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# Squarefree lexsegment ideals

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序. 本稿は, Annetta Aramova, Jürgen Herzog との共同研究である. 多項式環において, (斉次式, 特に) 単項式で生成されるイデアルがあったとき, その極小自由分解を構成し, ベッチ数列を計算することは, Hilbert, Macaulay の仕事に起源を有する伝統的な問題である. 単項式イデアルの理論では, lexsegment および stable と呼ばれるイデアルの類が重要である. １９９０年, Eliahou と Kervaire は stable イデアルの極小自由分解を具体的に構成することに成功した. 更に, Gröbner 基底の理論と Eliahou-Kervaire 分解を使って, １９９３年, Bigatti と Hulett は, それぞれ独立に, Hilbert 函数を固定したとき, ベッチ数列の上限は lexsegment イデアルで与えられることを示した. 他方, いわゆる squarefree な単項式が生成するイデアルは, 昨今, 可換代数と組合せ論の両面から盛んに研究されている. 本稿では, lexsegment および stable イデアルの squarefree 類似の考察を試みる.

**Abstract**

The squarefree analogue of lexsegment, strongly stable, and stable monomial ideals is studied. First, based on a combinatorial technique, it is proved that a minimal free resolution of a squarefree lexsegment ideal generated by monomials of the same degree is linear. Secondly, by means of the computation of the Koszul homology, we construct the explicit minimal free resolution of a squarefree stable ideal. On the other hand, a simple algebraic method how to construct the squarefree lexsegment ideal with the same Hilbert function of a given ideal generated by squarefree monomials is discussed. We conclude the present paper with a conjecture on the Betti numbers of an ideal generated by squarefree monomials.

## Introduction

The ideals generated by squarefree monomials have been studied from view points of both commutative algebra and combinatorics. Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in $v$ variables over a field $k$ with the standard grading, i.e., each $\deg x_i = 1$, and suppose that an ideal $I$ of $A$ is generated by squarefree monomials. We are interested in a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} A/I \longrightarrow 0$$

of $A/I$ over $A$. The above minimal free resolution is called $q$-linear if $\beta_{i_j} = 0$ for each $1 \le i \le h$ and for each $j \ne q+i-1$. Our original problem for organizing the present paper is as follows: Given arbitrary integers $d$, $q$ and $e$ with $q-1 \le e \le d$ and $q \ge 2$, construct an ideal $I$ generated by squarefree monomials with $\dim A/I = d$ and $\operatorname{depth} A/I = e$ such that $A/I$ has a $q$-linear resolution. To find a solution of this problem, the concept of squarefree lexsegment ideals, which is the formal analogue of lexsegment monomial ideals studied in, e.g., Bigatti [Big] and Hulett [Hul], is introduced in Section 1. When $I$ is generated by squarefree monomials, a topological formula to compute the Betti numbers $\beta_{i_j}$ is found by Hochster [Hoc]. By virtue of the formula, a minimal free resolution of $A/I$ over $A$ turns out to be $q$-linear if $I$ is a squarefree lexsegment ideal of degree $q$, see Theorem (1.4). Moreover, the required solution to our original problem is obtained in Corollary (1.6). We refer the reader to, e.g., [Bru–H$_1$], [Bru–H$_2$], [H$_2$] and [T–H] for some related topics.

In the theory of monomial ideal, there is the following hierarchy of ideals: lexsegment monomial ideals $\Rightarrow$ strongly stable monomial ideals $\Rightarrow$ stable monomial ideals. In Section 2, the squarefree analogue of stable ideals is defined and, based on the technique developed in [A–H], the explicit minimal free resolution of a squarefree stable ideal is construced, see Theorem (2.1) and Proposition (2.2). We find that the new resolution has the same formal structures as the classical Eliahou–Kervaire resolutions [E–K] of stable monomial ideals.

On the other hand, the main goal of Section 3 is to show that, given an ideal $I$ of $A$ generated by squarefree monomials, there exists a squarefree lexsegment ideal $J$ such that $A/I$ and $A/J$ have the same Hilbert function (cf. Theorem (3.5)). This result itself is well known in classical combinatorics and, in fact, is equivalent to the essential (and difficult) part of the so-called the Kruskal–Katona theorem, which give a complete characterization on the number of faces of simplicial complexes (see, e.g., [H$_1$, p. 18]). The benefit of our proof is to avoid tedious combinatorial technique and is based on simple algebraic results. Of course, there exists a quite short combinatorial proof of the Kruskal–Katona theorem, e.g., [Day], however, our argument enables the reader to understand the "higher" algebra behind the Kruskal–Katona theorem. We first define the squarefree analogue of strongly stable monomial ideals and show that, for an ideal $I$ generated by squarefree monomials, there exists a squarefree strongly stable ideal $I'$ such that $A/I$ and $A/I'$ have the same Hilbert function. This idea is borrowed from Kalai's work ([Kal$_1$] and [Kal$_2$]) on algebraic shifting with exterior algebras. We introduce instead the "indicator algebra" of an ideal $I$ of $A$ generated by (squarefree) monomials, which is defiend to be the quotient algebra $A/(I, x_1^2, x_2^2, \cdots, x_v^2)$. In the second step, based on the idea of Bigatti [Big], given a squarefree strongly stable ideal $I'$, we construct the squarefree lexsegment ideal $J$ such that $A/I'$ and $A/J$ have the same Hilbert function.

In Section 4 we first give an explicit formula to compute the Betti numbers of the indicator algebra of a squarefree strongly stable ideal. Let $I$ be an ideal generated by squarefree monomials and $J$ the squarefree lexsegment ideal such that $A/I$ and $A/J$ have the same Hilbert function. Then it may be conjectured that each Betti number $\beta_{i_j}$ of $A/I$ is less than or equal to that of $A/J$, the squarefree analogue of Bigatti–Hulett theorem on the upper bounds of Betti numbers of a given Hilbert function. It also can be expected that each Betti number $\beta_{i_j}$ of the indicator algebra of $I$ is less than or equal to that of $J$. Even though we have not yet proved these conjectures,we obtain some related results, see Theorem (4.4). We hope these conjetures will turn out to be true in the near future.

After distributing the first version of this manuscript, we learned that resolutions of similar ideals (called "lex-seg with holes" and "lex-seg plus powers") are studied in [C–E] independently.

We refer the reader to, e.g., [Bru–Her], [H$_1$], [Hoc] and [Sta] for the detailed information about combinatorial and algebraic background. Let **Z** denote the set of integers and **N** the set of nonnegative integers. We write $\sharp(X)$ for the cardinality of a finite set $X$.

## §1. Simplicial complexes with linear resolutions

A simplicial complex $\Delta$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of $V$ such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element $\sigma$ of $\Delta$ is called a *face* of $\Delta$. Set $d = \max\{\sharp(\sigma)\ ;\ \sigma \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim \Delta = d - 1$. Let $\tilde{H}_i(\Delta; k)$ denote the $i$-th reduced simplicial homology group of $\Delta$ with the coefficient field $k$. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & \text{if } \ i \ge 0 \\ k & \text{if } \ i = -1. \end{cases}$$

Given a face $\sigma$ of $\Delta$, we define the subcomplexes $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ to be

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta\ ;\ \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\};$$

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta\ ;\ \sigma \cup \tau \in \Delta\}.$$

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in $v$ variables over a field $k$. Here, we identify each $x_i \in V$ with the indeterminate $x_i$ of $A$. Define $I_\Delta$ to be the ideal of $A$ generated by all squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley–Reisner ring* of $\Delta$ over $k$. The Krull-dimension of $k[\Delta]$ is $\dim k[\Delta] = d$ $(= \dim \Delta + 1)$. Let $\operatorname{depth} k[\Delta]$ denote the depth of $k[\Delta]$. We easily see that if $I$ is an ideal of $A$ generated by squarefree monomials of degree $\ge 2$, then there exists a unique simplicial complex $\Delta$ on $V$ with $I = I_\Delta$.

In what follows, we consider $A$ to be the graded algebra $A = \bigoplus_{n\ge 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n\ge 0}(k[\Delta])_n$ as a graded module over $A$ with the quotient grading. We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n\in\mathbf{Z}}[A(j)]_n$ over $A$ with $[A(j)]_n := A_{n+j}$.

We study a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \qquad (1)$$

of $k[\Delta]$ over $A$. Here $h$ $(= v - \operatorname{depth} k[\Delta])$ is the homological dimension of $k[\Delta]$ over $A$ and $\beta_i = \beta_i^A(k[\Delta]) := \sum_{j\in\mathbf{Z}} \beta_{i_j}$ is the $i$-th Betti number of $k[\Delta]$ over $A$. It is known [Hoc, Theorem (5.1)] that

$$\beta_{i_j} = \sum_{W\subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k), \qquad (2)$$

where $\Delta_W$ is the simplicial complex

$$\Delta_W = \{\sigma \in \Delta\ ;\ \sigma \subset W\}$$

on the vertex set $W$. Thus, in particular, we have

$$\beta_i^A(k[\Delta]) = \sum_{W\subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

A minimal free resolution (1) is called *q-linear* if $\beta_{i_j} = 0$ for each $1 \le i \le h$ and for each $j \ne q + i - 1$. We say that $k[\Delta]$ has a $q$-linear resolution if a graded minimal free resolution of $k[\Delta]$ over $A$ is $q$-linear. If $k[\Delta] = A/I_\Delta$ has a $q$-linear resolution, then $I_\Delta$ is generated by square-free monomials of degree $q$; in particular, $\operatorname{depth} k[\Delta] \ge q - 1$ (see Lemma (1.2) below).

The purpose of this section is, given arbitrary integers $d$, $q$ and $e$ with $q - 1 \le e \le d$ and $q \ge 2$, to construct a simplicial complex $\Delta$ with $\dim k[\Delta] = d$ and $\operatorname{depth} k[\Delta] = e$ such that $k[\Delta]$ has a $q$-linear resolution.

We now introduce the concept of squarefree lexsegment ideals. Let $\binom{V}{q}$ denote the set of all squarefree monomials of degree $q \ge 1$ in the variables $V = \{x_1, x_2, \ldots, x_v\}$. We write $\le_{\text{lex}}$ for the lexicographic order on the finite set $\binom{V}{q}$, i.e., if $S = x_{i_1}x_{i_2}\cdots x_{i_q}$ and $T = x_{j_1}x_{j_2}\cdots x_{j_q}$ are squarefree monomials belonging to $\binom{V}{q}$ with $1 \le i_1 < i_2 < \cdots < i_q \le v$ and $1 \le j_1 < j_2 < \cdots < j_q \le v$, then $S <_{\text{lex}} T$ if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s > j_s$ for some $1 \le s \le q$. A nonempty set $\mathcal{M} \subset \binom{V}{q}$ is called a *squarefree lexsegment set of degree q* if $T \in \mathcal{M}$, $S \in \binom{V}{q}$ and $T \leq_{\text{lex}} S$ imply $S \in \mathcal{M}$. For example, if $v = 5$ and $q = 3$, then $\{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4\}$ is a squarefree lexsegment set of degree $q$. An ideal $I$ of $A = k[x_1, x_2, \ldots, x_v]$ is called a *squarefree lexsegment ideal of degree q* if $I$ is generated by the squarefree monomials belonging to a squarefree lexsegment set of degree $q$. More generally, we say that an ideal $I \subset A$ is *squarefree lexsegment ideal* if $I$ is generated by squarefree monomials and if, for every $1 \leq q \leq v$,

$$T \in I \cap \binom{V}{q},\ S \in \binom{V}{q} \text{ and } T <_{\text{lex}} S \text{ imply } S \in I.$$

It follows immediately that if $I \subset A$ is a squarefree lexsegment ideal of degree $q$ then $I$ is a squarefree lexsegment ideal.

First, we compute the Krull-dimension and the depth of the quotient algebra $A/I$ of a squarefree lexsegment ideal $I$ of degree $q$.

(1.1) PROPOSITION. *Let an ideal $I$ of $A = k[x_1, x_2, \ldots, x_v]$ be a squarefree lexsegment ideal of degree q which is generated by the squarefree monomials belonging to a squarefree lexsegment set $\mathcal{M} \subset \binom{V}{q}$. If $x_{\xi_1}x_{\xi_2}\cdots x_{\xi_q}$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_q \leq v$, is a unique minimal element (with respect to $<_{\text{lex}}$) of $\mathcal{M}$, then* $\dim A/I = v - \xi_1$.

*Proof.* First, we show that if $P = (x_{i_1}, x_{i_2}, \ldots, x_{i_r})$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, is a prime ideal of $A$ with $I \subset P$, then $r \geq \xi_1$. We may assume that $\xi_1 \geq 2$. Let $\mathcal{M} - \{x_1\}$ be the set of all squarefree monomials $T$ of $\mathcal{M}$ such that $x_1$ does not divide $T$. Then $\mathcal{M} - \{x_1\}$ is a squarefree lexsegment set of degree $q$ in the variables $x_2, x_3, \ldots, x_v$. Hence $I' = (I, x_1)/(x_1)$ is a squarefree lexsegment ideal of $A' = k[x_2, x_3, \ldots, x_v]$ of degree $q$ and $P' = (P, x_1)/(x_1)$ is a prime ideal of $A'$ with $I' \subset P'$. Thus, by induction, we have $r \geq \xi_1 - 1$ if $i_1 \geq 2$; and $r - 1 \geq \xi_1 - 1$ if $i_1 = 1$. Hence, $r \geq \xi_1$ as desired.

Now, since the ideal $I$ is contained in the prime ideal $(x_1, x_2, \ldots, x_{\xi_1})$ of $A$, the minimal height of prime ideals $P$ with $I \subset P$ is $\xi_1$. Hence, we have $\dim A/I = v - \xi_1$ as required. Q. E. D.

We give a standard technique to compute the depth of the Stanley–Reisner ring of a simplicial complex. Let $\Delta$ be a simplicial complex of dimension $d - 1$. The $i$-th *skeleton* $\Delta^{(i)}$, $0 \leq i \leq d - 1$, of a simplicial complex $\Delta$ is defined to be

$$\Delta^{(i)} = \{\sigma \in \Delta\ ;\ \sharp(\sigma) \leq i + 1\}.$$

Thus, in particular, $\dim \Delta = i$ and $\Delta^{(d-1)} = \Delta$. The following Lemma (1.2) can be found in, e.g., [Bru–Her, Exercise (5.1.23)].

(1.2) LEMMA. $\operatorname{depth} k[\Delta] = \max\{i + 1\ ;\ k[\Delta^{(i)}]$ *is Cohen–Macaulay*$\}$.

A simplicial complex $\Delta$ is called *pure* if every maximal face of $\Delta$ has the same cardinality. Note that $\Delta$ is pure if, e.g., $k[\Delta]$ is Cohen–Macaulay.

(1.3) PROPOSITION. *If an ideal $I$ of $A = k[x_1, x_2, \ldots, x_v]$ is a squarefree lexsegment ideal of degree $q \geq 2$ with $x_1 x_2 \cdots x_{q-1} x_v \in I$, then* $\operatorname{depth} A/I = q - 1$.

*Proof.* Let $\Delta$ be the simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ with $I = I_\Delta$. Since $I$ is generated by square-free monomials of degree $q$, every subset $\sigma$ of $V$ with $\sharp(\sigma) \leq q-1$ is a face of $\Delta$. Hence, $k[\Delta^{(q-2)}]$ is Cohen–Macaulay. Thus, by Lemma (1.2), $\operatorname{depth} A/I \geq q - 1$.

We now show that $\Delta^{(i)}$ with $i \geq q-1$ is not pure unless $I$ is generated by all squarefree monomials of degree $q$. If $x_{v-q+1} x_{v-q+2} \cdots x_v \notin I$, then $\sigma = \{x_{v-q+1}, x_{v-q+2}, \ldots, x_v\}$ is a face of $\Delta^{(i)}$ with $\sharp(\sigma) = q$. On the other hand, since $x_1 x_2 \cdots x_{q-1} x_j \in I$ for every $q \leq j \leq v$, $\tau = \{x_1, x_2, \ldots, x_{q-1}\}$ is a maximal face of $\Delta^{(i)}$ with $\sharp(\tau) = q-1$. Hence $\Delta^{(i)}$ is not pure; in particular, $\Delta^{(i)}$ is not Cohen–Macaulay for every $i \geq q-1$. Thus, again by Lemma (1.2), $\operatorname{depth} A/I \leq q-1$. Hence, $\operatorname{depth} A/I = q-1$ as required. Q. E. D.

We now come to the main theorem of this section.

(1.4) THEOREM. *Suppose that an ideal $I$ of $A = k[x_1, x_2, \ldots, x_v]$ is a squarefree lexsegment ideal of degree $q \geq 2$. Then, $A/I$ has a $q$-linear resolution.*

*Proof.* Let $\Delta$ denote the simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ with $I = I_\Delta$, and set $\Delta_1 = \Delta_{V - \{x_1\}}$, $\Delta_2 = \operatorname{star}_\Delta(\{x_1\})$ and $\Delta' = \operatorname{link}_\Delta(\{x_1\})$ Then, the ideal $I_{\Delta_1}$ of $A' = k[x_2, x_3, \ldots, x_v]$ is either a squarefree lexsegment ideal of degree $q$ or $I_{\Delta_1} = (0)$. Since the ideal $I_{\Delta'}$ of $A' = k[x_2, x_3, \ldots, x_v]$ is generated by those squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_{q-1}}$, $2 \leq i_1 < i_2 < \cdots < i_{q-1} \leq v$, with $x_1 x_{i_1} x_{i_2} \cdots x_{i_{q-1}} \in I$, $I_{\Delta'}$ is a squarefree lexsegment ideal of degree $q-1$.

By virtue of Eq. (2), what we must prove is $\tilde{H}_i(\Delta_W; k) = 0$ for every subset $W$ of $V$ and for each $i \neq q-2$. We employ the induction on $v$ and may assume that $\tilde{H}_i((\Delta_1)_W; k) = 0$ and $\tilde{H}_{i-1}(\Delta'_W; k) = 0$ for every subset $W$ of $V - \{x_1\}$ and for each $i \neq q-2$. Now, if $x_1 \notin W$, then $\Delta_W = (\Delta_1)_W$. Thus $\tilde{H}_i(\Delta_W; k) = 0$ for every subset $W$ of $V - \{x_1\}$ and for each $i \neq q-2$. Moreover, if $x_1 \in W$, then $(\Delta_2)_W$ is contractible; in particular, $\tilde{H}_i((\Delta_2)_W; k) = 0$ for every $i$. Thus, since $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 = \Delta'$, the reduced Mayer–Vietoris exact sequence

$$\cdots \longrightarrow \tilde{H}_i(\Delta'; k) \longrightarrow \tilde{H}_i(\Delta_1; k) \oplus \tilde{H}_i(\Delta_2; k) \longrightarrow \tilde{H}_i(\Delta; k) \longrightarrow \tilde{H}_{i-1}(\Delta'; k) \longrightarrow \cdots$$

guarantees that

$$\tilde{H}_{q-2}(\Delta_W; k) \cong \tilde{H}_{q-2}((\Delta_1)_{W - \{x_1\}}; k) \oplus \tilde{H}_{q-3}(\Delta'_{W - \{x_1\}}; k), \tag{3}$$

and $\tilde{H}_i(\Delta_W; k) = 0$ for each $i \neq q-2$. Q. E. D.

The above proof of theorem (1.4) enables us to compute the betti numbers of $A/I$ over $A$ for a squarefree lexsegment ideal $I$ of degree $q$. If $T = x_{i_1}x_{i_2}\cdots x_{i_r}$ is a squarefree monomial of $A = k[x_1, x_2, \ldots, x_v]$ with $1 \le i_1 < i_2 < \cdots < i_r \le v$, then we define $m(T)$ to be $i_r$, i.e., $m(T)$ is the greatest integer $i$ for which $x_i$ divides $T$.

(1.5) COROLLARY. *Let an ideal $I$ of $A = k[x_1, x_2, \ldots, x_v]$ be a squarefree lexsegment ideal of degree $q$ which is generated by the squarefree monomials belonging to a squarefree lexsegment set $\mathcal{M} \subset \binom{V}{q}$. Then, the i-th Betti number of $A/I$ over $A$ is*

$$\beta_i^A(A/I) = \sum_{T \in \mathcal{M}} \binom{m(T) - q}{i - 1}.$$

*Proof.* We inherit the same notation as in the proof of Theorem (1.4). Thanks to Eq. (3) together with the fact that $\Delta_W = (\Delta_1)_W$ if $x_1 \notin W$, we have

$$\beta_i^A(k[\Delta]) = \beta_i^A(k[\Delta_1]) + \beta_{i-1}^A(k[\Delta_1]) + \beta_i^A(k[\Delta']) \tag{4}$$

by Eq. (2). Let $\mathcal{N} \subset \binom{V - \{x_1\}}{q}$ denote the set of all squarefree monomials $T \in \mathcal{M}$ such that $x_1$ does not divide $T$. Thus $\mathcal{N} \subset \binom{V - \{x_1\}}{q}$ is a squarefree lexsegment set of degree $q$ and $I_{\Delta_1}$ is generated by the squarefree monomials belonging to the squarefree lexsegment set $\mathcal{N}$. Again, we employ the induction on $v$ and may assume that

$$\beta_i^A(k[\Delta_1]) = \sum_{T \in \mathcal{N}} \binom{m(T) - 1 - q}{i - 1};$$

$$\beta_{i-1}^A(k[\Delta_1]) = \sum_{T \in \mathcal{N}} \binom{m(T) - 1 - q}{i - 2}.$$

Hence,

$$\beta_i^A(k[\Delta_1]) + \beta_{i-1}^A(k[\Delta_1]) = \sum_{T \in \mathcal{N}} \binom{m(T) - q}{i - 1}. \tag{5}$$

On the other hand, the ideal $I_{\Delta'}$ of $A' = k[x_2, x_3, \ldots, x_v]$ is a squarefree lexsegment ideal of degree $q-1$ which is generated by those squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_{q-1}}$, $2 \le i_1 < i_2 < \cdots < i_{q-1} \le v$, with $x_1 x_{i_1} x_{i_2} \cdots x_{i_{q-1}} \in I$. Thus,

$$\beta_i^A(k[\Delta']) = \sum_{T \in \mathcal{M} - \mathcal{N}} \binom{(m(T) - 1) - (q - 1)}{i - 1}. \tag{6}$$

Now, the desired formula follows from Eqs. (4), (5) and (6). Q. E. D.

We conclude this section with the answer to our original problem for organizing the present paper.

(1.6) COROLLARY. *Given arbitrary integers $d$, $q$ and $e$ with $q-1 \le e \le d$ and $q \ge 2$, there exists a simplicial complex $\Delta$ with* $\dim k[\Delta] = d$ *and* $\operatorname{depth} k[\Delta] = e$ *such that $k[\Delta]$ has a $q$-linear resolution.*

*Proof.* Thanks to Propositions (1.1) and (1.3), there exists a squarefree lexsegment ideal $I'$ of degree $q$ in $A = k[x_1, x_2, \ldots, x_v]$ (for some $v$) such that $\dim A/I' = d - (e - q + 1)$ and $\operatorname{depth} A/I' = q - 1$. Let $B = A[y_1, \ldots, y_{e-q+1}]$ be the polynomial ring over $k$ in $v + e - q + 1$ variables and $I := I'B$. Then $\dim B/I = d$ and $\operatorname{depth} B/I = e$. Now, Theorem (1.4) guarantees that $A/I'$ has a $q$-linear resolution, thus $B/I$ has a $q$-linear resolution as required. Q. E. D.

## §2. Squarefree stable ideals and their resolutions

We present the concept of squarefree stable ideals $I$, which is the formal analogue of stable ideals, of the polynomial ring $A = k[x_1, x_2, \ldots, x_v]$ in $v$ variables over a field $k$, and discuss their explicit free resolutions. In the theory of monomial ideals, there is the following hierarchy of ideals:

lexsegment ideals $\Rightarrow$ strongly stable ideals $\Rightarrow$ stable ideals

When the base field $k$ is of characteristic 0, the generic initial ideals are exactly the strongly stable ideals. Combinatorially they are described as follows:

If $T \in I$ is a monomial, and $x_i$ divides $T$, then $(x_j T)/x_i \in I$ for all $j \le i$.

The support of a monomial $T \in A$ is $\operatorname{supp}(T) = \{i \; ; \; x_i \text{ divides } T\}$. Let us denote $m(T) = \max(T) := \max\{i \; ; \; i \in \operatorname{supp}(T)\}$ and $\min(T) := \min\{i \; ; \; i \in \operatorname{supp}(T)\}$. A stable ideal is defined by the following combinatorial property:

If $T \in I$ is a monomial, then $(x_j T)/x_{m(T)} \in I$ for all $j \le m(T)$.

If a monomial ideal $I$ is stable as above and, in addition, if $I$ is generated by squarefree monomials, then $I$ is nothing but the ideals $(x_1, x_2, \ldots, x_i)$ for $1 \le i \le v$.

We now come to the definition of squarefree stable ideals. Let $I$ be an ideal of $A = k[x_1, x_2, \ldots, x_v]$ which is generated by squarefree monomials. Then $I$ is called a *squarefree stable ideal* if, for every squarefree monomial $T \in I$, we have

$(x_j T)/x_{m(T)} \in I$ for each $j \le m(T)$ such that $x_j$ does not divide $T$.

The squarefree lexsegment ideals introduced in Section 1 are squarefree stable ideals. Hence all results of this section can be applied as well to squarefree lexsegment ideals.

The main goal of this section is to construct the explicit free resolutions of squarefree stable ideals, similar to the Eliahou–Kervaire resolutions of stable ideals.

It will turn out that the new resolutions have the same formal structure as the classical Eliahou–Kervaire resolutions. In order to describe our resolutions we need to introduce some more notation.

Let as before $I \subset A$ be an ideal generated by squarefree monomials. We write $G(I)$ for the unique minimal set of monomial generators of $I$, and $Q(I)$ for the (finite) set of all squarefree monomials in $I$. In particular, $G(I) \subset Q(I)$. Suppose now that $I$ is squarefree stable. Then it is immediately seen that, for every $S \in Q(I)$, there exists a unique pair $(T, T^*)$ of squarefree monomials in $A$ such that $T \in G(I)$, $S = TT^*$ and $\max(T) < \min(T^*)$. Thus, if we set $g(S) = T$, then we obtain a map $Q(I) \to G(I)$. Now, given $j \in \{1, 2, \ldots, v\}$ with $j \notin \operatorname{supp}(T)$ and $T \in Q(I)$, we set $T_j = g(x_j T)$ and $y(T)_j = (x_j T)/T_j$.

(2.1) THEOREM. *Suppose that $I \subset A$ is a squarefree stable ideal. Then $A/I$ has a minimal multigraded free $A$-resolution $(F, \partial)$ of the following form:*

(a) *Each $F_i$, $i > 0$, has a basis consisting of $f(\sigma; T)$ with $\sigma \subset \{1, 2, \ldots, v\}$ and $T \in G(I)$ such that $\sharp(\sigma) = i - 1$, $\max(\sigma) < m(T)$ and $\sigma \cap \operatorname{supp}(T) = \emptyset$;*

(b) *If $\epsilon_1, \epsilon_2, \ldots, \epsilon_v$ denotes the canonical basis of $\mathbf{Z}^v$, then $f(\sigma; T)$ is homogeneous of multidegree $\sum_{j \in \sigma} \epsilon_j + \sum_{j \in \operatorname{supp}(T)} \epsilon_j$;*

(c) *The differentials of the resolution are given by*

$$\partial_1(f(\emptyset; T)) = T,$$

*and by*

$$\partial_i(f(\sigma; T)) = \sum_{j \in \sigma} (-1)^{\alpha(\sigma, j)} (-x_j f(\sigma - \{j\}; T) + y(T)_j f(\sigma - \{j\}; T_j))$$

*for $i > 1$, where we set $\alpha(\sigma, j) = \sharp(\{i \in \sigma \,;\, i < j\})$.*

The proof of Theorem (2.1) is carried out in two steps. In the first step we determine cycles in the Koszul complex $K(x_1, x_2, \ldots, x_v; A/I)$ whose homology classes form a basis of the corresponding Koszul homology. This first step already gives us all the information to prove the assertions (a) and (b) as above. In the second step the differentials $\partial_i$ of $F$ are computed. This is done by using a technique developed in [A–H] which allows us to compute the differentials once the cycles (determined in step one) are known. This part of the proof is verbatim the same as that in [A–H] where the maps in the Eliahou–Kervaire resolutions were determined by this method. We omit its proof and refer the reader to [A–H] for the details.

We recall that $K_i(x_1, x_2, \ldots, x_v; A/I)$ is a free $A/I$-module with basis $e_\sigma$, $\sigma \subset \{1, 2, \ldots, v\}$, $\sharp(\sigma) = i$, where $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_i}$ for $\sigma = \{j_1, j_2, \ldots, j_i\}$, $j_1 < j_2 < \ldots < j_i$. The differential $d$ of $K$ is given by $d(e_\sigma) = \sum_{t \in \sigma} (-1)^{\alpha(\sigma, t)} x_t e_{\sigma - \{t\}}$.

We set $T' = T/x_{m(T)}$ for all $T \in G(I)$. It is further convenient to denote the image of a monomial $T \in A$ in any quotient ring of $A$ again by $T$. We will keep this convention throughout the present paper.

(2.2) PROPOSITION. *Let $I \subset A$ be a squarefree stable ideal. Then, for every $i > 0$, a basis of the homology classes of $H_i(x_1, x_2, \dots, x_v; A/I)$ is given by the homology classes of the cycles*

$$T'e_\sigma \wedge e_{m(T)}, \quad T \in G(I), \quad \sharp(\sigma) = i-1, \quad \max(\sigma) < m(T), \quad \sigma \cap \operatorname{supp}(T) = \emptyset.$$

*Proof.* A minimal free $A$-resolution of $A/I$ is multigraded; in other words, the differentials are homogeneous homomorphisms and, for each $i$, we have $F_i = \bigoplus_j A(-a_{ij})$ with $a_{ij} \in \mathbf{Z}^v$. Moreover, by virtue of [Hoc, Theorem (5.1)], all shifts $a_{ij}$ are squarefree, i.e., $a_{ij} \in \mathbf{Z}^v$ is of the form $\sum_{t\in\tau} \epsilon_t$, where $\tau$ is a subset of $\{1, 2, \dots, v\}$, and where, as before, $\epsilon_1, \epsilon_2, \dots, \epsilon_v$ is the canonical basis of $\mathbf{Z}^v$. Thus it follows that $H_i(x_1, x_2, \dots, x_v) := H_i(x_1, x_2, \dots, x_v; A/I)$ is multigraded $k$-vector space with $H_i(x_1, x_2, \dots, x_v)_a = 0$ if $a \in \mathbf{Z}^v$ is not squarefree. Hence, if we want to compute the homology module $H_i(x_1, x_2, \dots, x_v)$, it suffices to consider its squarefree multigraded components.

It is known (cf. [Bru–Her, Corollary (1.6.13)]) that, for each $0 < j < v$, there exists an exact sequence whose graded part for each $a \in \mathbf{Z}^v$ yields the long exact sequence of vector spaces

$$\begin{aligned} \cdots \quad &\xrightarrow{x_j} H_i(x_{j+1}, \dots, x_v)_a \longrightarrow H_i(x_j, \dots, x_v)_a \longrightarrow H_{i-1}(x_{j+1}, \dots, x_v)_{a-\epsilon_j} \\ &\xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_v)_a \longrightarrow H_{i-1}(x_j, \dots, x_v)_a \longrightarrow \cdots. \end{aligned}$$

We now show the following more precise result: For all $i > 0$, all $0 < j \le v$ and all squarefree $a \in \mathbf{Z}^v$, $H_i(x_j, \dots, x_v; A/I)_a$ is generated by the homology classes of the cycles

$$T'e_\sigma \wedge e_{m(T)}, \quad T \in G(I), \quad \sharp(\sigma) = i-1$$

with

$$j \le \min(\sigma), \quad \max(\sigma) < m(T), \quad \sigma \cap \operatorname{supp}(T) = \emptyset \quad \text{and} \quad \sigma \cup \operatorname{supp}(T) = a.$$

The proof is achieved by induction on $v - j$. The assertion is obvious for $j = v$. We now suppose that $j < v$. For such $j$, but $i = 1$, the assertion is again obvious. Hence we assume in addition that $i > 1$. We first claim that

$$H_{i-1}(x_{j+1}, \dots, x_v)_{a-\epsilon_j} \xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_v)_a$$

is the zero map. Since $a \in \mathbf{Z}^v$ is squarefree, the components of $a$ are either 0 or 1. If the $j$-th component of $a$ is 0, then $a - \epsilon_j$ has a negative component; hence

$H_{i-1}(x_{j+1},\ldots,x_v)_{a-\epsilon_j}=0$. Thus we may assume the $j$-th component of $a$ is 1. Then $a-\epsilon_j$ is squarefree and, by induction hypothesis, $H_{i-1}(x_{j+1},\ldots,x_n)_{a-\epsilon_j}$ is generated by the homology classes of cycles of the form $T'e_\sigma \wedge e_{m(T)}$ with $j \notin \mathrm{supp}(T)$. Such an element is mapped to the homology class of $T'x_je_\sigma \wedge e_{m(T)}$ in $H_{i-1}(x_{j+1},\ldots,x_v)_a$. However, since $I$ is stable, we have $T'x_j = 0$ as desired.

From these observations we deduce that we have short exact sequences

$$0 \longrightarrow H_i(x_{j+1},\ldots,x_v)_a \longrightarrow H_i(x_j,\ldots,x_v)_a \longrightarrow H_{i-1}(x_{j+1},\ldots,x_v)_{a-\epsilon_j} \longrightarrow 0$$

for all $i > 1$. The first map $H_i(x_{j+1},\ldots,x_v)_a \to H_i(x_j,\ldots,x_v)_a$ of the above exact sequence is simply induced by the natural inclusion map of the corresponding Koszul complexes, while the second map $H_i(x_j,\ldots,x_v)_a \to H_{i-1}(x_{j+1},\ldots,x_v)_{a-\epsilon_j}$ is a connecting homomorphism. Given the homology class of a cycle $z = T'e_\sigma \wedge e_{m(T)}$ in $H_{i-1}(x_{j+1},\ldots,x_v)_{a-\epsilon_j}$, it is easy to see that, up to a sign, the homology class of the cycle $T'e_j \wedge e_\sigma \wedge e_{m(T)}$ in $H_i(x_j,\ldots,x_v)_a$ is mapped to $[z]$. This guarantees all of our assertions as required. Q. E. D.

Let us draw some immediate consequences of Proposition (2.2). Recall that the $i$-th Betti number of a graded $A$-module $M$ is the nonnegative integer

$$\beta_i^A(M) = \dim_k \mathrm{Tor}_i^A(k,M)$$

and the *Poincaré series* of $M$ is the formal power series

$$P_M^A(\lambda) = \sum_{i=0}^{\infty} \beta_i^A(M)\lambda^i.$$

(2.3) COROLLARY. *Let $I \subset A$ be a squarefree stable ideal.*

(a) $\beta_i(I) = \sum_{T\in G(I)} \binom{m(T)-\deg(T)}{i}$ *for every $i \geq 0$. In particular, all the Betti numbers of $I$ are independent of the base field $k$.*

(b) $P_I^A(\lambda) = \sum_{T\in G(I)}(1+\lambda)^{m(T)-\deg(T)}$.

(2.4) COROLLARY. *Let $I \subset A$ be a squarefree stable ideal. Then*

$$\mathrm{depth}\, A/I = v - \max\{m(T)\ ;\ T \in G(I)\} + \max\{\deg(T)\ ;\ T \in G(I)\} - 1.$$

*In particular, if* $v = \max\{m(T)\ ;\ T \in G(I)\}$ *(which one may assume without loss of generality), then* $\mathrm{depth}\, A/I = \max\{\deg(T)\ ;\ T \in G(I)\} - 1$.

The following Corollary (2.5) generalizes Theorem (1.4) of Section 1.

(2.5) COROLLARY. *Let $I \subset A$ be a squarefree stable ideal and suppose that every element of $G(I)$ is of degree $q$. Then $A/I$ has a $q$-linear resolution.*

In the following Corollary (2.6) we describe another important numerical invariant of graded modules for squarefree stable ideals. For a graded $A$-module $M$, let $t_i(M)$ denote the maximal integer $a \in \mathbf{Z}$ with $\operatorname{Tor}_i^A(k, M)_a \neq 0$. We say that

$$\operatorname{reg}(M) = \max\{t_i(M) - i \; ; \; i \geq 0\}$$

is the *Castelnuovo–Mumford regularity* of $M$.

(2.6) COROLLARY. *Let $I \subset A$ be a squarefree stable ideal. Then*

$$\operatorname{reg}(I) = \max\{\deg(T) \; ; \; T \in G(I)\}.$$

*In particular, if* $v = \max\{m(T) \; ; \; T \in G(I)\}$, *then* $\operatorname{reg}(I) = \operatorname{depth} A/I + 1$.

Just as for stable monomial ideals (cf. [Pee]) we have

(2.7) COROLLARY. *Let $I \subset A$ be a squarefree stable ideal. Then $A/I$ is a Golod ring, and the residue class field $k$* of $A/I$ *has a resolution whose Poincaré series is given by*

$$P_k^{A/I}(\lambda) = \frac{(1+\lambda)^v}{1 - \sum_{T \in G(I)} (1+\lambda)^{m(T)-\deg(T)} \lambda^2}.$$

*Proof.* It follows from Proposition (2.2) that the product of any two cycles in $K(x_1, x_2, \ldots, x_v; A/I)$ is zero. Thus $A/I$ is trivially a Golod ring (see [G–L, Corollary (4.2.4)]). The Poincaré series of a Golod ring is given by

$$P_k^{A/I}(\lambda) = \frac{(1+\lambda)^v}{1 - P_I^A(\lambda)\lambda^2}.$$

Hence the required result follows from Corollary (2.3). Q. E. D.

## §3. Strongly stable simplicial complexes and the Kruskal–Katona theorem

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in $v$ variables over a field $k$ and $I$ an ideal of $A$ generated by squarefree monomials. The purpose of this section is to show that there exists a squarefree lexsegment ideal $J$ of $A$ such that $A/I$ and $A/J$ have the same Hilbert functions. A simplicial complex $\Delta$ is called a *lexsegment simplicial complex* if the defining ideal $I_\Delta$ of its Stanley–Reisner ring $k[\Delta]$ is a squarefree lexsegment ideal. In these terms the above result says that, given an arbitrary simplicial complex $\Delta$, there exists a unique lexsegment simplicial complex $\Delta'$ with the same $f$-vector of $\Delta$. This result itself is well known in classical combinatorics and, in fact, is equivalent to the essential (and difficult) part of so-called the Kruskal–Katona theorem, which give a complete characterization of the number of faces of simplicial complexes. See, e.g., [$H_1$, Chapter II] for a brief introduction about the Kruskal–Katona theorem. Our proof here is based on simple algebraic arguments and avoids tedious combinatorial considerations. It also has the benefit to show why, with respect to many properties, lexsegment simplicial complexes are extremal.

The proof will be accomplished in two steps. In the first step, we 'shift' a general simplicial complex to a strongly stable simplicial complex. This idea is borrowed from Kalai ([$Kal_1$] and [$Kal_2$]) who assigns to a simplicial complex a certain quotient ring of the exterior algebra. Here we introduce instead the so-called indicator algebra, which in characteristic 2 may as well be viewed as the quotient ring of an exterior algebra. In fact, for the shifting argument we assume characteristic 2. However, homologically we better understand the commutative indicator algebra. In the second step, we compare the strongly stable simplicial complexes with the lexsegment simplicial complexes by the similar technique as in Bigatti [Big].

We say that an ideal $I \subset A$ generated by squarefree monomials is a *squarefree strongly stable ideal* if, for all squarefee monomials $T \in I$, one has:

$$(x_jT)/x_i \in I \text{ for all } i \in \operatorname{supp}(T) \text{ and all } j \notin \operatorname{supp}(T) \text{ with } j < i.$$

In particular, any squarefree lexsegment ideal is squarefree strongly stable, and any squarefree strongly stable ideal is squarefree stable, so that by the results of Section 2, we know the resolution of these ideals. We say that a simplicial complex $\Delta$ is *strongly stable* if $I_\Delta$ is squarefree strongly stable.

We refer the reader to, e.g., [Bru–Her], [$H_1$], [Hoc] and [Sta] for the definitions and the detailed information about $f$-vectors and $h$-vectors of simplicial complexes.

Let $\Delta$ be a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$. We define the ideal $J_\Delta$ of $A = k[x_1, x_2, \ldots, x_v]$ to be

$$J_\Delta = (I_\Delta, x_1^2, x_2^2, \ldots, x_v^2).$$

The algebraic object which we attach with $\Delta$ is the quotient algebra

$$k\{\Delta\} = A/J_\Delta$$

of $A$, which we call the *indicator algebra* of a simplicial complex $\Delta$. The name is justified by the fact that $k\{\Delta\}$ is a multigraded $k$-algebra with $\dim_k k\{\Delta\}_a \leq 1$ for all $a \in \mathbf{N}^v$, and $\dim_k k\{\Delta\}_a = 1$ if and only if $a$ is squarefree with $\operatorname{supp}(a) \in \Delta$. Here we regard $\operatorname{supp}(a)$ to be a subset of $V$ in the obvious way. One immediate consequence of this observation is

(3.1) PROPOSITION. *Let $\Delta$ be a simplicial complex of dimension $d-1$ with* the $f$-*vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$. Then the Hilbert series $F(k\{\Delta\}, \lambda)$ of the indicator* *algebra* $k\{\Delta\}$ *of* $\Delta$ *is*

$$F(k\{\Delta\},\lambda)=\sum_{i=0}^{d} f_{i-1}\lambda^i,$$

*where, as usual,* $f_{-1}=1$. *In particular, since the h-vector determines the f-vector, and vice versa, it follows that the Hilbert function of the Stanley–Reisner ring and that of the indicator algebra of* $\Delta$ *determine each other.*

In what follows we define a certain operation on the indicator algebra $k\{\Delta\}$ of a simplicial complex. The output of this operation will be a new indicator algebra $k\{\Delta'\}$ of a suitable simplicial complex $\Delta'$. It will be clear from the construction that $\Delta$ and $\Delta'$ have the same $f$-vector. But the new simplicial complex $\Delta'$ is in a certain sense, which will be explained later, closer to a strongly stable simplicial complex than the original complex $\Delta$.

On the set of all monomials in $A$, we define the 'deglex order' as follows: Let $S=x_1^{a_1}x_2^{a_2}\cdots x_v^{a_v}$ and $T=x_1^{b_1}x_2^{b_2}\cdots x_v^{b_v}$ be monomials in $A$. Then $S<_{\text{deglex}}T$, if the first non-zero component of

$$\Big(\sum_{i=1}^{v} b_i-\sum_{i=1}^{v} a_i, b_1-a_1, b_2-a_2,\ldots,b_v-a_v\Big)$$

is positive.

If $g\in A$ is a non-zero polynomial, then we denote by $\text{in}(g)$ the largest (with respect to the deglex order) monomial occuring in $g$. One calls $\text{in}(g)$ the *initial monomial* of $g$. If $I\subset A$ is an ideal, then we write $\text{in}(I)$ for the ideal generated by all monomials $\text{in}(g)$ with $g\in I$. The ideal $\text{in}(I)$ is called the *initial ideal* of $I$. Let $T_1,T_2,\ldots,T_m$ be a set of monomial generators of $\text{in}(I)$, and choose $g_i\in I$ with $\text{in}(g_i)=T_i$ for each $1\le i\le m$. Then $g_1,g_2,\ldots,g_m$ is called a *Gröbner basis* of $I$.

We quote a few simple facts from Gröbner bases theory which can be found in standard references, e.g., [Eis], [Rob] or [Vas]. We restrict ourselves to graded ideals, and write $\mu(I)$ for the minimal number of generators of $I$.

(3.2) PROPOSITION. *Let* $I=\bigoplus_{n\ge 0} I_n$ *be a graded ideal of* $A$.

(a) *Any Gröbner basis of* $I$ *is a basis of* $I$. *In particular,* $\mu(I)\le\mu(\text{in}(I))$.

(b) *For all* $n\in\mathbf{N}$, *the elements* $\text{in}(g)$, $g\in I_n$, *generate the k-vector space* $\text{in}(I)_n$, *and if* $\text{in}(g_1),\text{in}(g_2),\ldots,\text{in}(g_m)$ *(with each* $g_i\in I_n$*) is a k-basis of* $\text{in}(I)_n$, *then* $g_1,g_2,\ldots,g_m$ *is a k-basis of* $I_n$. *In particular, the Hilbert function of* $A/I$ *and that of* $A/\text{in}(I)$ are the same.

We fix some $i$ and $j$ with $1\le j<i\le v$ and define an automorphism $\varphi:A\to A$, called an *elementary automorphism*, as follows:

$$\varphi(x_t)=\begin{cases} x_t & \text{if } t\ne i\\ x_j+x_i & \text{if } t=i.\end{cases}$$

Let $k\{\Delta\} = A/J_\Delta$ be the indicator algebra of a simplicial complex $\Delta$. If $T = x_{i_1}x_{i_2}\cdots x_{i_n}$ is a squarefree monomial in $J_\Delta$, then we set $c(T) = \sum_{j=1}^n i_j$, and $c_n(J_\Delta) = \sum_T c(T)$ where the sum is taken over all squarefree monomials $T \in J_\Delta$ of degree $n$.

(3.3) THEOREM. *Let $k$ be a field of characteristic 2, $k\{\Delta\} = A/J_\Delta$ the indicator algebra of a simplicial complex $\Delta$, and $\varphi : A \to A$ an elementary automorphism.*

(a) *$A/\mathrm{in}(\varphi(J_\Delta))$ is again the indicator algebra of some simplicial complex $\Delta'$.*

(b) *$\Delta$ and $\Delta'$ have the same $f$-vector.*

(c) *$c_n(J_{\Delta'}) \le c_n(J_\Delta)$ for all $n$.*

(d) *If $\Delta$ is not a strongly stable simplicial complex, then $\varphi$ can be chosen such that $c_n(J_{\Delta'}) < c_n(J_\Delta)$ for some $n$.*

*Proof.* We assume that $\varphi$ maps $x_i$ to $x_j + x_i$, $j < i$, and leaves all the other variables unchanged.

(a) Since the characteristic of $k$ is 2, we have $\varphi(x_i^2) = x_j^2 + x_i^2$. It follows that $\varphi((x_1^2, x_2^2, \dots, x_v^2)) = (x_1^2, x_2^2, \dots, x_v^2)$, and hence any monomial in $A$ which is not squarefree belongs to $\varphi(J_\Delta)$. We fix an integer $n \ge 0$. Then it follows that $\varphi(J_\Delta)_n$ has a $k$-basis consisting of the set of all monomials $\mathcal{L}_n$ of degree $n$ which are not squarefree and of the set of elements $\varphi(T)$, $T \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of all squarefree monomials $T \in (J_\Delta)_n$.

Let $T \in \mathcal{M}_n$; then $\varphi(T) = T$ if $i \notin \mathrm{supp}(T)$, and $\varphi(T) = (x_jT/x_i) + T$ if $i \in \mathrm{supp}(T)$. We write $\mathcal{M}_n(1)$ for the set of all monomials $T \in \mathcal{M}_n$ such that either $i \notin \mathrm{supp}(T)$, or $i \in \mathrm{supp}(T)$ and $j \in \mathrm{supp}(T)$, or $i \in \mathrm{supp}(T)$ and $x_jT/x_i \in \mathcal{M}_n$. Then the monomials in $\mathcal{L}_n \cup \mathcal{M}_n(1)$ together with the binomials $(x_jT/x_i) + T$ with $T \in \mathcal{M}_n(2)$ form a $K$-basis of $\varphi(J_\Delta)_n$, where $\mathcal{M}_n(2)$ is the set of all monomials $T \in \mathcal{M}_n$ such that $i \in \mathrm{supp}(T)$, $j \notin \mathrm{supp}(T)$ and $x_jT/x_i \notin \mathcal{M}_n$.

Now we take the initial forms of this particular $k$-basis of $\varphi(J_\Delta)_n$. This yields the monomials $T \in \mathcal{L}_n \cup \mathcal{M}_n(1)$ together with the monomials $x_jT/x_i$ with $T \in \mathcal{M}_n(2)$. Their number equals $\dim_k \varphi(J_\Delta)_n$, and they are all different. Since, by Proposition (3.2;b), one has $\dim_k \varphi(J_\Delta)_n = \dim_k \mathrm{in}(\varphi(J_\Delta))_n$, we conclude that these monomials form a basis of $\mathrm{in}(\varphi(J_\Delta))_n$. In particular, $\mathrm{in}(\varphi(J_\Delta))$, is generated by the elements $x_1^2, x_2^2, \dots, x_v^2$ together with some squarefree monomials. These squarefree monomials define a required simplicial complex $\Delta'$.

(b) Both operations – the application of $\varphi$ as well as taking initial forms – have no effect on the Hilbert function.

(c) When we compare the sum for $c_n(J_\Delta)$ and that for $c_n(\mathrm{in}(J_\Delta))$, then we see that all summands coincide except those which correspond to the monomials $T \in \mathcal{M}_n(2)$. In the first sum, each $c(T)$ with $T \in \mathcal{M}_n(2)$ has to be replaced by $c(x_jT/x_i)$ in the second sum. Since $c(x_jT/x_i) < c(T)$, the inequality follows.

(d) The proof of (c) as above enables us to see that we have the strict inequality if some $\mathcal{M}_n(2)$ is not empty. Of course $\mathcal{M}_n(2)$ not only depends on $J_\Delta$, but also on the choice of $i$ and $j$. If we suppose that $\Delta$ is not a strongly stable simplicial complex, then there exists an element $T \in \mathcal{M}_n$ such that for some $i \in \operatorname{supp}(T)$ and some $j < i$ with $j \notin \operatorname{supp}(T)$, one has $(x_j T)/x_i \notin \mathcal{M}_n$. For this choice of $i$ and $j$ we have $\mathcal{M}_n(2) \neq \emptyset$. Q. E. D.

(3.4) COROLLARY. *Given a simplicial complex $\Delta$, there exists a simplicial complex $\Delta'$ with the same $f$-vector of $\Delta$ such that the ideal $I_{\Delta'}$ is strongly stable.*

*Proof.* Let $c(\Delta)$ denote the sum $\sum_{n=0}^{v} c_n(J_\Delta)$. If $\Delta$ is not strongly stable, then by Theorem (2.3;d) there exists a simplicial complex $\Delta'$ with $c(\Delta') < c(\Delta)$. Thus induction on $c(\Delta)$ yields the assertion. Q. E. D.

Our next aim is to strengthen Corollary (3.4), and to prove the Kruskal–Katona theorem in the following form.

(3.5) THEOREM. *Given a simplicial complex $\Delta$, there exists a unique lexsegment simplicial complex $\Delta'$ with the same $f$-vector.*

Thanks to Corollary (3.4), it suffices to prove Theorem (3.5) only for strongly stable simplicial complexes. This part of the proof follows from the same line of arguments as the corresponding proof of Bigatti [Big]. We need to introduce some notation, and to prove some lemmata. A proof of Theorem (3.5) will begin after Corollary (3.10).

Let $\Gamma$ denote the simplicial complex which consists of all subsets of the vertex set $\{x_1, x_2, \ldots, x_v\}$. Then

$$k\{\Gamma\} = k[x_1, x_2, \ldots, x_v]/(x_1^2, x_2^2, \ldots, x_v^2),$$

and the set of all squarefree monomials forms a basis of $k\{\Gamma\}$. Instead of studying an ideal $J \subset k[x_1, x_2, \ldots, x_v]$ generated by squarefree monomials, we may as well study its image $I$ in $k\{\Gamma\}$, and give it the same attributes as $J$. Thus, for example, we call $I$ strongly stable (or lexsegment) if so is $J$ (in the squarefree sense).

The obvious characterization of strongly stable ideals in $k\{\Gamma\}$ stated below is required in the proof of Theorem (3.9).

(3.6) LEMMA. *Let $I$ be a monomial ideal in $k\{\Gamma\}$, and write $I = I' + I''x_v$ where $I'$ and $I''$ are generated by monomials in the variables $x_1, x_2, \ldots, x_{v-1}$. Then the following conditions are equivalent:*

(i) *$I$ is strongly stable;*

(ii) *$I'$ and $I''$ are strongly stable, and $I''(x_1, x_2, \ldots, x_{v-1}) \subset I'$.*

If $I \subset k\{\Gamma\}$ is a monomial ideal and $1 \leq i \leq v$, then we define $G(I;i)$, $m_i(I)$ and $m_{\leq i}(I)$ as follows:

$$G(I;i) = \{T \in G(I) \; ; \; m(T) = i\}, \quad m_i(I) = \sharp(G(I;i)), \quad m_{\leq i}(I) = \sum_{j \leq i} m_j(I).$$

(3.7) LEMMA. *Let $I \subset k\{\Gamma\}$ be a strongly stable ideal with all generators of degree $n(< v)$.*

(a) *We have the equalities*

$$\dim_k(I_{n+1}) = \sum_{i=1}^{v-1} m_i(I)(v-i) = \sum_{i=1}^{v-1} m_{\leq i}(I).$$

(b) *Let $I_{\langle n+1 \rangle}$ denote the ideal in $k\{\Gamma\}$ which is generated by all squarefree monomials in $I$ of degree $n+1$. Then, for all $i$, we have*

$$m_i(I_{\langle n+1 \rangle}) = m_{\leq i-1}(I).$$

*Proof.* (a) The second equality of the statement is obvious. In order to prove the first formula, we note that

(3.7.1) The map $G(I;i) \to G(I;i)x_j$, $T \mapsto Tx_j$, is injective for all $j > i$;

(3.7.2) The sets $G(I;i)x_j$, $j = i+1, \dots, v$, are pairwise disjoint.

The above fact (3.7.1) and (3.7.2) imply

$$\sharp(G(I;i)\{x_{i+1}, \dots, x_v\}) = m_i(I)(v-i).$$

Next we claim that

(3.7.3) If $j < i$, then $G(I;i)x_j \subset \bigcup_{t=1}^{i-1} G(I;t)x_i$.

In fact, let $T \in G(I;i)$. We may assume that $Tx_j \neq 0$. Then $j \notin \operatorname{supp}(T)$, and $Tx_j = (Tx_j/x_i)x_i$. Since $I$ is strongly stable, we have that $Tx_j/x_i \in I$. If $m(Tx_j/x_i) = t$, then clearly $t < i$ and $Tx_j/x_i \in G(I;t)$.

Since $G(I) = \bigcup_i G(I;i)$, the claim (3.7.3) implies that

$$G(I)\{x_1, x_2, \dots, x_v\} = \bigcup_i G(I;i)\{x_{i+1}, \dots, x_v\}.$$

Hence it suffices to show that

$$G(I;i)\{x_{i+1}, \dots, x_v\} \bigcap G(I;j)\{x_{j+1}, \dots, x_v\} = \emptyset$$

for all $i$ and $j$ with $i \neq j$. However, this is clear; because if $T \in G(I;i)\{x_{i+1},\ldots,x_v\}$, then we have $m(T/x_{m(T)}) = i$.

(b) From (3.7.3) we deduce that $G(I_{\langle n+1\rangle};i) = \bigcup_{t=1}^{i-1} G(I;t)x_i$. Hence, it follows that $m_i(I_{\langle n+1\rangle}) = \sharp(G(I_{\langle n+1\rangle};i)) = \sum_{t=1}^{i-1} m_t(I) = m_{\leq i-1}(I)$. Q. E. D.

For the proof of the next result we need some preparation. Let $n < v$ and write $\mathcal{N}_n$ for the set of all (squarefree) monomials of degree $n$ in $k\{\Gamma\}$. If $\mathcal{N} \subset \mathcal{N}_n$ we write $\min(\mathcal{N})$ for the smallest monomial $T \in \mathcal{N}$ (with respect to the lexicographic order). Furthermore we define a map $\alpha : \mathcal{N}_n \to \mathcal{N}_n$ by setting $\alpha(T) = T$, if $v \notin \operatorname{supp}(T)$, and $\alpha(T) = (x_jT)/x_v$ if $v \in \operatorname{supp}(T)$, where $j$ is the largest integer $< v$ which does not belong to $\operatorname{supp}(T)$.

(3.8) Lemma. (a) *The map $\alpha : \mathcal{N}_n \to \mathcal{N}_n$ is order preserving, that is, for $T, T' \in \mathcal{N}_n$, $T \leq_{\text{lex}} T'$, one has $\alpha(T) \leq_{\text{lex}} \alpha(T')$.*

(b) *Let $I = I' + I''x_v$ be a strongly stable ideal with generators of degree $n < v$, where $I'$ and $I''$ are generated by monomials in the elements $x_1, x_2, \ldots, x_{v-1}$. Then $\alpha(\min(G(I))) = \min(G(I'))$.*

*Proof.* (a) Let $T$ and $T'$ be two monomials of degree $n$ with $T \leq_{\text{lex}} T'$ and $m(T) = m(T') = v$, say $T = x_{i_1}\cdots x_{i_{n-1}}x_v$ and $T' = x_{i'_1}\cdots x_{i'_{n-1}}x_v$ with $1 \leq i_1 < i_2 < \cdots < i_{n-1} < v$ and $1 \leq i'_1 < i'_2 < \cdots < i'_{n-1} < v$. Then there exists an integer $t$ with $1 \leq t \leq n-1$ such that $i_1 = i'_1, \ldots, i_{t-1} = i'_{t-1}$ and $i_t > i'_t$. Let $j$ be the largest integer $< v$ which is not in $\operatorname{supp}(T)$, and define $j'$ similarly for $T'$. Since $i_t > i'_t$, there is at least one 'gap' in the sequence $i'_t, \ldots, i'_{n-1}, v$. Thus $j' > i'_t$. Hence if $j \geq i_t$, then the first indices of the factors of $\alpha(T)$ and $\alpha(T')$ in which they differ are again $i_t$ and $i'_t$, and the inequality is preserved. On the other hand, if $j < i_t$, then we must have

$$T = x_{i_1}\cdots x_{i_{t-1}}x_{v-n+t}x_{v-n+t+1}\cdots x_{v-1}x_v,$$

and $j = i_t - 1 = v - n + t - 1$ since $i_{t-1} = i'_{t-1} < i'_t < i_t$. That is, the factors 'after' $x_{i_{t-1}}$ have the highest possible indices. It is then obvious that $\alpha(T) \leq_{\text{lex}} \alpha(T')$ as desired. By the similar way one treats the case $m(T') < m(T) = v$, while if $m(T) < m(T') = v$ one has $\alpha(T) = T \leq_{\text{lex}} T' \leq_{\text{lex}} \alpha(T')$.

(b) It follows from the above result (a) that $\alpha(\min(G(I))) \leq_{\text{lex}} \alpha(\min(G(I'))) = \min(G(I'))$ since $\min(G(I)) \leq_{\text{lex}} \min(G(I'))$. On the other hand, since $I$ is strongly stable, $\alpha(\min(G(I))) \in G(I')$, which implies the reverse inequality. Q. E. D.

(3.9) Theorem. *Let $I$ and $J$ be monomial ideals in $k\{\Gamma\}$ with generators in degree $n$. Suppose that $I$ is strongly stable, that $J$ is lexsegment, and that $\dim_k J_n \leq \dim_k I_n$. Then*

$$m_{\leq i}(J) \leq m_{\leq i}(I)$$

*for all $i$.*

*Proof.* We proceed by induction on $v$, the number of variables. The inequality $m_{\leq v}(J) \leq m_{\leq v}(I)$ is just our hypothesis. In order to prove it for $i < v$, we write $J = J' + J''x_v$ and $I = I' + I''x_v$ with $J'$, $J''$, $I'$ and $I''$ ideals generated by monomials in $x_1, x_2, \dots, x_{v-1}$. It is clear that $J'$ is lexsegment, and that $I'$ is strongly stable. Hence if we show that $\dim_k(J')_n \leq \dim_k(I')_n$, we may apply our induction hypothesis, and the assertion follows immediately.

It may be assumed that $I'$ and $I''$ are lexsegment. In fact, let $I^*$ (resp. $I^{**}$) be the lexsegment ideal generated by monomials in $x_1, x_2, \dots, x_{v-1}$ of degree $n$ (resp. $n-1$) such that $\dim_k(I^*)_n = \dim_k(I')_n$ (resp. $\dim_k(I^{**})_{n-1} = \dim_k(I'')_{n-1}$) and set $\tilde{I} = I^* + I^{**}x_v$. Then $\tilde{I}$ is also strongly stable. To see this we apply Lemma (3.6), and hence have to show that $I^{**}(x_1, x_2 \dots, x_{v-1}) \subset I^*$. By Lemma (3.7) and our induction hypothesis for $I^*$ and $I^{**}$, we have

$$\begin{aligned} \dim_k(I^{**})_{n-1}(x_1, x_2 \dots, x_{v-1}) &= \sum_{i=1}^{v-2} m_{\leq i}(I^{**}) \leq \sum_{i=1}^{v-2} m_{\leq i}(I'') \\ &= \dim_k(I'')_{n-1}(x_1, x_2 \dots, x_{v-1}). \end{aligned}$$

Since $I$ is strongly stable, Lemma (3.6) implies that

$$\dim_k(I'')_{n-1}(x_1, x_2, \dots, x_{v-1}) \leq \dim_k(I')_n = \dim_k(I^*)_n.$$

Thus $I^{**}(x_1, x_2 \dots, x_{v-1}) \subset I^*$ since $I^{**}(x_1, x_2 \dots, x_{v-1})$ and $I^*$ both are lexsegment ideals.

Recall that we are in the following situation: $J = J' + J''x_v$ lexsegment, and $I = I' + I''x_v$ strongly stable as before, but in addition $I'$ and $I''$ lexsegment. Assuming $\dim_k J_n \leq \dim_k I_n$, we want to show that $\dim_k(J')_n \leq \dim_k(I')_n$. Thanks to Lemma (3.8) we have

$$\min(G(I')) = \alpha(\min(G(I))) \leq_{\text{lex}} \alpha(\min(G(J))) = \min(G(J')).$$

Since the ideals $J'$ and $I'$ are lexsegment, the required inequality follows. Q. E. D.

The above Theorem (3.9) together with Lemma (3.7) guarentees

(3.10) COROLLARY. *Let $I$ and $J$ be monomial ideals in $k\{\Gamma\}$ with generators of degree $n$. Suppose that $I$ is strongly stable, that $J$ is lexsegment, and that* $\dim_k J_n \leq \dim_k I_n$. *Then*

$$\dim_k J_{n+1} \leq \dim_k I_{n+1}.$$

We are now in the position to give a proof of Theorem (3.5). As we already remarked, we may assume that $\Delta'$ is strongly stable. Let $I = I_\Delta k\{\Gamma\}$, and suppose that $I_n \neq 0$ while $I_{n-1} = 0$. We choose the lexsegment ideal $J$ generated by (squarefree) monomials of degree $n$ such that $\dim_k J_n = \dim_k I_n$. It follows from

Corollary (3.10) that $\dim_k J_{n+1} \leq \dim_k I_{n+1}$. Since $J_{n+1}$ is spanned by a lexsegment, we may add some monomials of degree $n+1$ to $J$ in the lexicographic order such that the new ideal $J'$ is a lexsegment ideal, and such that $\dim_k (J')_t = \dim_k I_t$ for $t = n, n+1$. Again $\dim_k (J')_{n+2} \leq \dim_k I_{n+2}$, and we may proceed as before. Hence a simple induction argument concludes the proof of the existence. The arguments in this proof also show that two lexsegment ideals with the same Hilbert function must coincide, and this yields the uniqueness. Q. E. D.

(3.11) COROLLARY *Let $\Delta$ be a simplicial complex, and $\Delta'$ the unique lexsegment simplicial complex with the same $f$-vector. Then $\mu(I_\Delta) \leq \mu(I_{\Delta'})$*

*Proof.* The proof of Theorem (3.5) shows that this inequality holds if $\Delta$ is strongly stable. On the other hand, if we compare a general simplicial complex with the corresponding strongly stable simplicial complex as constructed in Corollary (3.4), then the same inequality for the number of generators holds. This follows from the proof of Theorem (3.3) and from Proposition (3.2;b). Q. E. D.

## §4. Betti numbers of indicator algebras of strongly stable simplicial complexes

We now describe the Koszul homology of the indicator algebra $k\{\Delta\}$ of a strongly stable simplicial complex $\Delta$ and compute the Betti numbers of $k\{\Delta\}$. Let $\Delta$ denote a simplicial complex on the vertex set $V = \{x_1, x_2, \dots, x_v\}$ and set $I = I_\Delta$. If $\tau$ is a subset of $\{1, 2, \dots, v\}$, then we define

$$I_\tau = I : x_\tau \quad \text{where} \quad x_\tau = \prod_{i \in \tau} x_i.$$

Thus, in particular, $I_\emptyset = I$. Notice that $I_\tau$ is generated by the monomials

$$T/\gcd(x_\tau, T), \qquad T \in G(I).$$

Hence, $I_\tau \neq (1)$ if and only if $\{x_i \,;\, i \in \tau\} \in \Delta$. We identify $\tau \subset \{1, 2, \dots, v\}$ with $\{x_i \,;\, i \in \tau\} \subset V$ and write, e.g., $\tau \in \Delta$ instead of $\{x_i \,;\, i \in \tau\} \in \Delta$.

(4.1) PROPOSITION. *Let $\Delta$ be a strongly stable simplicial complex. Then a basis of $H_i(x_1, x_2, \dots, x_v; k\{\Delta\})$ is given by the homology classes of the cycles*

$$T' e_\sigma \wedge e_{m(T)} \wedge x_\tau e_\tau, \qquad T \in G(I_\tau),$$

*where $\sigma$, $\tau$ and $\mathrm{supp}(T)$ are pairwise disjoint, $\sharp(\sigma \cup \tau) = i - 1$, $\max(\sigma) < m(T)$ and $\tau \in \Delta$, and furthermore of the cycles $x_\tau e_\tau$ for all $\tau \in \Delta$ with $\sharp(\tau) = i$.*

*Proof.* Let $R_j$ denote $k[x_1, x_2, \dots, x_v]/(I_\Delta, x_1^2, x_2^2, \dots, x_j^2)$ for each $0 \leq j \leq v$. Thus, $R_0 = k[\Delta]$ and $R_v = k\{\Delta\}$. By induction on $j$ we prove the statement as follows: The Koszul homology $H_i(R_j) := H_i(x_1, x_2, \dots, x_v; R_j)$ is given by the homology classes of the cycles stated in Proposition (4.1), satisfying the additional condition $\max(\tau) \le j$. We write $\mathcal{Z}_j$ for this set of cycles. Let $I = I_\Delta$.

If $j = 0$, then $\tau = \emptyset$; thus the assertion follows from Proposition (2.2). Let $j > 0$ and assume that our claim is true for $j - 1$. We then consider the exact sequence

$$0 \longrightarrow (x_j^2) \longrightarrow R_{j-1} \longrightarrow R_j \longrightarrow 0.$$

As in Section 2, we write $\epsilon_1, \epsilon_2, \dots, \epsilon_v$ for the canonical basis of $\mathbf{Z}^v$. Then

$$(x_j^2) \simeq R^*(-2\epsilon_j),$$

where

$$\begin{aligned} R^* &= k[x_1, x_2, \dots, x_v]/((I, x_1^2, \dots, x_{j-1}^2) : x_j^2) \\ &= k[x_1, x_2, \dots, x_v]/(I_{\{j\}}, x_1^2, \dots, x_{j-1}^2). \end{aligned}$$

Hence, we have the long exact homology sequence

$$\cdots \longrightarrow H_i(R^*)(-2\epsilon_j) \longrightarrow H_i(R_{j-1}) \longrightarrow H_i(R_j) \longrightarrow \cdots.$$

We claim that the homomorphism $H_i(R^*)(-2\epsilon_j) \to H_i(R_{j-1})$ is zero for all $i \ge 0$. In fact, if we choose $a \in \mathbf{Z}^n$, then we obtain the map

$$H_i(R^*)_{a-2\epsilon_j} \longrightarrow H_i(R_{j-1})_a.$$

We may assume that $H_i(R_{j-1})_a \neq 0$. By our induction hypothesis we know the cycles of $H_i(R_{j-1})$. In particular, we conclude that the $j$-th component of $a$ is $\le 1$. But this implies that $a - 2\epsilon_j \notin \mathbf{N}^v$, hence $H_i(R^*)_{a-2\epsilon_j} = 0$ as desired. Thus, for all $i > 0$, we obtain the exact sequence

$$0 \longrightarrow H_i(R_{j-1}) \longrightarrow H_i(R_j) \longrightarrow H_{i-1}(R^*)(-2\epsilon_j) \longrightarrow 0.$$

Again, by assumption of induction, the homology classes of the cycles $\mathcal{Z}_{j-1}$ form a basis of $H_i(R_{j-1})$. Thus the homology classes of these cycles form a part of a basis of $H_{i-1}(R_j)$. The rest of a basis of $H_i(R_j)$ is formed by the preimages of a basis of $H_{i-1}(R^*)(-2\epsilon_j)$. If $I_{\{j\}} = (1)$, then $R^* = 0$, and the assertion follows, since then there is no $\tau$ with $\max(\tau) = j$ and $I_\tau \neq (1)$, so that $\mathcal{Z}_{j-1} = \mathcal{Z}_j$. If $I_{\{j\}} \neq (1)$, then $I_{\{j\}}$ is a strongly stable ideal in the ring $k[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_v]$. Thus we may apply our induction hypothesis, and find that $H_{i-1}(R^*)$ is generated by the homology classes of all the cycles described in Proposition (4.1), satisfying the additional conditions $\sigma, \tau \subset \{1, \dots, j-1, j+1, \dots, v\}$, $\max(\tau) \le j - 1$ and $\sharp(\sigma \cup \tau) = i - 2$. Let $\mathcal{Z}$ denote this set of cycles. Then, for every $z \in \mathcal{Z}$, the element $z \wedge x_j e_j$ is a cycle in $H_i(R_j)$ whose homology class is mapped to the homology class $\pm[z]$ in $H_{i-1}(R^*)$. Thus the homology classes of the cycles $\mathcal{Z}_{j-1}$ together with the homology classes of the cycles $z \wedge x_j e_j$, $z \in \mathcal{Z}$, form a basis of the cycles of $H_i(R_j)$. This proves our assertion, since $\mathcal{Z}_j = \mathcal{Z}_{j-1} \cup \{z \wedge x_j e_j \ ; \ z \in \mathcal{Z}\}$. Q. E. D.

Let us consider the simple example $I = I_\Delta = (x_1 x_2)$. Then $I_{\{1\}} = (x_2)$, $I_{\{2\}} = (x_1)$ and $I_{\{1,2\}} = (1)$. Thus $H_1$ is generated by the homology classes of the cycles:

$$x_1 e_2, \quad x_1 e_1, \quad \text{and} \quad x_2 e_2,$$

and $H_2$ by

$$e_2 \wedge x_1 e_1, \quad \text{and} \quad e_1 \wedge x_2 e_2.$$

(4.2) COROLLARY. *Let $\Delta$ be a strongly stable simplicial complex with $f$-vector $(f_0, f_1, \ldots, f_{d-1})$. Then*

$$\beta_i(J_\Delta) = \sum_{\tau \in \Delta} ( \sum_{T \in G(I_\tau)} \binom{m(T) - \deg T - |\tau|_T}{i - \sharp(\tau)} ) + f_i,$$

*where $|\tau|_T$ is the number of elements of $\tau$ which are less than $m(T)$. In particular, all the Betti numbers of $J_\Delta$ do not depend on the characteristic of the field.*

Let $\Delta$ be an arbitrary simplicial complex. Since the ideals $I_\Delta$ and $J_\Delta$ are graded ideals, they have a graded free resolution, and we may, as in Section 1, discuss the graded Betti numbers $\beta_{i_j}$ of these ideals. Let $\Delta''$ denote the lexsegment simplicial complex with same $f$-vector of $\Delta$.

(4.3) CONJECTURE.

$$\beta_{i_j}(I_\Delta) \leq \beta_{i_j}(I_{\Delta''}) \quad \text{and} \quad \beta_{i_j}(J_\Delta) \leq \beta_{i_j}(J_{\Delta''})$$

*for all $i$ and $j$.*

The first inequalities of Conjecture (4.3) is the squarefree version of Bigatti–Hulett theorem on the upper bounds for the Betti numbers of a given Hilbert function. On the other hand, the second inequalities of Conjecture (4.3) have an interesting combinatorial consequence as follows. Recall from, e.g., [H1] or [Bru–Her] that for given positive integers $a$ and $i$, there exists a unique representation of $a$ of the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$
$$a_i > a_{i-1} > \cdots > a_j \geq j \geq 1.$$

Then, we define

$$\partial_{i-1}(a) = \binom{a_i}{i-1} + \binom{a_{i-2}}{i-2} + \cdots + \binom{a_j}{j-1}.$$

The second inequalities of Conjecture (4.3) imply the following: Let $\Delta$ be an arbitrary simplicial complex with $f$-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ and write $n_i(\Delta)$ for the number of maximal faces $\sigma$ of $\Delta$ with $\sharp(\sigma) = i+1$. Then

$$n_{i-1}(\Delta) \leq f_{i-1} - \partial_i(f_i)$$

for every $1 \leq i \leq d-1$ (and $n_{d-1}(\Delta) = f_{d-1}$). Moreover, if $\Delta$ is lexsegment, then the equality holds for every $1 \leq i \leq d-1$.

We conclude this paper with some results related with Conjecture (4.3).

(4.4) THEOREM. *Let $\Delta$ be an arbitrary simplicial complex, $\Delta'$ a strongly stable simplicial complex with same $f$-vector as $\Delta$, and $\Delta''$ the lexsegment simplicial complex with same $f$-vector as $\Delta$. Then, for every $i$ and $j$, we have the inequality*

$$\beta_{i_j}(I_{\Delta'}) \leq \beta_{i_j}(I_{\Delta''}).$$

*Moreover, if the base field is of characteristic* 0 *or* 2, *then, for every $i$ and $j$, we have the inequality*

$$\beta_{i_j}(J_{\Delta}) \leq \beta_{i_j}(J_{\Delta'}).$$

*Proof.* First, we prove the second inequality for $\beta_{i_j}(J_\Delta)$ and $\beta_{i_j}(J_{\Delta'})$. The Betti numbers may depend on the field characteristic. To express this dependence we write, e.g., $\beta^k_{i_j}$ for $\beta_{i_j}$. Quite generally, if $I$ is an ideal whose generators have integer coefficients, then $\beta^k_{i_j}(I) \leq \beta^{k'}_{i_j}(I)$ for any two fields with $\operatorname{char}(k) = 0$ and $\operatorname{char}(k') > 0$. In view of this fact, it suffices to show the inequality in the case of characteristic 2. Now, Theorem (3.3) guarantees that $J_{\Delta'}$ is obtained from $J_\Delta$ by taking initial forms, which implies the desired inequality. See, e.g., [B–H–V].

We next give a proof of the first inequality for $\beta_{i_j}(I_{\Delta'})$ and $\beta_{i_j}(I_{\Delta''})$. Let $I$ be a squarefree strongly stable ideal and set

$$G(I)_n = \{T \in G(I) \ ; \ \deg T = n\}.$$

Then it follows from Theorem (2.1) (or Proposition (2.2)) that

$$\beta_{i_j}(I) = \sum_{T \in G(I)_{j-i+1}} \binom{m(T) - j + i - 1}{i}.$$

We split this sum into a difference of two sums $C$ and $D$. Let $I_{\langle n \rangle}$ denote the ideal in $k\{\Gamma\}$ which is generated by all (squarefree) monomials in $I$ of degree $n$. Then

$$G(I)_{n+1} = G(I_{\langle n+1 \rangle}) - G(I_{\langle n \rangle})\{x_1, x_2, \ldots, x_v\},$$

and hence $\beta_{i_j}(I) = C - D$, where

$$C = \sum_{T \in G(I_{\langle j-i+1 \rangle})} \binom{m(T) - j + i - 1}{i},$$

and $D$ is the same sum taken over all monomials $T \in G(I_{\langle j-i\rangle})\{x_1, x_2, \ldots, x_v\}$. Then, with the same notation as in Section 3, we have

$$\begin{aligned}
C &= \sum_{t=1}^{v} \sum_{T \in G(I_{\langle j-i+1\rangle};t)} \binom{t-j+i-1}{i} \\
&= \sum_{t=1}^{v} m_t(I_{\langle j-i+1\rangle}) \binom{t-j+i-1}{i} \\
&= \sum_{t=1}^{v} (m_{\leq t}(I_{\langle j-i+1\rangle}) - m_{\leq t-1}(I_{\langle j-i+1\rangle})) \binom{t-j+i-1}{i} \\
&= m_{\leq v}(I_{\langle j-i+1\rangle}) \binom{v-j+i-1}{i} \\
&\quad + \sum_{t=1}^{v-1} (m_{\leq t}(I_{\langle j-i+1\rangle}) [\binom{t-j+i-1}{i} - \binom{(t+1)-j+i-1}{i}] \\
&= m_{\leq v}(I_{\langle j-i+1\rangle}) \binom{v-j+i-1}{i} \\
&\quad - \sum_{t=1}^{v-1} m_{\leq t}(I_{\langle j-i+1\rangle}) \binom{t-j+i-1}{i-1}.
\end{aligned}$$

Furthermore, it follows from Lemma (3.7;b) that

$$D = \sum_{t=1}^{v} m_{\leq t-1}(I_{\langle j-i\rangle}) \binom{t-j+i-1}{i}.$$

Now let $J$ denote the squarefree lexsegment ideal with the same Hilbert function of $I$. Then, the number of generators of $I_{\langle n\rangle}$ and $J_{\langle n\rangle}$ are equal for all $n$. Thus, $m_{\leq v}(I_{\langle n\rangle}) = m_{\leq v}(J_{\langle n\rangle})$, and it follows from Theorem (3.9) that $m_{\leq i}(J_{\langle n\rangle}) \leq m_{\leq i}(I_{\langle n\rangle})$ for all $i$. Therefore, if we compare the above expression for $\beta_{i_j}(I)$ with that of $\beta_{i_j}(J)$, the required inequality follows. Q. E. D.

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