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p 進単位球体の中の凸体

(Convex sets in the p-adic open ball)

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序　文

複素単位球体 $B_n := \{(z_1, \cdots, z_r) \in \mathbf{C}^r ; |z_1|^2 + \cdots + |z_r|^2 < 1\}$ は有界対称領域の代表的なもので、ユニタリ変換群に似た線形群 $U(1,r)$ が推移的に作用している。この作用を自明な部分で割った $U(1,r)/U(1)$ が B_n の自己同型群であることは古典的によく知られている。

この複素単位球体の p 進解析における類似として栗原 [K] および Mustafin [M] により p 進単位球体 $\mathcal{P}(\Delta)$ が p 進整数環 $\mathbf{Z}_p$ 上の形式スキームとして構成された。この形式スキームの構造は $(\mathbf{Q}_p)^{r+1}$ の中の階数 $r+1$ の部分自由 $\mathbf{Z}_p$ 加群の全体から作られた Bruhat-Tits 複体 Δ と深く結び付いている。Mumford が $p=r=2$ の場合の $\mathcal{P}(\Delta)$ を用いて、射影平面と同じベッチ数を持つ一般型の代数曲面、すなわち擬射影平面を構成したことはよく知られている。(cf.[Mum])

$\mathcal{P}(\Delta)$ には線形群 $\mathrm{PGL}(r+1, \mathbf{Q}_p)$ が効果的に作用している。Mustafin の論文には、この群が $\mathcal{P}(\Delta)$ の自己同型群であることが書かれているが、証明があまりに簡

略的なため確認しにくくなっている。最近、京都大学の加藤文元と私の擬射影平面に関する共同研究の中で、この Mustafin の結果 [M, Prop.4.2] を使う必要が生じたので、念のために、その証明をここに書くことにする。

この証明のために、形式スキーム $\mathcal{P}(\Delta)$ の再構成を行った。基本的には Mustafin のものと同じであるが、Bruhat-Tits 複体の頂点集合Δ_0 の部分集合についてその凸性を定義して、Δ_0 を有限凸集合の増大列の和にすることにより、$\mathcal{P}(\Delta)$ の元になるスキーム $\mathcal{X}(\Delta)$ を $\mathbf{Z}_p$ 上の射影空間 $\mathbf{P}^r_{\mathbf{Z}_p}$ から始まる blowing-up の列の極限として構成した。これはある意味で p 進単位球体をコンパクトな凸部分集合の増大列の極限として表したことになる。このことは $\mathcal{P}(\Delta)$ の自己同型が線形変換であることを漸近的に示すために有効である。

第 1 節の射影空間の線形部分空間による blowing-up についての結果は、後の節の補助となるものであるが、単独でも結構おもしろいと思う。

1 Blowing-ups of a projective space

In this section, we fix a field k of an arbitrary characteristic.

For a projective space P over k, we denote by $\Sigma(P)$ the set of proper k-linear subspaces of P. A finite subset $\Phi \subset \Sigma(P)$ is said to be *intersection closed* if $P_\alpha, P_\beta \in \Phi$ implies $P_\alpha \cap P_\beta = \emptyset$ or $P_\alpha \cap P_\beta \in \Phi$.

Let Φ be an intersection closed finite subset of $\Sigma(P)$. We define a modification $p : \sigma(\Phi)P \to P$ inductively as follows.

For each integer $0 \leq i < r := \dim P$, let $\Phi_i := \{P_\alpha \in \Phi \ ; \ \dim P_\alpha \leq i\}$. We define $\sigma(\Phi_0)P$ to be the blowing-up of P at the points in Φ_0. For an integer $0 \leq d \leq r-2$, assume that $\sigma(\Phi_d)P$ is already defined. Then $\sigma(\Phi_{d+1})P$ is defined to be the blowing-up of $\sigma(\Phi_d)P$ at the union of proper transforms of $P_\alpha \in \Phi_{d+1} \setminus \Phi_d$.

Thus we get $\sigma(\Phi)P = \sigma(\Phi_{r-1})P$. By the construction, the centers of the blowing-ups are always nonsingular. Hence $\sigma(\Phi_d)P$ is also nonsingular for all d. In particular, $\sigma(\Phi)P = \sigma(\Phi_{r-1})P \to \sigma(\Phi_{r-2})P$ is an isomorphism.

Let $|\Phi|$ be the union of the linear subspaces of P which belong to Φ. We are going to describe the divisor $p^{-1}(|\Phi|) \subset \sigma(\Phi)P$. For the convenience of notation, we denote the elements of Φ with indexes as P_λ while we set $P_1 := P$. For distinct elements $P_\alpha, P_\beta \in \Phi \cup \{P_1\}$ with $P_\alpha \subset P_\beta$, we denote by $P_{\beta/\alpha}$ the projective space parametrizing linear subspaces $P' \subset P_\beta$ with $P_\alpha \subset P'$ and $\dim P' = \dim P_\alpha + 1$. There exists a natural projection $P_\beta \setminus P_\alpha \to P_{\beta/\alpha}$. This is an $\mathbf{A}^d$-bundle for $d := \dim P_\alpha + 1$. In particular, we get a natural morphism $P \setminus |\Phi| \to \prod_{P_\lambda \in \Phi} P_{1/\lambda}$.

Proposition 1.1 *The scheme $\sigma(\Phi)P$ is naturally isomorphic to the closure of the image of the morphism*

$$P \setminus |\Phi| \longrightarrow P \times \big(\prod_{P_\lambda \in \Phi} P_{1/\lambda} \big) \,.$$

Proof. This is well-known, if Φ consists of a single element. In this proof, we denote by Γ_Φ the closure of the image. For each $P_\lambda \in \Phi$, let I_λ be the sheaf of ideals defining $P_\lambda \subset P$. Since $\Gamma_{\{P_\lambda\}}$ is the blowing-up of P at P_λ, a morphism of varieties $f : X \to P$ factors $\Gamma_{\{P_\lambda\}}$ if $f^{-1}I_\lambda \subset \mathcal{O}_X$ is invertible [H, II,Prop.7.14]. In the construction of $\sigma(\Phi)P$, the connected components of the centers of the blowing-ups $\sigma(\Phi_{d'+1})P \to \sigma(\Phi_{d'})P$ are inside or completely outside of the proper transform of P_λ in $\sigma(\Phi_{d'})P$ for $d' < d := \dim P_\lambda$. Hence the inverse image of I_λ becomes invertible after by the blowing-up $\sigma(\Phi_{d+1})P \to \sigma(\Phi_d)P$ (cf. Lemma 2.5). In particular, the morphism $p : \sigma(\Phi)P \to P$ factors $\Gamma_{\{P_\lambda\}}$ for all $P_\lambda \in \Phi$. Hence we get a morphism $a_\Phi : \sigma(\Phi)P \to \Gamma_\Phi$.

We prove that a_Φ is an isomorphism by induction on the number of elements in Φ. The assetion is trivially true for $\Phi = \emptyset$. Assume $\Phi \neq \emptyset$ and let $P_\alpha \in \Phi$ be an element of maximal dimension. Since P_α is not the intersection of other elements of Φ, $\Phi' := \Phi \setminus \{P_\alpha\}$ is also intersection closed. By the assumption of the induction, $a_{\Phi'} : \sigma(\Phi')P \to \Gamma_{\Phi'}$ is an isomorphism. It is sufficient to show that the morphism $\Gamma_\Phi \to \Gamma_{\Phi'} \simeq \sigma(\Phi')P$ factors $\sigma(\Phi)P$. Let Y be the proper transform of P_α in $\sigma(\Phi')P$. Then the inverse image of I_α by $\sigma(\Phi')P \to P$ is the ideal of the union of Y and the exceptional divisors corresponding to $P_\beta \in \Phi'$ with $P_\beta \subset P_\alpha$. Since $\Gamma_\Phi \to P$ factors $\Gamma_{\{P_\alpha\}}$, the inverse image of I_α in Γ_Φ is invertible. Hence that of the ideal of Y in Γ_Φ is also invertible. By the universal property of blowing-up, the birational map $\Gamma_\Phi \to \sigma(\Phi)P$ is regular, since $\sigma(\Phi)P$ is the blowing-up of $\sigma(\Phi')P$ at Y. *q.e.d.*

For each $P_\alpha \in \Phi$, we denote by D_α the proper transform of P_α in $\sigma(\Phi)P$, i.e., the closure of the inverse image of the generic point of P_α in $\sigma(\Phi)P$. Then D_α is a nonsingular prime divisor by the construction of $\sigma(\Phi)P$. Furthermore, since the centers of the blowing-ups are always transversal with the exceptional divisors, we get the following proposition.

Proposition 1.2 *The reduced subscheme $p^{-1}(|\Phi|)$ of $\sigma(\Phi)P$ is a simple normal crossing divisor with the set of irreducible components $\{D_\alpha\ ;\ P_\alpha \in \Phi\}$.*

Note that, if P_α is of codimension one, then D_α is birational to P_α, and hence is not an exceptional divisor of the morphism $\sigma(\Phi)P \to P$.

Let P_α be an element of Φ. We set

$$\Phi^\alpha := \{P_\lambda \in \Phi\ ;\ P_\lambda \subset P_\alpha, P_\lambda \neq P_\alpha\}\ .$$

This is an intersection closed subset of $\Sigma(P_\alpha)$. On the other hand, we set

$$\Phi(\alpha) := \{P_\lambda \in \Phi \,;\, P_\alpha \subset P_\lambda\}$$

and

$$\Phi_\alpha := \{P_{\lambda/\alpha} \in \Phi \,;\, P_\lambda \in \Phi(\alpha) \setminus \{P_\alpha\}\}\,.$$

Then Φ_α is an intersection closed subset of $\Sigma(P_{1/\alpha})$.

Proposition 1.3 *The prime divisor D_α is naturally isomorphic to $\sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi_\alpha)P_{1/\alpha}$.*

Proof. Consider the projections

$$p_1 : \sigma(\Phi)P \longrightarrow P \times (\prod_{P_\lambda \in \Phi\setminus\Phi(\alpha)} P_{1/\lambda})$$

and

$$p_2 : \sigma(\Phi)P \longrightarrow \prod_{P_\lambda \in \Phi(\alpha)} P_{1/\lambda}\,.$$

The image of the first projection is equal to $\sigma(\Phi \setminus \Phi(\alpha))P$. The proper transform of P_α in $\sigma(\Phi \setminus \Phi(\alpha))P$ is equal to $\sigma(\Phi^\alpha)P_\alpha$. Hence we have $p_1(D_\alpha) = \sigma(\Phi^\alpha)P_\alpha$. On the other hand, since $P_{1/\lambda} = P_{(1/\alpha)/(\lambda/\alpha)}$ for $P_\lambda \in \Phi(\alpha) \setminus \{P_\alpha\}$, the image of the second projection is equal to $\sigma(\Phi_\alpha)P_{1/\alpha}$. Hence D_α is contained in the product $\sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi_\alpha)P_{1/\alpha}$. These are equal since both of them are irreducible of dimension $\dim P-1$.

q.e.d.

When a point $x \in \sigma(\Phi)P$ is contained in D_α, we set $x^\alpha := p_1(x)$ and $x_\alpha := p_2(x)$, and we write $x = (x^\alpha, x_\alpha)$.

Lemma 1.4 *Assume that $P_\alpha, P_\beta \in \Phi$ satisfy neither $P_\alpha \subset P_\beta$ nor $P_\beta \subset P_\alpha$. Then $D_\alpha \cap D_\beta = \emptyset$.*

Proof. Let $P_\gamma := P_\alpha \cap P_\beta$. By the projection of $\sigma(\Phi)P$ to $P_{1/\gamma}$, the divisors D_α and D_β are mapped to disjoint subspaces $P_{\alpha/\gamma}$ and $P_{\beta/\gamma}$, respectively. *q.e.d.*

Let P_α, P_β be distinct elements of Φ with $P_\alpha \subset P_\beta$. Recall that Φ^β is an intersection closed subset of $\Sigma(P_\beta)$. We set $\Phi^\beta_\alpha := (\Phi^\beta)_\alpha$. This is an intersection closed subset of $\Sigma(P_{\beta/\alpha})$.

Proposition 1.5 *For $P_\alpha, P_\beta \in \Phi$ with $P_\alpha \subset P_\beta$, the intersection $D_\alpha \cap D_\beta$ is naturally isomorphic to*

$$\sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi^\beta_\alpha)P_{\beta/\alpha} \times \sigma(\Phi_\beta)P_{1/\beta}$$

Proof. We denote the product variety by Z in this proof. By Proposition 1.3, the proper transform of $P_\alpha \subset P_\beta$ in $\sigma(\Phi^\beta)P_\beta$ is $\sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi^\beta_\alpha)P_{\beta/\alpha}$, while that of $P_{\beta/\alpha} \subset P_{1/\alpha}$ in $\sigma(\Phi_\alpha)P_{1/\alpha}$ is $\sigma(\Phi^\beta_\alpha)P_{\beta/\alpha} \times \sigma(\Phi_\beta)P_{1/\beta}$. Hence we have natural inclusion maps

$$\phi_1 : Z \to D_\beta \simeq \sigma(\Phi^\beta)P_\beta \times \sigma(\Phi_\beta)P_{1/\beta} \ .$$

and

$$\phi_2 : Z \to D_\alpha \simeq \sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi_\alpha)P_{1/\alpha}$$

We can check that Z is equally embedded in $P \times (\prod_{P_\lambda \in \Phi} P_{1/\lambda})$ by these inclusion maps. Actually, the composites of both ϕ_i with the projections from $\sigma(\Phi)P$ to

$$P \times (\prod_{P_\lambda \in \Phi \setminus \Phi(\alpha)} P_{1/\lambda}) \ , \quad \prod_{P_\lambda \in \Phi(\alpha) \setminus \Phi(\beta)} P_{1/\lambda} \text{ and } \prod_{P_\lambda \in \Phi(\beta)} P_{1/\lambda}$$

are equal to the composites of the projections from Z to the three components and the canonical embeddings to the above three varieties, respectively. Hence Z is a subvariety of $\sigma(\Phi)P$ contained in $D_\alpha \cap D_\beta$.

If a point $x = (x^\beta, x_\beta) \in D_\beta$ is containd in D_α, then the projection of x^β in $\sigma(\Phi^\alpha)P$ is in $\sigma(\Phi^\alpha)P_\alpha$. Hence $x^\beta \in \sigma(\Phi^\alpha)P_\alpha \times \sigma(\Phi^\beta_\alpha)P_{\beta/\alpha}$. This implies $x \in Z$. Hence $Z = D_\alpha \cap D_\beta$. *q.e.d.*

More generally, we get the following theorem.

Theorem 1.6 *Let $P_{\alpha_1}, P_{\alpha_2}, \cdots, P_{\alpha_d}$ be distinct elements of Φ with*

$$P_{\alpha_1} \subset P_{\alpha_2} \subset \cdots \subset P_{\alpha_d}\,.$$

Then $D_{\alpha_1} \cap \cdots \cap D_{\alpha_d}$ is naturally isomorphic to

$$\sigma(\Phi^{\alpha_1})P_{\alpha_1} \times \sigma(\Phi^{\alpha_2}_{\alpha_1})P_{\alpha_2/\alpha_1} \times \cdots \times \sigma(\Phi^{\alpha_d}_{\alpha_{d-1}})P_{\alpha_d/\alpha_{d-1}} \times \sigma(\Phi_{\alpha_d})P_{1/\alpha_d}\,.$$

Proof. We prove the theorem by induction on $\dim P$. Note that $d \leq \dim P$ and the theorem is true for $\dim P \leq 2$. For $1 \leq i \leq d-1$, $D_{\alpha_i} \cap D_{\alpha_d}$ is equal to

$$\sigma(\Phi^{\alpha_i})P_{\alpha_i} \times \sigma(\Phi^{\alpha_d}_{\alpha_i})P_{\alpha_d/\alpha_i} \times \sigma(\Phi_{\alpha_d})P_{1/\alpha_d}$$

by Proposition 1.5. Let D'_{α_i} be the proper transform of P_{α_i} in $\sigma(\Phi^{\alpha_d})P_{\alpha_d}$. Then we have

$$D_{\alpha_i} \cap D_{\alpha_d} = D'_{\alpha_i} \times \sigma(\Phi_{\alpha_d})P_{1/\alpha_d}$$

by Proposition 1.3. Hence

$$D_{\alpha_1} \cap \cdots \cap D_{\alpha_d} = (D'_{\alpha_1} \cap \cdots \cap D'_{\alpha_{d-1}}) \times \sigma(\Phi_{\alpha_d})P_{1/\alpha_d}\,.$$

We get the theorem, since

$$D'_{\alpha_1} \cap \cdots \cap D'_{\alpha_{d-1}} = \sigma(\Phi^{\alpha_1})P_{\alpha_1} \times \sigma(\Phi^{\alpha_2}_{\alpha_1})P_{\alpha_2/\alpha_1} \times \cdots \times \sigma(\Phi^{\alpha_d}_{\alpha_{d-1}})P_{\alpha_d/\alpha_{d-1}}$$

by the assumption of the induction. *q.e.d.*

Let x be a point of $\sigma(\Phi)P$. Then $F_x := \{P_\alpha \in \Phi : x \in D_\alpha\}$ forms a flag by Lemma 1.4. Let $F_x = (P_{\alpha_1} \subset \cdots \subset P_{\alpha_d})$ and

$$x = (x^{\alpha_1}, x^{\alpha_2}_{\alpha_1}, \cdots, x^{\alpha_d}_{\alpha_{d-1}}, x_{\alpha_d})$$

with respect to the product description of $D_{\alpha_1} \cap \cdots \cap D_{\alpha_d}$. Since x is not in the other D_α, we have

$$\begin{aligned} x^{\alpha_1} &\in P_{\alpha_1} \setminus |\Phi^{\alpha_1}| \\ x^{\alpha_{i+1}}_{\alpha_i} &\in P_{\alpha_{i+1}/\alpha_i} \setminus |\Phi^{\alpha_{i+1}}_{\alpha_i}| \quad \text{for } i = 1, \cdots, d-1 \text{ , and} \\ x_{\alpha_d} &\in P_{1/\alpha_d} \setminus |\Phi_{\alpha_d}| \; . \end{aligned}$$

Thus we get the following theorem.

Theorem 1.7 *The scheme $\sigma(\Phi)P$ has a stratification*

$$\sigma(\Phi)P = \coprod_F X_F$$

of locally closed subschemes where $F = (P_{\alpha_1} \subset \cdots \subset P_{\alpha_d})$ runs over all flags in Φ including the empty flag $()$. Each X_F consists of points x with $F_x = F$ and is naturally isomorphic to

$$(P_{\alpha_1} \setminus |\Phi^{\alpha_1}|) \times (P_{\alpha_2/\alpha_1} \setminus |\Phi^{\alpha_2}_{\alpha_1}|) \times \cdots \times (P_{\alpha_d/\alpha_{d-1}} \setminus |\Phi^{\alpha_d}_{\alpha_{d-1}}|) \times (P_{1/\alpha_d} \setminus |\Phi_{\alpha_d}|) \; ,$$

which is understood to be $P \setminus |\Phi|$ if F is the empty flag.

2 Mustafin's scheme

Let R be a complete discrete valuation ring with the finite residue field k. Let K be the quotient field of R. We fix a generator t of the maximal ideal of R.

Let r be a non-negative integer and V the K-vector space $KX_0 \oplus \cdots \oplus KX_r$ with the basis $\{X_0, \cdots, X_r\}$.

Since R is PID and $\dim_K V = r+1$, finitely generated R-submodules of V are free of rank at most $r+1$. Let $\tilde{\Delta}_0$ be the set of free R-submodules $M \subset V$ of the maxiaml rank.

We denote by Δ_0 the quotient of $\tilde{\Delta}_0$ by the equivalence relation defined by

$$M \sim M' \Leftrightarrow \exists a \in K^\times, M' = aM$$

for $M, M' \in \tilde{\Delta}_0$, where $K^\times := K \setminus \{0\}$. The class containing M is denoted by $[M]$. We denote by ρ_K the natural map $\tilde{\Delta}_0 \to \Delta_0$.

A subset $S \subset \Delta_0$ is defined to be a simplex if $\rho_K^{-1}(S)$ is totally ordered. It is easy to see that a simplex has at most $r+1$ elements and the set of the simplexes forms a simplicial complex. This complex is called the Bruhat-Tits complex.

For $\alpha \in \Delta_0$, we denote by $M_\alpha \in \tilde{\Delta}_0$ an element which represents α. The choice of M_α is not unique and depends on the case. In genaral, for a given subset $S \subset \Delta_0$, the choice of M_α is free for the first $\alpha \in S$. A choice of M_β for the other $\beta \in S$ is called maximal in M_α, if $M_\alpha \supset M_\beta$ and $tM_\alpha \not\supset M_\beta$.

For α, β in Δ_0, we denote by $[\alpha, \beta]_K$ the subset

$$\{[M_\alpha + aM_\beta] \ ; \ a \in K^\times\}$$

of Δ_0, and call it the interval with the ends α, β. This definition does not depend on the choice of the representatives M_α, M_β.

The number of elements of $[\alpha, \beta]_K$ is equal to $d(\alpha, \beta) + 1$, where $d(\alpha, \beta)$ is the nonnegative integer defined by

$$d(\alpha, \beta) := \min\{n \in \mathbf{Z} \ ; \ t^n M_\alpha \subset M_\beta\} - \max\{m \in \mathbf{Z} \ ; \ t^m M_\alpha \supset M_\beta\}$$

(cf.[M, §1]). If M_β was taken to be maximal in M_α, then

$$[\alpha,\beta]_K = \{[t^n M_\alpha + M_\beta] \; ; n = 0, \cdots, d(\alpha,\beta)\} .$$

A subset $S \subset \Delta_0$ is said to be convex if $[\alpha,\beta]_K \subset S$ for every pair (α,β) of elements of S.

It is easy to see that a subset $S \subset \Delta_0$ is convex if and only if $M, M' \in \rho_K^{-1}(S)$ implies $M + M' \in \rho_K^{-1}(S)$. In particular, the simplexes of the Bruhat-Tits complexes are convex. This definition of convexity in Δ_0 is not exactly equal to that of Mustafin [M, §1].

Lemma 2.1 *(1) Let α be an element of Δ_0. For a nonnegative integer N,*

$$\{\beta \in \Delta_0 \; ; d(\alpha,\beta) \leq N\}$$

is a finite convex subset of Δ_0.

(2) If S, S' are convex subsets of Δ_0, then so is $S \cap S'$.

(3) Any convex subset of Δ_0 is the union of an increasing family of finite convex subsets of Δ_0.

Proof. (1) and (2) are easy. (3) is a consequence of (1) and (2). *q.e.d.*

For each $\alpha \in \Delta_0$, we set

$$\mathbf{P}(\alpha) := \operatorname{Proj} \operatorname{Sym}_R(M_\alpha) ,$$

where $\operatorname{Sym}_R(M_\alpha)$ is the symmetric R-algebra of the free R-module M_α. If we take an R-basis $\{Y_0, \cdots, Y_r\}$ of M_α, then $\mathbf{P}(\alpha)$ is equal to the projective space $\mathbf{P}_R^r$ with the homogeneous coordinate $(Y_0 : \cdots : Y_r)$. This definition does not depend on the choice of M_α, since an equivalent R-module has the basis $\{t^c Y_0, \cdots, t^c Y_r\}$ for an integer c. The fiber $\mathbf{P}(\alpha)_0$ over the closed point of $\operatorname{Spec} R$ is an r-dimensional projective space over the finite field k.

For all $\alpha \in \Delta_0$, $\mathbf{P}(\alpha)$ has the generic fiber $\mathbf{P}^r_K = \operatorname{Proj} K[X_0, \cdots, X_r]$. In particular, the function field of $\mathbf{P}(\alpha)$ is always

$$K(\frac{X_1}{X_0}, \cdots, \frac{X_r}{X_0}) .$$

We treat many integral R-schemes with this function field. Two such schemes are identified if the canonical birational map is isomorphic.

Let α, β be elements of Δ_0 with $d(\alpha, \beta) = 1$. If M_β is maximal in M_α then $M_\alpha \supset M_\beta \supset tM_\alpha$. The vector subspace $M_\beta/tM_\alpha \subset M_\alpha/tM_\alpha$ define a linear subspace $P_{\beta/\alpha}$ of $\mathbf{P}(\alpha)_0$ whose codimension is equal to $\dim_k(M_\beta/tM_\alpha)$. However $P_{\beta/\alpha}$ defined here is not equal to $P_{\beta/\alpha}$ defined for projective spaces P_α, P_β with $P_\alpha \subset P_\beta$ in Section 1, we use this notation because there is a good compatibility in these definitions.

For a pair (α, β) of elements of Δ_0, we define the directed length $\operatorname{len}(\alpha, \beta)$ by

$$\operatorname{len}(\alpha, \beta) := \operatorname{length}(M_\alpha/M_\beta)$$

for M_α and M_β maximal in M_α. Namely, if

$$M_\alpha/M_\beta \simeq R/(t^{e_1}) \oplus \cdots \oplus R/(t^{e_r}) ,$$

then $\operatorname{len}(\alpha, \beta) = e_1 + \cdots + e_r$. Clearly, this is greater than or equal to $d(\alpha, \beta) = \max\{e_1, \cdots, e_r\}$. In particular,

$$\{\beta \in \Delta_0 \,;\, \operatorname{len}(\alpha, \beta) \leq N\}$$

is finite for any α and an integer N.

If $d(\alpha, \beta) = 1$, then the equalities $\operatorname{len}(\alpha, \beta) = \dim_k P_{\beta/\alpha} + 1$ and $\operatorname{len}(\alpha, \beta) + \operatorname{len}(\beta, \alpha) = r + 1$ hold.

An ordering $\{\alpha = \alpha_1, \alpha_2, \cdots\}$ of the elements of a convex subset S of Δ_0 is said to be *convex* if

$$S_d := \{\alpha_1, \alpha_2, \cdots, \alpha_{d-1}, \alpha_d\}$$

is convex for every $1 \leq d < {}^{\#}S$.

Lemma 2.2 *Let α be an arbitrary element of a convex subset S of Δ_0. Let $\{\alpha = \alpha_1, \alpha_2, \cdots\}$ be an ordering of the elements of S such that $i < j$ implies* $\mathrm{len}(\alpha, \alpha_i) \leq \mathrm{len}(\alpha, \alpha_j)$ *for any positive integers i, j. Then this ordering is convex.*

Proof. We take M_α and $\{M_\beta\ ;\ \beta \in S \setminus \{\alpha\}\}$ so that M_β is maximal in M_α for every β. For $i, j \leq d$, an element $\gamma \in [\alpha_i, \alpha_j]_K \subset S$ is represented by $M_{\alpha_i} + t^s M_{\alpha_j}$ for some $s \in \mathbf{Z}$. By exchanging i, j, if necessary, we may assume $s \geq 0$. Then, since $M_{\alpha_i} + t^s M_{\alpha_j} \subset M_\alpha$, $\mathrm{len}(\alpha, \gamma) < \mathrm{len}(\alpha, \alpha_i)$ if $\gamma \neq \alpha_i$. Hence $\gamma = \alpha_p$ is in S_d by the rule of the ordering. *q.e.d.*

Lemma 2.3 *Let S be a convex subset of Δ_0, T a finite convex subset of S with a convex ordering $\{\alpha_1, \cdots, \alpha_c\}$. Then, there exists a convex ordering $\{\alpha_1, \alpha_2, \cdots\}$ of the elements of S which is an extension of that of T.*

Proof. By Lemma 2.1, (3), it is sufficient to show the following.

In the situation of the lemma, assume further that S is finite and $S \setminus T$ is nonempty. Then there exsits $\delta \in S \setminus T$ such that $T \cup \{\delta\}$ is convex.

We take an element $\gamma \in S \setminus T$. Then $T \vee \{\gamma\} := \bigcup_{\alpha \in T} [\alpha, \gamma]_K$ is a convex subset of S. Let T' be a minimal convex subset of S which contains T as a proper subset. Let δ be an element of $T' \setminus T$. Then $T' = T \vee \{\delta\}$ by the minimality of T'. It is easy to see that $(T \vee \{\delta\}) \setminus \{\delta\}$ is convex. Hence $T' = T \cup \{\delta\}$, again by the minimality of T'. *q.e.d.*

For a finite subset S of Δ_0, we denote by $\bigvee_{\alpha \in S} \mathbf{P}(\alpha)$ the integral R-scheme obtained by taking the closure of the diagonal embedding

$$\operatorname{Proj} K[X_0, \cdots, X_r] \to \prod_{\alpha \in S} \mathbf{P}(\alpha)$$

to the R-scheme (cf.[M, §2]). When S is a convex finite subset of Δ_0, we denote $\bigvee_{\alpha \in S} \mathbf{P}(\alpha)$ simply by $\mathbf{P}(S)$.

Lemma 2.4 *Let α, β be elements of Δ_0 with $d(\alpha, \beta) = 1$. Then the blowing-up of $\mathbf{P}(\alpha)$ at $P_{\beta/\alpha}$ is equal to that of $\mathbf{P}(\beta)$ at $P_{\alpha/\beta}$. Furthermore, this R-scheme is equal to $\mathbf{P}(\{\alpha, \beta\})$.*

For the proof, see [M, Prop.2.1].

This lemma implies that there exists a projection map

$$\mathbf{P}(\alpha)_0 \setminus P_{\beta/\alpha} \longrightarrow P_{\alpha/\beta}.$$

For an element α of a convex set S, we set

$$\Phi(S, \alpha) := \{P_{\beta/\alpha}\ ;\ \beta \in S, d(\alpha, \beta) = 1\}\ .$$

Then the convexity of S implies that $\Phi(S, \alpha)$ is an intersection closed subset of $\Sigma(\mathbf{P}(\alpha)_0)$. We set

$$B(S, \alpha) := \sigma(\Phi(S, \alpha))\mathbf{P}(\alpha)_0$$

in the notation of Section 1.

If $\alpha, \beta \in \Delta_0$ and $d(\alpha, \beta) = 1$, then we set

$$\Phi(S)_\alpha^\beta := \{P_{\gamma/\alpha}\ ;\ \gamma \in S, M_\alpha \supset {}^\exists M_\gamma \supset M_\beta, \gamma \neq \alpha, \beta\}\ ,$$

where M_β is maximal in M_α. This is an intersection closed subset of $\Sigma(P_{\beta/\alpha})$. We also use the notation $P_{\alpha/\alpha} := \mathbf{P}(\alpha)_0$ and

$$\Phi(S)_\alpha^\alpha := \{P_{\gamma/\alpha}\ ; \gamma \in S, M_\alpha \supset {}^{\exists}M_\gamma \supset tM_\alpha, \gamma \neq \alpha\} .$$

The following lemma is checked easily (cf.[M, §2,Lem.]).

Lemma 2.5 *Let Y, Z be irreducible closed regular subschemes of a regular scheme X defined by the ideals I_Y and I_Z, respectively. For the blowing-up $p : X' \to X$ at Y, let Y' the exceptional divisor and Z' the proper transform of Z.*

(1) *If $Y \subset Z$, then $p^{-1}I_Z = I_{Z'} \otimes \mathcal{O}_{X'}(-Y')$, where $I_{Z'}$ is the ideal defining Z'.*

(2) *If $Y \cap Z$ is either empty or regular equidimensional of dimension* $\dim Y + \dim Z - \dim X$, *then $p^{-1}I_Z = I_{Z'}$.*

Theorem 2.6 *Let $S \neq \emptyset$ be a convex finite subset of Δ_0. Then* (1) *$\mathbf{P}(S)$ is a regular R-scheme.* (2) *The closed fiber $\mathbf{P}(S)_0$ is a reduced simple normal crossing divisor with the components*

$$\{B(S,\alpha)\ ; \alpha \in S\} .$$

(3) *For a subset $T \subset S$, the intersection*

$$\bigcap_{\alpha \in T} B(S,\alpha)$$

is nonempty if and only if T is a simplex of Δ_0. If $T = \{\alpha_0, \cdots, \alpha_d\}$ and

$$M_{\alpha_0} \supset \cdots \supset M_{\alpha_d} \supset tM_{\alpha_0} ,$$

the intersection is naturally isomorphic to

$$\sigma(\Phi(S)_{\alpha_0}^{\alpha_1})P_{\alpha_1/\alpha_0} \times \cdots \times \sigma(\Phi(S)_{\alpha_{d-1}}^{\alpha_d})P_{\alpha_d/\alpha_{d-1}} \times \sigma(\Phi(S)_{\alpha_d}^{\alpha_0})P_{\alpha_0/\alpha_d} .$$

(4) *Let* δ *be an element of* $\Delta_0 \setminus S$ *such that* $S' := S \cup \{\delta\}$ *is convex. Then* $\mathbf{P}(S')$ *is equal to the blowing-up of* $\mathbf{P}(S)$ *at* $\sigma(\Phi(S)^{\delta}_{\alpha})P_{\delta/\alpha} \subset B(S, \alpha)$, *where* $\alpha \in S$ *is the element such that* $\mathrm{len}(\delta, \alpha)$ *is minimal.*

Proof. Note that $d(\alpha, \delta) = 1$ in (4). In fact, if we take M_α maximal in M_δ, then $M_\alpha + tM_\delta$ represents an element of S by the convexity. Then $M_\alpha \supset tM_\delta$ by the minimalty of $\mathrm{len}(\delta, \alpha)$.

We prove the theorem by induction on the number N of the elements of S. If $S = \{\alpha\}$, then $B(S, \alpha) = \mathbf{P}(\alpha)_0$. Hence (1), (2), (3) are trivially true, while (4) is a consequence of Lemma 2.4.

Assume that $N > 1$ and the assertion is true if we replace S by its proper convex subset. Let $\{\alpha_1, \cdots, \alpha_N\}$ be a convex ordering of the elements of S. Then, by the assumption of the induction, $\mathbf{P}(S)$ is a succession of blowing-ups at nonsingular centers starting from $\mathbf{P}(\alpha_1)$. In particular, we have (1). Since the each center is contained in a component of the divisor, and transversely intersects other components, the union of the proper transform of $\mathbf{P}(\alpha_1)_0$ and the exceptional divisors is a simple normal crossing divisor.

We show that the proper transform of $\mathbf{P}(\alpha)_0$ in $\mathbf{P}(S)$ is isomorphic to $B(S, \alpha)$ for each $\alpha \in S$. By Lemma 2.3, we can take the convex ordering $\{\alpha_1, \cdots, \alpha_N\}$ so that $\alpha_1 = \alpha$,

$$\{\beta \in S \,;\, d(\alpha, \beta) = 1\} = \{\alpha_2, \cdots, \alpha_c\}$$

and

$$\{\gamma \in S \,;\, d(\alpha, \gamma) > 1\} = \{\alpha_{c+1}, \cdots, \alpha_N\}$$

for an integer c. Furthermore, we may assume that the ordering of S_c is defined by the directed lengths as in Lemma 2.2. Then, by the process of the blowing-ups, the proper transform of $\mathbf{P}(\alpha)_0$ in $\mathbf{P}(S_c)$ is equal to $B(S,\alpha) = \sigma(\Phi(S,\alpha))\mathbf{P}(\alpha)_0$ in view of the construction of $\sigma(\Phi)P$ in Section 1. It is of multiplicity one, since so is $\mathbf{P}(\alpha)_0$. The centers of the blowing-ups $\mathbf{P}(S_{i+1}) \to \mathbf{P}(S_i)$ do not intersect $B(S,\alpha)$ for $i \geq c$ by (4) of the induction assumption. Hence we get (2) for α. Here, we know also that $B(S,\alpha) \cap B(S,\gamma) = \emptyset$ if $d(\alpha,\gamma) > 1$.

If $d(\alpha,\beta) = 1$, then the intersection $B(S,\alpha) \cap B(S,\beta)$ is the proper transform of $P_{\beta/\alpha}$ in $B(S,\alpha)$. Hence (3) follows from Theorem 1.6 applied for $\mathbf{P}(\alpha)_0$ and its linear subspaces

$$P_{\alpha_1/\alpha_0}, \cdots, P_{\alpha_d/\alpha_0} \in \Sigma(\mathbf{P}(\alpha)_0) .$$

Now we prove the last assertion (4) of the theorem. Let Y be the blown-up scheme of $\mathbf{P}(S)$. We take a convex ordering $\{\alpha = \alpha_1, \cdots, \alpha_N\}$ of the elements of S as in the proof of (2). Let I be the ideal sheaf of $\mathcal{O}_{\mathbf{P}(\alpha)}$ defining $P_{\delta/\alpha}$, and I' the ideal of $\mathcal{O}_{\mathbf{P}(S)}$ defining the center $\sigma(\Phi(S)^{\delta}_{\alpha})P_{\delta/\alpha} \subset \mathbf{P}(S)$. By the minimality of $\mathrm{len}(\delta,\alpha)$, $d(\delta,\beta) = 1$ and $\beta \in S$ imply that $M_\delta \supset M_\alpha \supset M_\beta \supset tM_\delta$, where M_α and M_β are maximal in M_δ. We see that the blowing-up $\mathbf{P}(S_{i+1}) \to \mathbf{P}(S_i)$ is the case (1) of Lemma 2.5 if $d(\delta,\alpha_{i+1}) = 1$, while it is the case (2) if $(\delta,\alpha_{i+1}) > 1$, for the ideal of the proper transforms of $P_{\delta/\alpha}$. Hence the inverse image of I to $\mathcal{O}_{\mathbf{P}(S)}$ is the tensor product of I' and the invertible sheaf $\mathcal{O}_{\mathbf{P}(S)}(-E)$, where E is the union of $B(S,\alpha_i)$'s whose image in $\mathbf{P}(\alpha)$ is contained in $P_{\delta/\alpha}$. Since the morphism $\mathbf{P}(S') \to \mathbf{P}(\alpha)$ factors $\mathbf{P}(\{\alpha,\delta\})$, the inverse image of I to $\mathcal{O}_{\mathbf{P}(S')}$ is invertible. Hence the inverse image of I' to $\mathcal{O}_{\mathbf{P}(S')}$ is also invertible. By the universality of blowing-up, the birational map $\mathbf{P}(S') \to Y$ is regular. Conversely, since the inverse image of I' to $\mathcal{O}_Y$ is invertible, so is the inverse image of I to $\mathcal{O}_Y$. Hence the birational map $Y \to \mathbf{P}(\{\alpha,\delta\})$ is regular. Since $S' = S \cup \{\alpha,\delta\}$ and Y dominates $\mathbf{P}(S)$, the birational map $Y \to \mathbf{P}(S')$ is also regular. Hence $Y = \mathbf{P}(S')$. *q.e.d.*

By (4) of this theorem, we get the following corollary.

Corollary 2.7 *Let $\{\alpha_1, \cdots, \alpha_N\}$ be a convex ordering of the elements of a finite convex subset S of Δ_0. Then the sequence of morphisms*

$$\mathbf{P}(S) = \mathbf{P}(S_N) \to \cdots \to \mathbf{P}(S_2) \to \mathbf{P}(S_1) = \mathbf{P}(\alpha_1)$$

is the succession of blowing-ups at nonsingular centers.

Since $\mathbf{P}(S)_0 \subset \mathbf{P}(S)$ is a simple normal crossing divisor, it has a stratification induced by the intersections of the irreducible components.

Theorem 2.8 *Let S be a convex finite subset of Δ_0, and $\Sigma(S)$ the set of simplexes $T = \{\alpha_0, \cdots, \alpha_d\} \in \Delta$ whose vertices are in S. Then the k-scheme $\mathbf{P}(S)_0$ has a stratification*

$$\mathbf{P}(S)_0 = \coprod_{T \in \Sigma(S)} X(T)$$

of locally closed subschemes $X(T)$ consisting of the points x with

$$\{\alpha \in S \,;\, x \in B(S, \alpha)\} = T\ .$$

Each $X(T)$ for $T = \{\alpha_0, \cdots, \alpha_d\}$ with

$$M_{\alpha_0} \supset \cdots \supset M_{\alpha_d} \supset tM_{\alpha_0}$$

is naturally isomorphic to

$$(P_{\alpha_1/\alpha_0} \setminus |\Phi_{\alpha_0}^{\alpha_1}|) \times (P_{\alpha_2/\alpha_1} \setminus |\Phi_{\alpha_1}^{\alpha_2}|) \times \cdots \times (P_{\alpha_d/\alpha_{d-1}} \setminus |\Phi_{\alpha_{d-1}}^{\alpha_d}|) \times (P_{\alpha_0/\alpha_d} \setminus |\Phi_{\alpha_d}^{\alpha_0}|)\ ,$$

where we abbreviate $\Phi_\alpha^\beta := \Phi(S)_\alpha^\beta$.

Proof. This is a consequence of Theorem 2.6,(3) and Theorem 1.7. *q.e.d.*

We are ready to reconstruct the R-scheme $\mathcal{X}(S)$ and the formal R-scheme $\mathcal{P}(S)$ of Mustafin for an infinite convex subset $S \subset \Delta_0$. For the definition of formal schemes, see [EGA1, §10]. We say simply "formal R-scheme" instead of "formal Spf(R)-scheme".

Let $\alpha_0 := [RX_0 + \cdots + RX_r]$. We may assume $\alpha_0 \in S$ by exchanging the K-basis of V, if necessary. We define an ordering

$$\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$$

of the elemenets of S in the following rule.

If $i < j$, then

(1) $d(\alpha_0, \alpha_i) < d(\alpha_0, \alpha_j)$ or

(2) $d(\alpha_0, \alpha_i) = d(\alpha_0, \alpha_j)$ and $\mathrm{len}(\alpha_0, \alpha_i) \leq \mathrm{len}(\alpha_0, \alpha_j)$.

Then this ordering is convex, i.e., $S_k := \{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ is convex for any positive integer k. The R-scheme $\mathcal{X}(S)$ is defined as the limit of the infinite sequence of blowing-ups

$$\cdots \quad \to \mathbf{P}(S_3) \to \mathbf{P}(S_2) \to \mathbf{P}(S_1) \to \mathbf{P}(\alpha_0)\,.$$

More precisely, $\mathcal{X}(S)$ is described as follows.

For each nonnegative integer s, let N_s be the integer such that $i \leq N_s$ if and only if $d(\alpha_0, \alpha_i) \leq s$. We define

$$U_s := \mathbf{P}(S_{N_s}) \setminus \Big(\bigcup_{i=N_s+1}^{N_{s+1}} \tilde{P}_{\alpha_i/\beta_i} \Big)\,,$$

where β_i is the element of S_{N_s} with the minimal $\mathrm{len}(\alpha_i, \beta_i)$ for each i, and $\tilde{P}_{\alpha_i/\beta_i}$ is the proper transform of P_{α_i/β_i} in $\mathbf{P}(S_{N_s})$. Then U_s is outside of the centers of the blowing-ups $\mathbf{P}(S_{i+1}) \to \mathbf{P}(S_i)$ for all $i \geq N_s$. Hence

$$U_0 \subset U_1 \subset U_2 \subset \cdots$$

is a sequence of open immersions of R-schemes with the common generic fiber $\operatorname{Proj} K[X_0, \cdots, X_r]$. Then $\mathcal{X}(S)$ is defined to be $\bigcup_{s=0}^{\infty} U_s$.

$\mathcal{X}(S)$ is an R-scheme locally of finite type with the function field

$$K(\frac{X_1}{X_0}, \cdots, \frac{X_r}{X_0}).$$

The formal R-scheme $\mathcal{P}(S)$ is defined to be the formal completion of $\mathcal{X}(S)$ along the closed fiber $\mathcal{X}(S)_0$ over $\operatorname{Spec} R$.

$\mathcal{X}(\Delta_0)$ and $\mathcal{P}(\Delta_0)$ are also denoted by $\mathcal{X}(\Delta)$ and $\mathcal{P}(\Delta)$, respectively. $\mathcal{P}(\Delta)$ is known as the p-adic unit ball of Kurihara and Mustafin.

3 Proof of Mustafin's proposition

Let k be a finite field. Let Σ be the set of all proper k-linear subspaces of $\mathbf{P}^r_k$. We set $B := \sigma(\Sigma)\mathbf{P}^r_k$ and denote by A the union of D_α for all $P_\alpha \in \Sigma$ in the notation of Section 1. The total exceptional divisor $E \subset B$ of the projection $p : B \to \mathbf{P}^r_k$ is the union of D_α for P_α of codimension greater than one.

Lemma 3.1 *Let k'/k be a field extension. We set $B' = B \otimes_k k'$ and $A' = A \otimes_k k'$. Then the natural homomorphism*

$$\operatorname{PGL}(n, k) \to \operatorname{Aut}(B', A')$$

is an isomorphism, where $\operatorname{Aut}(B', A')$ *is the group of k'-automorphisms of B' which maps A' to itself.*

Proof. We prove this lemma by induction on r. Let ϕ be an element of the group $\mathrm{Aut}(B', A')$. If $r = 1$, then $B' = \mathbf{P}^1_{k'}$ and A' is the set of k-rational points. Hence, ϕ is a k-rational linear automorphism. Set $E' := E \otimes_k k' \subset B'$. For $r \geq 2$, it is sufficient to show that $\phi(E') = E'$. In fact, $B' \setminus E'$ is isomorphic to the open subset of $\mathbf{P}^r_{k'}$ whose complement F is the union of k-rational linear subspaces of codimension two. Hence $\mathrm{Pic}(B' \setminus E') \simeq \mathbf{Z}$ and ϕ induces an automorphism of the homogeneous coordinate ring of $\mathbf{P}^r_{k'}$. Since $F = p(E') \subset \mathbf{P}^r_{k'}$ is mapped to itself by ϕ, it is a k-rational linear automorphism.

For $r = 2$, the components of A' are nonsingular rational curves with the self-intersection numbers $-q$ or -1, where $q := |k|$. It is an exceptional divisor if and only if the number is -1. Hence $\phi(E') = E'$.

Assume $r > 2$. Each point x of A' is called i-ple for the number i of irreducible components of A' which contains x. Since A' is a simple normal crossing divisor, it is at most r-ple. For an i-ple point, the i linear subspaces of $\mathbf{P}^r_k$ corresponding to the components form a flag of length i (cf. Theorem 1.7). Let D_α be a component of A' associated to $P_\alpha \in \Sigma$. Then the number of r-ple points on D_α is equal to that of full-length k-rational flags which contains P_α as a member. The number of r-ple points on D_α is calculated easily to be

$$\prod_{i=1}^{s} \frac{q^{i+1}-1}{q-1} \prod_{i=1}^{r-1-s} \frac{q^{i+1}-1}{q-1},$$

where $s := \dim P_\alpha$. Since this number is invariant by ϕ, $\phi(D_\alpha) = D_\beta$ for a $P_\beta \in \Sigma$ with $\dim P_\beta = s$ or $r - 1 - s$.

Since D_α is in E if and only if $\dim P_\alpha < r-1$, it is sufficient to deny the possibility that there are $P_\alpha, P_\beta \in \Sigma$ with $\dim P_\alpha = 0$, $\dim P_\beta = r - 1$ and $\phi(D_\alpha) = D_\beta$.

Suppose that there were such P_α and P_β. By Proposition 1.3, there are natural isomorphisms $D_\alpha \simeq P_\alpha \times \sigma(\Sigma_\alpha)P_{1/\alpha}$ and $D_\beta \simeq \sigma(\Sigma^\beta)P_\beta$, where P_α is a single point. By the assumption of the induction, the isomorphism $D_\alpha \simeq D_\beta$ induced by ϕ descends to a linear isomorphism $\bar{\phi} : P_{1/\alpha} \to P_\beta$ of projective spaces. We may replace k' by its algebraic closure, in order to take a sufficiently general line $\ell_\alpha \subset P_{1/\alpha}$. We set $\ell_\beta := \bar{\phi}(\ell_\alpha)$, and let $\ell'_\alpha \subset D_\alpha$ and $\ell'_\beta \subset D_\beta$ be the proper transforms of ℓ_α and ℓ_β, respectively. We shall compare the intersection numbers $D_\alpha \cdot \ell'_\alpha$ and $D_\beta \cdot \ell_\beta$.

Since ℓ_α in $P_\alpha \times P_{1/\alpha} \subset \sigma(P_\alpha)\mathbf{P}^r_{k'}$ does not intersect the centers of the nontrivial blowing-ups, the intersection number $D_\alpha \cdot \ell'_\alpha$ is equal to $(P_\alpha \times P_{1/\alpha}) \cdot \ell_\alpha = -1$. On the other hand, ℓ_β in $P_\beta \subset \mathbf{P}^r_{k'}$ intersects k-rational hyperplanes of P_β which are going to be the centers of the blowing-ups. Since there are $(q^r - 1)/(q-1)$ such hyperplanes, the intersection number $D_\beta \cdot \ell_\beta$ is equal to $1 - (q^r - 1)/(q-1) = -(q + \cdots + q^{r-1})$. This is a contradiction since these intersection numbers must be equal. *q.e.d.*

Now, we come back to the notation of §2.

We define $\alpha_0 \in \Delta_0$ by $M_{\alpha_0} := RX_0 + \cdots + RX_r$.

Since the generic fibers of the R-schemes $\mathcal{X}(\Delta)$ and $\mathbf{P}(\alpha_0)$ are both equal to $\mathbf{P}^r_K$, there exists a birational map

$$\lambda : \mathcal{X}(\Delta) \to \mathbf{P}(\alpha_0)\ .$$

The following lemma is clear by our construction of $\mathcal{X}(\Delta)$.

Lemma 3.2 *The birational map λ is regular.*

The restriction of λ to the closed fibiers

$$\lambda_0 : \mathcal{X}(\Delta)_0 \to \mathbf{P}(\alpha_0)_0$$

is a morphism of k-schemes.

We denote simply by $B(\alpha)$ the component $B(\alpha, \Delta_0)$ of $\mathcal{X}(\Delta)_0$. Let ϕ be an automorphism of the k-scheme $\mathcal{X}(\Delta)_0$. We denote also by ϕ the induced automorphism of the complex Δ. For $\alpha \in \Delta_0$, we denote by ϕ_α the isomorphism $B(\alpha) \to B(\phi(\alpha))$ induced by ϕ. By Lemma 3.1, there exists an isomorphism $\bar{\phi}_\alpha : \mathbf{P}(\alpha)_0 \to \mathbf{P}(\phi(\alpha))_0$ such that the diagram

$$\begin{array}{ccc} B(\alpha) & \xrightarrow{\phi_\alpha} & B(\phi(\alpha)) \\ p_\alpha \downarrow & & \downarrow p_{\phi(\alpha)} \\ \mathbf{P}(\alpha)_0 & \xrightarrow{\bar{\phi}_\alpha} & \mathbf{P}(\phi(\alpha))_0 \end{array}$$

is commutative, where p_α and $p_{\phi(\alpha)}$ are the natural projections.

For $\alpha \in \Delta_0$, we set $\Delta_0(\alpha) := \{\beta \in \Delta_0 \,;\, d(\alpha, \beta) = 1\}$.

Lemma 3.3 *Let ϕ be an automorphism of $\mathcal{X}(\Delta)_0$ and α an element of Δ_0. Then, for every $\beta \in \Delta_0(\alpha)$, we have $\phi(\beta) \in \Delta_0(\phi(\alpha))$ and*

$$\bar{\phi}_\alpha(P_{\alpha/\beta}) = P_{\phi(\alpha)/\phi(\beta)} \; .$$

Proof. Note that $\beta \in \Delta_0$ with $\beta \neq \alpha$ is in $\Delta_0(\alpha)$ if and only if $B(\alpha) \cap B(\beta) \neq \emptyset$. Since ϕ is an isomorphism, the last condition is equivalent to $B(\phi(\alpha)) \cap B(\phi(\beta)) \neq \emptyset$. Hence $\phi(\beta) \in \Delta_0(\phi(\alpha))$ if and only if $\beta \in \Delta_0(\alpha)$. The equality follows from $P_{\alpha/\beta} = p_\alpha(B(\alpha) \cap B(\beta))$ and $P_{\phi(\alpha)/\phi(\beta)} = p_{\phi(\alpha)}(B(\phi(\alpha)) \cap B(\phi(\beta)))$. *q.e.d.*

By this lemma, $\dim_k P_{\alpha/\beta} = \dim_k P_{\phi(\alpha)/\phi(\beta)}$ for any pair (α, β) of elements of Δ_0 with $d(\alpha, \beta) = 1$. In particular, ϕ preserves the directed lengths of Δ.

Lemma 3.4 *Let ϕ_0 be a k-automorphism of $\mathcal{X}(\Delta)_0$ such that the restriction to the component $B(\alpha_0)$ is the identitiy map. Then the equality $\lambda_0 = \lambda_0 \cdot \phi_0$ holds.*

Proof. Let α be an element of Δ_0. We prove the equality $\lambda_0 = \lambda_0 \cdot \phi_0$ on the irreducible component $B(\alpha)$ by the induction on $d := \mathrm{len}(\alpha_0, \alpha)$. It is sufficient to show the equality for the generic point of $B(\alpha)$. The assertion is trivial for $d = 0$.

We assume $d > 0$. Let $S := \{\beta \in \Delta_0 \,;\, \mathrm{len}(\alpha_0, \beta) < d\}$ and $S' := S \cup \{\alpha\}$. Then S and S' are convex. Let $\beta \in S$ be the elemenet with the minimal $\mathrm{len}(\alpha, \beta)$. We have $d(\alpha, \beta) = 1$ similarly as in the proof of Theorem 2.6. Then the natural morphism $\mathbf{P}(S') \to \mathbf{P}(S)$ is the blowing-up of $\mathbf{P}(S)$ at the nonsingular center $\sigma(\Phi(S)^{\alpha}_{\beta})P_{\alpha/\beta}$ contained in the component $B(S, \beta)$ of the closed fiber $\mathbf{P}(S)_0$. By the assumption of the induction, the equality $\lambda_0 = \lambda_0 \cdot \phi_0$ holds on $\bigcup_{\alpha \in S} B(\alpha) \subset \mathcal{X}(\Delta)_0$. Since ϕ preserves the directed lengths of the complex Δ, $\phi(S) = S$.

By Lemma 3.3, the isomorphism $\bar{\phi}_\alpha : \mathbf{P}(\alpha)_0 \to \mathbf{P}(\phi(\alpha))_0$ induces an isomorphism

$$\phi'_\alpha : B(S, \alpha) \to B(S, \phi(\alpha))$$

for every $\alpha \in S$. Since $\mathbf{P}(S) = \bigcup_{\alpha \in S} B(S, \alpha)$, we get an automorphism of $\mathbf{P}(S)_0$. The natural morphism

$$p_S : \bigcup_{\alpha \in S} B(\alpha) \to \mathbf{P}(S)_0$$

induced by the morphism $\mathcal{X}(\Delta)_0 \to \mathbf{P}(S)_0$ is compatible with these automorphisms, since it is compatible on each component. Since p_S is birational on each components, the equality $\lambda_0 = \lambda_0 \cdot \phi_0$ holds on $\mathbf{P}(S)_0$. Hence it is sufficient to show the commutativity of the diagram

$$\begin{array}{ccccc} B(S', \alpha) & \to & \sigma(\Phi(S)^{\alpha}_{\beta})P_{\beta/\alpha} & \subset & \mathbf{P}(S)_0 \\ \downarrow & & \downarrow & & \\ B(\phi(S'), \phi(\alpha)) & \to & \sigma(\Phi(S)^{\phi(\alpha)}_{\phi(\beta)})P_{\phi(\beta)/\phi(\alpha)} & \subset & \mathbf{P}(S)_0 \end{array}$$

induced by ϕ. This follows from the fact that the horizontal morphisms are the projective space bundle maps generically defined by the linear systems $|\tilde{H} - \tilde{P}_{\beta/\alpha}|$

and $|\phi(\tilde{H}) - \phi(\tilde{P}_{\beta/\alpha})|$, where $\tilde{H}$ is the total transform of a hyperplane H of $\mathbf{P}(\alpha)_0$ in $B(S',\alpha)$ and $\tilde{P}_{\beta/\alpha}$ is the proper transform of $P_{\beta/\alpha} \subset \mathbf{P}(\alpha)_0$ in $B(S',\alpha)$ which is also equal to $B(S',\alpha) \cap B(S',\beta)$. *q.e.d.*

Let $\mathcal{P}(\alpha_0)$ be the formal R-scheme defined by taking the completion of $\mathbf{P}(\alpha_0)$ along $\mathbf{P}(\alpha_0)_0$. We denote by $\hat{\lambda}$ the induced morphism of the formal schemes $\mathcal{P}(\Delta) \to \mathcal{P}(\alpha_0)$. Note that the morphism of the base topological spaces of $\hat{\lambda}$ is equal to that of $\lambda_0 : \mathcal{X}(\Delta)_0 \to \mathbf{P}(\alpha_0)_0$.

Lemma 3.5 *The natural homomorphism of $\mathcal{O}_{\mathcal{P}(\alpha_0)}$-algebras*

$$\hat{\lambda}^* : \mathcal{O}_{\mathcal{P}(\alpha_0)} \to \hat{\lambda}_* \mathcal{O}_{\mathcal{P}(\Delta)}$$

is an isomorphism.

Proof. Let U_0 be a nonempty open subscheme of $\mathbf{P}(\alpha_0)_0$, and let U be the open formal subscheme of $\mathcal{P}(\alpha_0)$ with the base space equal to that of U_0.

Let g be an element of $\Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$. Then there exists a nonnegative integer c such that $t^{-c}g$ is regular and its restiction to $\mathbf{P}(\alpha_0)_0$ is nonzero. Hence $\hat{\lambda}^*(t^{-c}g)$ is nonzero. Since t is not a zero-devisor in $\Gamma(\hat{\lambda}^{-1}(U), \mathcal{O}_{\mathcal{P}(\Delta)})$, $\hat{\lambda}^*(g)$ is also nonzero. Hence the homomorphism is injective.

Let f be an element of the R-algebra $\Gamma(\hat{\lambda}^{-1}(U), \mathcal{O}_{\mathcal{P}(\Delta)})$. It is sufficient to show that f comes from an element of $\Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$.

We set $f_0 := f$. Let $p : B(\alpha_0) \to \mathbf{P}(\alpha_0)_0$ be the natural birational morphism. Then $p^{-1}(U_0) \subset \lambda_0^{-1}(U_0)$, since p is the restriction of λ_0. Since $p_* \mathcal{O}_{B(\alpha_0)} = \mathcal{O}_{\mathbf{P}(\alpha_0)}$, there exists an element $\bar{a}_0 \in \Gamma(U_0, \mathcal{O}_{\mathbf{P}(\alpha_0)_0})$ such that $f_0|_{p^{-1}(U_0)} = p^*(\bar{a}_0)$. Since $\bar{a}_0$ is a rational function of the projective space $\mathbf{P}(\alpha_0)_0$, it has a lifting $a_0 \in \Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$.

Since the fiber $\lambda_0^{-1}(x)$ is a connected scheme with complete components for every closed point x of U_0, the restriction of $f_0 - \hat{\lambda}^*(a_0)$ to the reduced scheme $\phi^{-1}(U_0)$ is zero. Hence $f_0 - \hat{\lambda}^*(a_0) = tf_1$ for some $f_1 \in \Gamma(\hat{\lambda}^{-1}(U), \mathcal{O}_{\mathcal{P}(\Delta)})$. Similarly, there exists $a_1 \in \Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$ such that $f_1 - \hat{\lambda}^*(a_1) = tf_2$ for some f_2. Repeating this process, we get a sequence $a_0, a_1, \cdots$ of elements of $\Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$ such that

$$f - \hat{\lambda}^*(a_0 + ta_1 + \cdots + t^d a_d) \in t^{d+1}\Gamma(\hat{\lambda}^{-1}(U), \mathcal{O}_{\mathcal{P}(\Delta)})$$

for each nonnegative integer d. Hence $f = \hat{\lambda}^*(g)$ for $g = \sum_{i=0}^{\infty} t^i a_i \in \Gamma(U, \mathcal{O}_{\mathcal{P}(\alpha_0)})$.

q.e.d.

Let ϕ be an automorphism of the formal R-scheme $\mathcal{P}(\Delta)$ which is identity on the subscheme $B(\alpha_0)$. Since the base reduced scheme of $\mathcal{P}(\Delta)$ is equal to $\mathcal{X}(\Delta)_0$, ϕ induces a k-automorphism ϕ_0 of $\mathcal{X}(\Delta)_0$. For an open formal subscheme U of $\mathcal{P}(\alpha_0)$, Lemma 3.4 implies $\phi(\hat{\lambda}^{-1}(U)) = \hat{\lambda}^{-1}(U)$. Hence ϕ induces an automorphism of R-algebra $\Gamma(\hat{\lambda}^{-1}(U), \mathcal{O}_{\mathcal{P}(\Delta)})$. By Lemma 3.5, we get an automorphism $\bar{\phi}$ of the formal R-scheme $\mathcal{P}(\alpha_0)$ which is identity on the subscheme $\mathbf{P}(\alpha_0)_0$.

Now we identify $\mathbf{P}(\alpha_0)$ with $\mathbf{P}^r_R$ and we set $\mathcal{P}^r_R := \mathcal{P}(\alpha_0)$.

Lemma 3.6 *The natural homomorphism*

$$\mathrm{Aut}_R\,\mathbf{P}^r_R = \mathrm{PGL}(r+1, R) \longrightarrow \mathrm{Aut}_R\,\mathcal{P}^r_R$$

is an isomorphism.

Proof. A nontrivial automorphism of $\mathbf{P}^r_R$ induces a nontrivial automorphism of $\mathbf{P}^r_{R/(t^m)}$ for a sufficiently large m. Hence the homomorphism $\mathrm{Aut}_R\,\mathbf{P}^r_R \to \mathrm{Aut}_R\,\mathcal{P}^r_R$ is injective. Let ϕ be an element of $\mathrm{Aut}_R\,\mathcal{P}^r_R$. For each positive integer m, let ϕ_m be the induced automorphism of $\mathbf{P}^r_{R/(t^m)}$. Then ϕ_m is represented by a matrix $(a^{(m)}_{i,j}) \in \mathrm{GL}(r+1, R/(t^m))$. By the surjectivity of the homomorphism $(R/(t^{m+1}))^\times \to (R/(t^m))^\times$, we can choose the matrices so that they are compatible with the reductions $R/(t^{m+1}) \to R/(t^m)$ for every m. For each (i,j), let $a_{i,j} \in R$ be the projective limit of $a^{(m)}_{i,j}$ for m. Then $A := (a_{i,j})$ defines an element of $\mathrm{Aut}_R \mathbf{P}^r_R$ which induces the automorphism ϕ. *q.e.d.*

Lemma 3.7 *Let g^* be the automorphism of $\mathcal{P}(\Delta)$ induced by $g \in \mathrm{GL}(r+1, R)$. If the automorphism of Δ induced by g^* is the identity, then g is a constant matrix.*

Proof. We denote also by g the R-automorphism of M_{α_0} defined by g. Let M be an R-submodule of M_{α_0} of rank $r+1$. By the condition, $g(M) = t^c M$ for an integer c. Since M_{α_0}/M and $M_{\alpha_0}/g(M)$ are isomorphic, they have same length as R-modules. Hence $g(M) = M$.

For any nonzero element $x \in M_{\alpha_0}$, the R-module Rx is the intersection of M's which contain Rx. Hence $g(Rx) = Rx$, i.e., $g(x) = ux$ for a unit element u. Let $g(X_i) = u_i X_i$ for $i = 0, \cdots, r$. Since $g(X_0 + \cdots + X_r) = u(X_0 + \cdots + X_r)$ for a unit u and g is linear, we have $u_0 = \cdots = u_r = u$. Hence g is equal to the constant matrix uI_{r+1}. *q.e.d.*

The following theorem is equivalent to [M, Prop.4.2].

Theorem 3.8 *The natural homomorphism*

$$\mathrm{PGL}(r+1, K) \to \mathrm{Aut}_R \mathcal{P}(\Delta)$$

is an isomorphism.

Proof. The injectivity of the homomorphism follows from Lemma 3.7. Let ρ be an automorphism of $\mathcal{P}(\Delta)$. Since $\mathrm{PGL}(r+1, K)$ acts transitively on Δ and the homomorphism $\mathrm{PGL}(r+1, R) \to \mathrm{PGL}(r+1, k)$ is surjective, there exists $g \in \mathrm{PGL}(r+1, K)$ such that $\phi := g^{-1} \cdot \rho$ is identity on $B(\alpha_0)_0$. As we remarked after Lemma 3.5, ϕ induces an automorphism $\bar{\phi}$ of $\mathcal{P}^r_R$. By Lemma 3.6, ϕ is represented by an element $h \in \mathrm{PGL}(r+1, R)$. Hence $\rho = g \cdot h \in \mathrm{PGL}(r+1, K)$. *q.e.d.*

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