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<th>CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE INVARIANT $\chi$(Bimodules in Operator Algebras)</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 936: 46-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60031">http://hdl.handle.net/2433/60031</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE INVARIANT $\chi$

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In this note, we shall present a tensor-categorical interpretation of the invariant $\chi$ for subfactors, which can be applied to compute the invariant for group-subgroup subfactors.

Automorphisms in Subfactors

Given a subfactor $N \subset M$, set

$$Aut(M, N) = \{ \theta \in Aut(M); \theta(N) = N \},$$

$$Int(M, N) = \{ Ad u \in Aut(M, N); u \text{ is a unitary in } N \}.$$  

Each $\theta \in Aut(M, N)$ is inductively extended to automorphisms of the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset \ldots$$

by

$$\theta(e_i) = e_i, \quad i = 1, 2, 3, \ldots$$

and hence induces

$$Loi(\theta) = \text{the family of induced automorphisms on } N' \cap M \subset N' \cap M_1 \subset \ldots.$$  

Remark. We can use automorphisms on $M' \cap M_1 \subset M' \cap M_1 \subset \ldots$ as well, which contains the equivalent information.

Theorem (Popa, Loi). Let $M, N$ be AFD $II_1$-factors and $N \subset M$ be amenable. Then for $\theta \in Aut(M, N)$,

(i) $\theta$ is centrally trivial iff $\theta$ is inner at some $M_k$, i.e., $\exists 0 \neq u \in M_k$ such that $\theta(x)u = ux$ for $x \in M$.

(ii) $Loi(\theta) = 1$ iff $\theta \in Int(M, N)$.

According to Y. Kawahigashi, we define the group

$$\chi(M, N) = \frac{Cnt(M, N) \cap Int(M, N)}{Int(M, N)}$$

as the $\chi$-invariant for subfactors.
Theorem (Kawahigashi). For subfactors of index $< 4$

\[ \chi = \begin{cases} 
Z_2 & \text{for } A_{2n+1} \ (n \geq 2) \text{ and } E_6, \\
Z_2 \oplus Z_2 & \text{for } A_3, \\
Z_3 \oplus Z_3 & \text{for } D_4, \\
0 & \text{otherwise}. 
\end{cases} \]

Interpretations with bimodules

For a factor $N$, the correspondence

\[ \alpha \in Aut(N) \mapsto X_\alpha = NL^2(N)\alpha_N \]

induces

\[ X_\alpha^* = X_{\alpha^{-1}}, \]
\[ X_\alpha \otimes^N X_\beta = X_{\alpha\beta}. \]

Here the right $N$-action in $X_\alpha$ is modified by $\alpha$ compared to the standard bimodule $L^2(N)$.

The bimodule $X_\alpha$ satisfies $X_\alpha \otimes X_\alpha^* \cong NL^2(N)_N$. The converse is not always true: Let $R$ be an AFD $\text{II}_1$-factor, $\epsilon$ be a non-trivial projection in $R$ and $\nu : R \to \epsilon Re$ be an isomorphism. Then the bimodule $X = \epsilon eL^2(R)\epsilon$ gives an example, where the right action is induced from the isomorphism $\varphi$.

Theorem. Let $X$ be an $N-N$ bimodule such that $X \otimes X^* \cong L^2(N)$ and consider one of the following cases.

(i) $N$ is properly infinite.

(ii) $X$ is a descendent of an irreducible bimodule $Z$ of finite index.

Then $\exists \alpha \in \text{Aut}(N)$ such that $X \cong X_\alpha$.

The bimodule $X_\alpha = NL^2(N)\alpha_N$ is simply denoted by $\alpha$ in the following.

For a bimodule $_AX_B$ and $\alpha \in \text{Aut}(A)$, $\beta \in \text{Aut}(B)$, set

\[ \alpha X \beta = \alpha \otimes X \otimes \beta \]

and

\[ \text{Out}(A) \times_X \text{Out}(B) = \{ ([\alpha], [\beta]) \in \text{Out}(A) \times \text{Out}(B) ; \alpha X \cong X \beta \}, \]

a subgroup of $\text{Out}(A) \times \text{Out}(B)$.

Theorem (Kosaki, Choda-Kosaki). For $Z = NL^2(M)_M$ with $N \subset M$ irreducible, we have

(i) $\text{Out}(N) \times_Z \text{Out}(M) \cong \text{Aut}(M, N)/\text{Int}(M, N)$.

(ii) $\theta$ is inner at some $M_k$ iff $M L^2(M)\theta_M$ appears in $Z^*Z \cdots Z^*Z (= (Z^*Z)^k)$.

Thus (ii) Use $M_3 = \text{End}(ZZ^*Z_M)$ and the Frobenius reciprocity

\[ \text{Hom}(\theta Z^*ZZ^*, ZZ^*Z) \cong \text{Hom}(L^2(M)\theta, (Z^*Z)^3) \]

for example. \qed
Theorem (Goto). \( \text{Loi}(\theta) = 1 \) implies \([\theta] = ([\alpha], [\beta]) \in \text{Aut}(M, N)/\text{Int}(M, N) \) is in the center of fusion algebra, i.e.,
\[
\alpha X \cong X \alpha, \quad \beta Y \cong Y \beta, \quad \alpha Z' \cong Z' \beta
\]
for descendants \( X_N, Y_M \) and \( Z'_M \) of \( Z \).
Conversely, the centrality in the fusion algebra forces the triviality of \( \text{Loi} \) invariant as long as the principal graph (or the dual principal graph) is multiplicity-free.

Theorem. The following are equivalent.
(i) \( \text{Loi}(\theta) = 1 \).
(ii) For each bimodule \( X \) in the tensor category generated by \( Z \), we can find an isomorphism \( I_X : X \to \theta X \theta^{-1} \) such that \( I_X^{*} = \overline{I_X} \), \( I_{XY} = I_X \otimes I_Y \), and the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{I_X} & \theta X \theta^{-1} \\
\downarrow T & & \downarrow \theta T \theta^{-1} \\
Y & \xrightarrow{I_Y} & \theta Y \theta^{-1}
\end{array}
\]
commutes for \( T \in \text{Hom}(X, Y) \).

Applications to \( \chi(G, H) \)

For a subgroup \( H \subset G \) of a finite group \( G \) with an outer action on an AFD \( \text{II}_1 \)-factor, set
\[
\chi(G, H) = \chi(R \rtimes G, R \rtimes H).
\]
According to [KY], irreducible bimodules generated by \( R \rtimes H \mathrm{L}^2(R \rtimes G) R \rtimes G \) are parametrized by
\[
R \rtimes G - R \rtimes G : \quad \hat{G}
\]
\[
R \rtimes H - R \rtimes G : \quad \hat{H}
\]
\[
R \rtimes H - R \rtimes H : \quad \coprod_{\delta \in H \backslash G/H} H \hat{a}^{-1}.
\]
Note that the tensor category of \( R \rtimes H - R \rtimes H \) bimodules contains the Tannaka dual of \( H \) as a subcategory. With this description, we can deduce
\[
\text{Cnt}(M, N)/\text{Int}(M, N) \cong \Xi \times (N_G(H)/H),
\]
where
\[
\Xi = \{(\chi, \eta) \in H^* \times G^* ; \chi = \eta|_H \}
\]
and \( H^* \) and \( G^* \) refer to the group of 1-dimensional representations.
Taking the restriction of centralizing morphisms to the Tannaka dual of \( H \), we can deduce the following.
Theorem. We have

\[ \Xi \times Z(G)H/H \subset \chi(G, H) \subset \Xi \times \{ \dot{c} \in C_G(H)H/H; \dot{c} \text{ acts trivially on } H \backslash G/H \}, \]

where \( C_G(H) \) denotes the centralizer of \( H \) in \( G \) and \( \dot{a} \in N_G(H)/H \) acts on \( H \backslash G/H \) by

\[ HgH \mapsto Haga^{-1}H. \]

Corollary.

(i) \( \chi(G, \{e\}) \cong G^* \times Z(G) \).

(ii) \( \chi(A \times H, H) \cong \{ (\chi, \eta) \in A^* \times (A \times H)^*; \chi = \eta|_H \} \times A^H \), where \( A \) is an abelian group and \( A^H = \{ a \in A; hah^{-1} = a, \text{ for all } h \in H \} \).

(iii) \( \chi(S_n, S_k) \cong \mathbb{Z}_2 \).

(iv) \( \chi(A_n, A_k) = \{e\} \).

References


