<table>
<thead>
<tr>
<th>Title</th>
<th>CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE INVARIANT $\chi$(Bimodules in Operator Algebras)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>YAMAGAMI, SHIGERU</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 936: 46-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60031">http://hdl.handle.net/2433/60031</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE IN Variant $\chi$

YAMAGAMI Shigeru

In this note, we shall present a tensor-categorical interpretation of the invariant $\chi$ for subfactors, which can be applied to compute the invariant for group-subgroup subfactors.

Automorphisms in Subfactors

Given a subfactor $N \subset M$, set

$$\text{Aut}(M, N) = \{\theta \in \text{Aut}(M); \theta(N) = N\},$$

$$\text{Int}(M, N) = \{\text{Ad} u \in \text{Aut}(M, N); u \text{ is a unitary in } N\}.$$

Each $\theta \in \text{Aut}(M, N)$ is inductively extended to automorphisms of the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

by

$$\theta(e_i) = e_i, \quad i = 1, 2, 3, \cdots$$

and hence induces

$$\text{Loi}(\theta) = \text{the family of induced automorphisms on } N' \cap M \subset N' \cap M_1 \subset \cdots.$$

Remark. We can use automorphisms on $M' \cap M_1 \subset M' \cap M_1 \subset \cdots$ as well, which contains the equivalent information.

Theorem (Popa, Loi). Let $M, N$ be AFD II$_1$-factors and $N \subset M$ be amenable. Then for $\theta \in \text{Aut}(M, N)$,

(i) $\theta$ is centrally trivial iff $\theta$ is inner at some $M_k$, i.e., $\exists 0 \neq u \in M_k$ such that $\theta(x)u = ux$ for $x \in M$.

(ii) $\text{Loi}(\theta) = 1$ iff $\theta \in \overline{\text{Int}}(M, N)$.

According to Y. Kawahigashi, we define the group

$$\chi(M, N) = \frac{\text{Cnt}(M, N) \cap \overline{\text{Int}}(M, N)}{\text{Int}(M, N)}$$

as the $\chi$-invariant for subfactors.
Theorem (Kawahigashi). For subfactors of index $<4$,
\[ \chi = \begin{cases} 
\mathbb{Z}_2 & \text{for } A_{2n+1} \ (n \geq 2) \text{ and } E_6, \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } A_3, \\
\mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{for } D_4, \\
0 & \text{otherwise.}
\end{cases} \]

Interpretations with bimodules

For a factor $N$, the correspondance
\[ \alpha \in Aut(N) \mapsto X_\alpha = N L^2(N) \alpha_N \]
induces
\[ X_\alpha^* = X_{\alpha^{-1}} \]
\[ X_\alpha \otimes^N X_\beta = X_{\alpha\beta}. \]
Here the right $N$-action in $X_\alpha$ is modified by $\alpha$ compared to the standard bimodule $L^2(N)$.

The bimodule $X_\alpha$ satisfies $X_\alpha \otimes X_\alpha^* \cong N L^2(N)_N$. The converse is not always true: Let $R$ be an AFD $\text{II}_1$-factor, $e$ be a non-trivial projection in $R$ and $v : R \to eRe$ be an isomorphism. Then the bimodule $X = reL^2(R)R$ gives an example, where the right action is induced from the isomorphism $\varphi$.

Theorem. Let $X$ be an $N-N$ bimodule such that $X \otimes X^* \cong L^2(N)$ and consider one of the following cases.
(i) $N$ is properly infinite.
(ii) $X$ is a descendent of an irreducible bimodule $Z$ of finite index.
Then $\exists \alpha \in Aut(N)$ such that $X \cong X_\alpha$.

The bimodule $X_\alpha = N L^2(N) \alpha_N$ is simply denoted by $\alpha$ in the following.

For a bimodule $AX_B$ and $\alpha \in Aut(A)$, $\beta \in Aut(B)$, set
\[ \alpha X \beta = \alpha \otimes X \otimes \beta \]
and
\[ Out(A) \times_X Out(B) = \{ ([\alpha], [\beta]) \in Out(A) \times Out(B); \alpha X \cong X \beta \}, \]
a subgroup of $Out(A) \times Out(B)$.

Theorem (Kosaki, Choda-Kosaki). For $Z = N L^2(M)_M$ with $N \subset M$ irreducible, we have
(i) $Out(N) \times_Z Out(M) \cong Aut(M,N)/Int(M,N)$.
(ii) $\theta$ is inner at some $M_k$ iff $M L^2(M) \theta_M$ appears in $Z^*Z \cdots Z^*Z (= (Z^*Z)^k)$.

\[ \therefore \] (ii) Use $M_3 = \text{End}(ZZ^*Z_M)$ and the Frobenius reciprocity
\[ \text{Hom}(\theta Z^*ZZ^*, Z^*ZZ^*) \cong \text{Hom}(L^2(M)\theta, (Z^*Z)^3) \]
for example. \(\square\)
Theorem (Goto). \( \text{Loi}(\theta) = 1 \) implies \( [\theta] = ([\alpha], [\beta]) \in \text{Aut}(M, N)/\text{Int}(M, N) \) is in the center of fusion algebra, i.e.,

\[
\alpha X \cong X\alpha, \quad \beta Y \cong Y\beta, \quad \alpha Z' \cong Z'\beta
\]

for descendants \( N X_N, M Y_M \) and \( N Z'_M \) of \( Z \).

Conversely the centrality in the fusion algebra forces the triviality of \( \text{Loi} \) invariant as long as the principal graph (or the dual principal graph) is multiplicity-free.

Theorem. The following are equivalent.

(i) \( \text{Loi}(\theta) = 1 \).

(ii) For each bimodule \( X \) in the tensor category generated by \( Z \), we can find an isomorphism \( I_X : X \to \theta X \theta^{-1} \) such that \( I_{X^*} = I_X, I_{XY} = I_X \otimes I_Y \), and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{I_X} & \theta X \theta^{-1} \\
T \downarrow & & \downarrow \theta T \theta^{-1} \\
Y & \xrightarrow{I_Y} & \theta Y \theta^{-1}
\end{array}
\]

commutes for \( T \in \text{Hom}(X, Y) \).

Applications to \( \chi(G, H) \)

For a subgroup \( H \subset G \) of a finite group \( G \) with an outer action on an AFD II\(_1\) factor, set

\[
\chi(G, H) = \chi(R \rtimes G, R \rtimes H).
\]

According to [KY], irreducible bimodules generated by \( R \rtimes H L^2(R \rtimes G)_{R \rtimes G} \) are parametrized by

\[
\begin{align*}
R \rtimes G & \rtimes R \rtimes G : \quad \hat{G} \\
R \rtimes H \rtimes R \rtimes G & : \quad \hat{H} \\
R \rtimes H \rtimes R \rtimes H & : \quad \prod_{a \in H \setminus G/H} H \hat{a} H^{-1}.
\end{align*}
\]

Note that the tensor category of \( R \rtimes H \rtimes R \rtimes H \) bimodules contains the Tannaka dual of \( H \) as a subcategory. With this description, we can deduce

\[
\text{Cnt}(M, N)/\text{Int}(M, N) \cong \Xi \times (N_G(H)/H),
\]

where

\[
\Xi = \{(\chi, \eta) \in H^* \times G^* ; \chi = \eta|_H\}
\]

and \( H^* \) and \( G^* \) refer to the group of 1-dimensional representations.

Taking the restriction of centralizing morphisms to the Tannaka dual of \( H \), we can deduce the following.
Theorem. We have
\[ \Xi \times Z(G)H/H \subset \chi(G, H) \subset \Xi \times \{ \dot{c} \in C_G(H)H/H; \dot{c} \text{ acts trivially on } H \backslash G/H \}, \]
where \( C_G(H) \) denotes the centralizer of \( H \) in \( G \) and \( \dot{a} \in N_G(H)/H \) acts on \( H \backslash G/H \) by
\[ HgH \mapsto Haga^{-1}H. \]

Corollary.
(i) \( \chi(G, \{e\}) \cong G^* \times Z(G) \).
(ii) \( \chi(A \times H, H) \cong \{ (\chi, \eta) \in A^* \times (A \times H)^*; \chi = \eta|_H \} \times A^H \), where \( A \) is an abelian group and \( A^H = \{ a \in A; hah^{-1} = a, \text{ for all } h \in H \} \).
(iii) \( \chi(S_n, S_k) \cong \mathbb{Z}_2 \).
(iv) \( \chi(A_n, A_k) = \{e\} \).

References