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CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE INVARIANT χ

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In this note, we shall present a tensor-categorical interpretation of the invariant χ for subfactors, which can be applied to compute the invariant for group-subgroup subfactors.

Automorphisms in Subfactors

Given a subfactor $N \subset M$, set

$$Aut(M, N) = \{\theta \in Aut(M); \theta(N) = N\},$$

$$Int(M, N) = \{Ad\, u \in Aut(M, N); u \text{ is a unitary in } N\}.$$

Each $\theta \in AUt(M, N)$ is inductively extended to automorphisms of the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

by

$$\theta(e_i) = e_i, \quad i = 1, 2, 3, \cdots$$

and hence induces

$Loi(\theta) =$ the family of induced automorphisms on $N' \cap M \subset N' \cap M_1 \subset \cdots$.

Remark. We can use automorphisms on $M' \cap M_1 \subset M' \cap M_1 \subset \cdots$ as well, which contains the equivalent information.

Theorem (Popa, Loi). *Let M, N be AFD II_1-factors and $N \subset M$ be amenable. Then for $\theta \in Aut(M, N)$,*
(i) *θ is centrally trivial iff θ is inner at some M_k, i.e., $\exists 0 \neq u \in M_k$ such that $\theta(x)u = ux$ for $x \in M$.*
(ii) *$Loi(\theta) = 1$ iff $\theta \in \overline{Int(M, N)}$.*

According to Y. Kawahigashi, we define the group

$$\chi(M, N) = \frac{Cnt(M, N) \cap \overline{Int(M, N)}}{Int(M, N)}$$

as the χ-invariant for subfactors.

Theorem (Kawahigashi). *For subfactors of index* < 4,

$$\chi = \begin{cases} \mathbb{Z}_2 & \text{for } A_{2n+1}\ (n \geq 2) \text{ and } E_6, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } A_3, \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{for } D_4, \\ 0 & \text{otherwise.} \end{cases}$$

Interpretations with bimodules

For a factor N, the correspondance

$$\alpha \in Aut(N) \rightsquigarrow X_\alpha = {}_N L^2(N) \alpha_N$$

induces

$$X_\alpha^* = X_{\alpha^{-1}}$$
$$X_\alpha \otimes^N X_\beta = X_{\alpha\beta}.$$

Here the right N-action in X_α is modified by α compared to the standard bimodule $L^2(N)$.

The bimodule X_α satisfies $X_\alpha \otimes X_\alpha^* \cong {}_N L^2(N)_N$. The converse is not always true: Let R be an AFD II_1-factor, e be a non-trivial projection in R and $v : R \to eRe$ be an isomorphism. Then the bimodule $X = {}_R eL^2(R)_R$ gives an example, where the right action is induced from the isomorphism φ.

Theorem. *Let X be an N-N bimodule such that $X \otimes X^* \cong L^2(N)$ and consider one of the following cases.*
(i) *N is properly infinite.*
(ii) *X is a descendent of an irreducible bimodule Z of finite index.*
Then $\exists\, \alpha \in Aut(N)$ such that $X \cong X_\alpha$.

The bimodule $X_\alpha = {}_N L^2(N) \alpha_N$ is simply denoted by α in the following.
For a bimodule ${}_A X_B$ and $\alpha \in Aut(A)$, $\beta \in Aut(B)$, set

$$\alpha X \beta = \alpha \otimes X \otimes \beta$$

and

$$Out(A) \times_X Out(B) = \{([\alpha], [\beta]) \in Out(A) \times Out(B); \alpha X \cong X\beta\},$$

a subgroup of $Out(A) \times Out(B)$.

Theorem (Kosaki, Choda-Kosaki). *For $Z = {}_N L^2(M)_M$ with $N \subset M$ irreducible, we have*
(i) $Out(N) \times_Z Out(M) \cong Aut(M, N)/Int(M, N)$.
(ii) *θ is inner at some M_k iff ${}_M L^2(M) \theta_M$ appears in $Z^* Z \cdots Z^* Z$ $(= (Z^* Z)^k)$.*

$\because$) (ii) Use $M_3 = End(ZZ^*Z_M)$ and the Frobenius reciprocity

$$Hom(\theta Z^* Z Z^*, Z^* Z Z^*) \cong Hom(L^2(M)\theta, (Z^* Z)^3)$$

for example. $\square$

Theorem (Goto). *$Loi(\theta) = 1$ implies $[\theta] = ([\alpha], [\beta]) \in Aut(M,N)/Int(M,N)$ is in the center of fusion algebra, i.e.,*

$$\alpha X \cong X\alpha, \quad \beta Y \cong Y\beta, \quad \alpha Z' \cong Z'\beta$$

for descendants ${}_N X_N$, ${}_M Y_M$ and ${}_N Z'_M$ of Z.

Conversely the centrality in the fusion algebra forces the triviality of Loi invariant as long as the principal graph (or the dual principal graph) is multiplicity-free.

Theorem. *The following are equivalent.*

(i) $Loi(\theta) = 1$.

(ii) *For each bimodule X in the tensor category generated by Z, we can find an isomorphism $I_X : X \to \theta X \theta^{-1}$ such that $I_{X^*} = \overline{I_X}$, $I_{XY} = I_X \otimes I_Y$, and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{I_X} & \theta X \theta^{-1} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \theta T \theta^{-1}} \\ Y & \xrightarrow[I_Y]{} & \theta Y \theta^{-1} \end{array}$$

commutes for $T \in Hom(X,Y)$.

Applications to $\chi(G,H)$

For a subgroup $H \subset G$ of a finite group G with an outer action on an AFD II_1-factor, set

$$\chi(G,H) = \chi(R \rtimes G, R \rtimes H).$$

According to [KY], irreducible bimodules generated by ${}_{R\rtimes H}L^2(R \rtimes G)_{R\rtimes G}$ are parametrized by

$$\begin{aligned} R \rtimes G\text{-}R \rtimes G : &\quad \widehat{G} \\ R \rtimes H\text{-}R \rtimes G : &\quad \widehat{H} \\ R \rtimes H\text{-}R \rtimes H : &\quad \coprod_{\dot{a} \in H\backslash G/H} \widehat{H \cap aHa^{-1}}. \end{aligned}$$

Note that the tensor category of $R \rtimes H$-$R \rtimes H$ bimodules contains the Tannaka dual of H as a subcategory. With this description, we can deduce

$$Cnt(M,N)/Int(M,N) \cong \Xi \times (N_G(H)/H),$$

where

$$\Xi = \{(\chi, \eta) \in H^* \times G^*; \chi = \eta|_H\}$$

and H^* and G^* refer to the group of 1-dimensional representations.

Taking the restriction of centralizing morphisms to the Tannaka dual of H, we can deduce the following.

Theorem. *We have*

$$\Xi \times Z(G)H/H \subset \chi(G,H) \subset \Xi \times \{\dot{c} \in C_G(H)H/H; \dot{c} \text{ acts trivially on } H\backslash G/H\},$$

where $C_G(H)$ denotes the centralizer of H in G and $\dot{a} \in N_G(H)/H$ acts on $H\backslash G/H$ by

$$HgH \mapsto Haga^{-1}H.$$

Corollary.
(i) $\chi(G,\{e\}) \cong G^* \times Z(G)$.
(ii) $\chi(A \rtimes H, H) \cong \{(\chi,\eta) \in A^* \times (A \rtimes H)^*; \chi = \eta|_H\} \times A^H$, *where A is an abelian group and $A^H = \{a \in A; hah^{-1} = a,$ for all $h \in H\}$.*
(iii) $\chi(S_n, S_k) \cong \mathbb{Z}_2$.
(iv) $\chi(A_n, A_k) = \{e\}$.

References

[CK] M. Choda and H. Kosaki, *Strongly outer actions for an inclusion of factors*, J. Funct. Anal. **122** (1994), 315–332.

[Go] S. Goto, *Commutativity of automorphisms of subfactors modulo inner automorphisms*, Proc. Amer. Math. Soc., to appear.

[Ka] Y. Kawahigashi, *Centrally trivial automorphisms and an analogue of Connes' $\chi(M)$ for subfactors*, Duke Math. J. **71** (1993), 93–118.

[Ko] H. Kosaki, *Automorphisms in the irreducible decompositions of sectors*, Quantum and non-commutative analysis (H. Araki et al., eds.), Kluwer Academic, 1993, pp. 305–316.

[GSTC] S. Yamagami, *Group symmetry in tensor categories*, preprint (1995).