. Because of the property of the special trip, $T_2 = A\otimes B\uparrow$, $T_3 = A\uparrow$; and there exist an integer $i < n$, and formulas D and C, such that each T_j $(3 \le j \le i)$ is a point in the subgraph G_A and $T_i = D\downarrow$, and $T_{i+1} = A\downarrow$, $T_{i+2} = B\uparrow$, $T_{i+3} = C\uparrow$, each T_j $(i+3 \le j \le n-1)$ is a point in the subgraph G_B and $T_n = B\downarrow$. Therefore there exist Γ and Δ such that $\Sigma \equiv \Delta, \Gamma, A\otimes B$, where Γ are the terminal edges in G_A and Δ are the terminal edges in G_B. Moreover,

the part of the special trip $A\downarrow, T_3, \cdots, T_i$ gives a special trip on a marked D-R graph G_A satisfying $S_{G_A}(T_i) \equiv \Gamma, A$, and the part of the special trip $B\downarrow, T_{i+2}, \cdots, T_{n-1}$ gives a special trip on a marked D-R graph G_B satisfying $S_{G_B}(T_{n-1}) \equiv \Delta, B$. Thus both graphs G_A and G_B satisfy the stack condition. The rest of the proof follows from the induction hypothesis applied to G_A and G_B.
(Cut.) Similar to the case of tensor. □

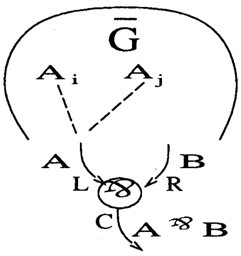
6 Strong Planity Implies Stack Condition.

In order to establish the equivalence between the non-commutative proof nets and the three characterizations, we finally prove that the strong planity implies the stack condition.

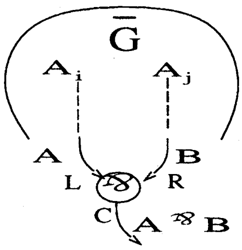
Definition 6.1 *An edge A is said to be unilaterally connected to an edge B, if there is a directed path from the edge A to the edge B.*

Lemma 6.2 *Assume that a strongly planar D-R graph G with $A_1, \cdots, A_n$, satisfies the switching condition. If $1 \leq i < j \leq n$, then in a closure $\bar{G}$ of G, there exists a par-link such that the edge A_i is unilaterally connected to its L-edge and the edge A_j is unilaterally connected to its R-edge.*

Proof. By Proposition 3.8, we may assume that a clockwisely directed plane graph drawing $\bar{G}$ with a single terminal edge, being an associative par instance of $A_1, \cdots, A_n$. We prove the lemma by induction on the number of formulas in the associative par instance, whose in-edges, the edges A_i and A_j are unilaterally connected to. Assume that the edges A_i and A_j are unilaterally connected to $A \wp B$. By removing the par-link connecting the edges A and B as the L-edge and the R-edge, we have 2 cases: (I) Assume that the both edges A_i and A_j are unilaterally connected to a single edge A or B. Each A and B is an associative par instance of a proper subsequence of $A_1, \cdots, A_n$. By the induction hypothesis, the claim holds. (II) Otherwise, the par-link we just removed is the one for the lemma.



(I)

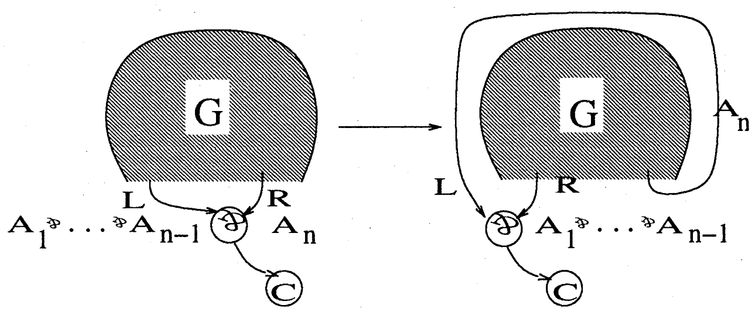


(II)

□

Lemma 6.3 *If a D-R graph with a sequence $\Sigma = A_1, \cdots, A_n$ is strongly planar, so is a D-R graph with a shift of Σ.*

Proof. It suffices to construct a clockwisely directed graph drawing with $A_n \wp (A_1 \wp \cdots \wp A_{n-1})$ as a single terminal edge. By Proposition 3.8, there exists a clockwisely directed plane graph with $(A_1 \wp \cdots \wp A_{n-1}), A_n$. Now we connect by a par-link the edges $A_1 \wp \cdots \wp A_{n-1}$ and A_n, with R-edge and L-edge, respectively. Since the graph drawing is a finite figure, we obtain a clockwisely directed plane graph drawing with the single terminal edge $A_n \wp (A_1 \wp \cdots \wp A_{n-1})$.



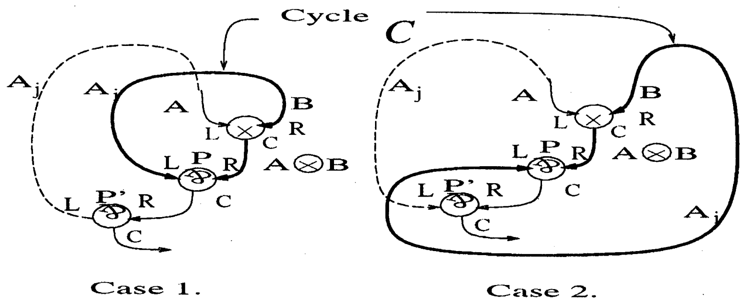
In the lemma below,a *splitting formula* in a D-R graph G satisfying the switching condition is a formula $A \otimes B$ found in Splitting Lemma ([4]); such that the removal of the tensor-link between A and B splits D-R graph G into two separate D-R graphs G_A and G_B, whose edges are connected to edges A and B respectively, both satisfying the switching condition.

Lemma 6.4 *Assume a strongly planar D-R graph G with $A_1, \cdots, A_n, A \otimes B$ satisfies the switching condition, and that $A \otimes B$ is a splitting formula, and that D-R graphs G_A and G_B are defined as above. Let A_i be the edge which belongs to graph G_B; and assume that any edge $A_j (i < j)$ belongs to G_A. Then any edge $A_j (j \leq i)$ belongs G_B.*

We note that Splitting Lemma only guarantees the switching condition of D-R graphs G_A and G_B: Their strong planity will be shown in Lemma 6.5.
Proof. By Proposition 3.8, there exists a clockwisely directed plane graph drawing $\bar{G}$, with single terminal node labeled an associative par instance of $A_1, \cdots, A_n, A \otimes B$.

By Lemma 6.2, in the graph drawing $\bar{G}$, there exists a par-link P such that the edge A_i is unilaterally connected to its L-edge and edge $A \otimes B$ is unilaterally connected to its R-edge. On the other hand, by the assumption, edge A_i and edge B are connected in G. Hence there must be a cycle C connecting B, $A \otimes B$ and A_i in the graph $\bar{G}$. Then again by Lemma 6.2, for any edge $A_j (i < j)$, there exists a par-link P' such that edge A_j is unilaterally connected to its L-edge and edge A_i is unilaterally connected to its R-edge.

Hence edge A_j and edge A are not located on the same side of the cycle C: If A is inside of the cycle, then the C-edge of P is not, and vice versa. Therefore edges A_j and A are only connected via a tensor-link between A and B in plane graph drawing $\bar{G}$. Thus edge A_j belongs to the graph G_B.



How edges are located in Lemma 6.4.

□

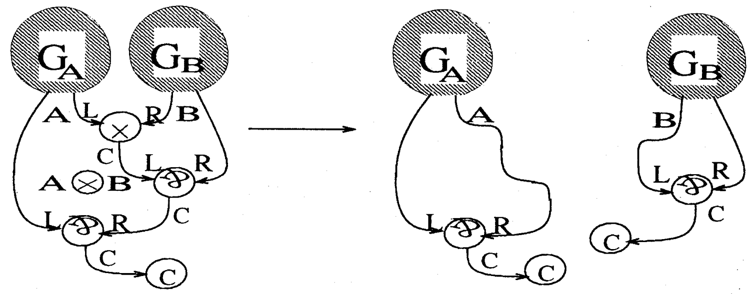
Let us make two remarks on the figures above. The figure in Case 1 represents the case when the single conclusion of a closure of the graph G is located outside of the cycle C; and the one in Case 2 represents the case otherwise.

We draw the figures in both cases that edge A_i is connected by the link P first, and then by the link P'. However the order of the connection is inessential by Proposition 3.8. Thus the same argument above holds in a picture where the edge A_i is connected by the link P' first, and then by the link P.

In what follows, when we can uniformly argue in a proof independently of the location of the conclusion nodes of a graph, we simply use a canonical figure where the conclusion nodes are spread outside of the graph.

Lemma 6.5 *Assume that a strongly planar D-R graph G with Σ satisfies the switching condition, and that $A \otimes B$ is a splitting formula. Let G_A and G_B be D-R graphs obtained from G by removing the tensor-link between A and B, whose edges are connected to edge A, and are connected to edge B, respectively. Then there are sequences Γ and Δ of terminal edges in G, such that (1) the edges in Γ belongs to G_A and the edges in Δ belongs to G_B, (2) D-R graphs G_A with Γ, A and G_B with B, Δ are strongly planar. (3) $\Gamma, A \otimes B, \Delta$ is a shift of Σ.*

Proof. We apply Lemma 6.4 to $\Sigma, A \otimes B$, and obtain Δ and Γ. A plane drawing of the graph G contains a subdivision of a plane drawing of each graph G_A and G_B. An example of the splitting can be shown in the following figure.



□

If the splitting formula is a cut one, then we have the following.

Lemma 6.6 *Assume a strongly planar D-R graph G with $D, A_1, \cdots, A_n$ satisfies the switching condition, and that $\perp$ in G is a splitting formula, and that G_A and $G_{A^\perp}$ be a graph obtained from G by removing the cut-link between A and $A^\perp$, whose edges are connected to edge A and are connected to edge $A^\perp$, respectively. Let A_i be the edge which belongs to graph G_A; and assume that any edge $A_j(i < j)$ belongs to $G_{A^\perp}$. Then any edge $A_j(j \leq i)$ belongs G_A.*

Proof. Again, we consider a clockwisely directed plane graph drawing $\bar{G}$, with a single terminal edge labeled by an associative par instance of $D, A_1, \cdots, A_n$. Let a formula A_{j_1} be the edge connected to edge $A^\perp$,such that any edge $A_j(j < j1)$ belongs to G_A. By Lemma 6.2, for any edge $A_j(j < i)$, in the graph drawing $\bar{G}$, there exists a par-link P such that the edge D is unilaterally connected to its L-edge and the edge A_j is unilaterally connected to its R-edge. Again by Lemma 6.2, there exists a par-link P' such that the edge A_j is unilaterally connected to its L-edge and the edge A_i is unilaterally connected to its R-edge. Finally there exists a par-link P'' such that the edge A_i is unilaterally connected to its L-edge and the edge A_{j_1} is unilaterally connected to its R-edge by Lemma 6.2. Hence there must be a cycle C' connecting D, A and A_i. Clearly the edges A_j and $A^\perp$ are not located on the same side of the cycle C'. The rest of the proof goes similarly to Lemma 6.4.

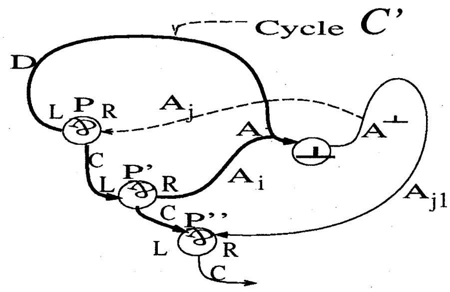
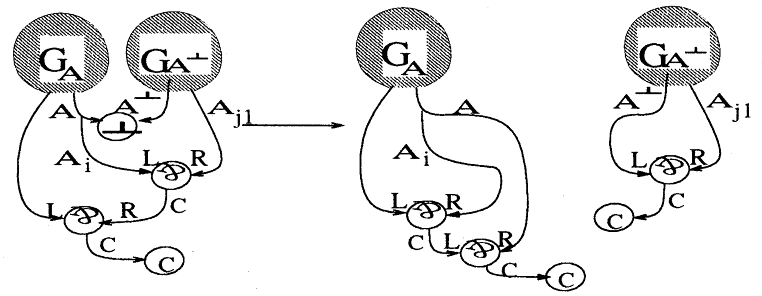


Fig. 6. How edges are located in Lemma 6.6.

□

Lemma 6.7 *Assume that a strongly planar D-R graph G with $A_1, \cdots, A_n$ satisfies the switching condition, and that $\perp$ in G be a splitting formula. Let G_A and $G_{A^\perp}$ be D-R graphs obtained from G by removing the cut-link between A and $A^\perp$, whose edges are connected to edge A, and are connected to edge $A^\perp$, respectively. Then there are sequences Γ and Δ of terminal edges in G, such that (1) the edges in Γ belong to G_A and the edges in Δ belong to $G_A^\perp$, (2) D-R graphs G_A with Γ, A and $G_A^\perp$ with $A^\perp, \Delta$ are strongly planar. (3) Γ, Δ is a shift of $A_1, \cdots, A_n$.*

Proof. We apply Lemma 6.6 to $A_1, \cdots, A_n$. Let A_i be the edge belonging to G_A, such that any edge $A_j (i < j)$ belongs to $G_{A^\perp}$. Let A_{j_1} be the edge belonging to $G_{A^\perp}$, such that any edge $A_j (j < j1)$ belongs to G_A. Because of the strongly planity of the graph G, the terminal edge and the cut-link must be on the same side of the cycle through edges A and A_i. Hence the edge A and the terminal edge can be connected without crossing the graph G_A itself. Let Γ be $A_i, \cdots, A_n, A_1, \cdots, A_{j_1-1}$, and Δ be $A_{j_1-1}, \cdots, A_{i-1}$.

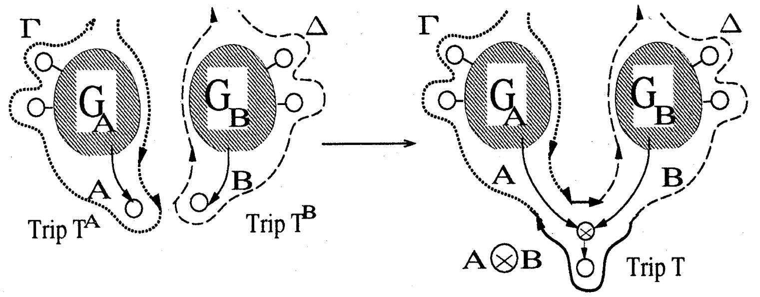


□

Theorem 6.8 *Assume that a D-R graph G satisfies the switching condition. If G is strongly planar with Σ, then G with Σ satisfies the stack condition.*

Proof. We prove by induction on the inductive structure of the proof net.
(Axiom.) Clear.
Assume the graph G is a non-commutative proof net with terminal edges Σ. *(Par.)* Let Σ be $\Gamma, A\wp B, \Delta$. Then graph G is strongly planar with $\Gamma, A\wp B, \Delta$, iff a graph G', obtained by removing the par-link between A and B, is strongly planar with Γ, A, B, Δ by the definition of strong planity. By induction hypothesis, the graph G' with Γ, A, B, Δ satisfies the stack condition. Now we show the stack condition on G follows. Let C be the rightmost formula in Δ. By the stack condition of G', a special trip $T'_1, \cdots, T'_m$ on G' with $T'_1 = C\downarrow$ satisfies $S_{G'}(T'_m) = \Gamma, A, B, \Delta$. We construct a special trip $T_1, \cdots, T_n$ on G such that $S_G(T_n) = \Gamma, A\wp B, \Delta$. We follow the same trip up to $A\downarrow$; let $T'_i = A\downarrow$.

We define $T_j = T'_j$ $(j \leq i)$: Then $A\wp B \downarrow$ follows; $T_{i+1} = A\wp B \downarrow$: Then $A\wp B \uparrow$ follows; $T_{i+2} = A\wp B \uparrow$: Finally the trip is continued with $A \uparrow$ and the rest of the trip is the same as on G'; Thus we define $T_j = T'_{j-2}$ $(i+3 \leq j \leq n)$. Now we show that $S_G(T_n) = \Gamma, A\wp B, \Delta$. Because $S_G(T_j) = S_G(T'_j)$ for $j \leq i$, $S_G(T_i) = S_G(T'_i) = B, \Delta$. Since $T_{i+1} = A\wp B \downarrow$, $S_G(T_{i+1}) = \Delta$. And then $S_G(T_{i+2}) = A\wp B, \Delta$, because $T_{i+2} = A\wp B \uparrow$. Since $T_j = T'_{j-2}$ $(i+3 \leq j \leq n)$, the claim holds.
(Tensor.) We may assume there is no par-link in Σ, whose C-edge is a terminal one. By Splitting Lemma ([4]), we moreover may assume the tensor-link $A\otimes B$ is added last. By removing the tensor-link, Lemmas 6.3 and 6.5 imply that we obtain strongly planar graphs G_A with Γ, A and G_B with B, Δ, respectively, where $A\otimes B, \Delta, \Gamma$ is a shift of Σ. By induction hypothesis, both graphs G_A with Γ, A and G_B with B, Δ satisfy the stack condition. Let $T_1^A, \cdots, T_{n_1}^A$ be a special trip on G_A starting from $T_1^A = A \downarrow$, $T_2^A = A \uparrow$ and $S_{G_A}(T_{n_1}^A) = \Gamma, A$. Similarly let $T_1^B, \cdots, T_{n_2}^B$ be a special trip on G_B starting from $T_1^B = B \downarrow$, $T_2^B = B \uparrow$ and $S_{G_B}(T_{n_2}^B) = B, \Delta$. We define a special trip $T_1, \cdots, T_n$ on G as: $A\otimes B \uparrow$, followed by $T_2^A, \cdots, T_{n_1}^A$, $A \downarrow$, $T_2^B, \cdots, T_{n_2}^B, B \downarrow$, $A\otimes B \downarrow$. We can easily show that $S_G(T_n) = A\otimes B, \Delta, \Gamma$. Proposition 5.3 implies the stack condition on G with Σ.



(Cut.) Similar to the case of tensor, but use Lemmas 6.3 and 6.7. □

Theorem 6.9 *(Characterization theorem with respect to the marked D-R graph for MN-CLL) A marked D-R graph is a non-commutative proof net iff it satisfies the switching condition and (1) it is strongly planar, or (2) it satisfies the long trip condition, or (3) it satisfies the stack condition.*

Proof. By Theorems 3.9, 4.14, 5.5 and 6.8. □

Let us explain the relationship between the strong planity of the marked D-R graphs and the planity of the D-R graphs: Clearly from the definition, a strongly planar marked D-R graph is a planar D-R graph by simply forgetting the labels L and R in the graph. As for the reverse direction, a planar D-R graph in general has more than one plane graph drawings; and distinct plane D-R graph drawings correspond to distinct non-commutative proof nets, respectively. Thus the next characterization theorem on D-R graph drawing cannot be improved to a theorem on planar D-R graphs. Note that we can define a *closure* of a D-R graph G by means of (unmarked) par-links as in Definition 3.3.

Theorem 6.10 *(Characterization theorem with respect to the D-R graph drawing for MNCLL) A plane D-R graph drawing with one terminal edge, satisfying the switching condition, represents a unique non-commutative proof net modulo the mirror images.*

Proof. We can uniformly label any plane D-R graph drawing by L, R and C clockwisely or label it counter-clockwisely: Or we can label the plane graph drawing clockwisely or label its mirror image clockwisely also: Both ways yield the same two non-commutative proof nets from the plane graph drawing. □

Theorem 6.11 *(Characterization theorem with respect to the D-R graph drawing for Abruci's MNLL) A plane D-R graph drawing with one terminal edge, satisfying the switching condition, represents a unique proof net for MNLL of Abruci [2].*

Proof. By uniformly labeling any plane D-R graph drawing as in the previous theorem, we can make it a non-commutative proof net. Now we consider its canonical drawing such as in Fig. 1, where we can talk about left and right: On each axiom-link in the non-commutative proof net, we put formulas A on the left edge and $A^{\perp}$ on the right edge. This make the plane graph drawing into a proof net for MNLL. □

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