数理解析研究所講究録
936 巻 1996 年 32-45

TWO TOPICS ON FLATNESS IN THE PARAGROUP THEORY

NOBUYA SATO (UNIVERSITY OF TOKYO)
佐藤 信哉 (東大数理)

1. Introduction

Subfactor theory has much progressed since its initiation by V. F. R. Jones ([J]) and has much similarity with other mathematical and physical structures like rational conformal field theory (in the sense of Moore and Seiberg), exactly solvable lattice models, topological quantum field theory and quantum groups.

In the progress of subfactor theory, A. Ocneanu presented his striking theory "*paragroups*" in 1985. It is well known that a subfactor has Jones index and principal and dual principal graph as their invariants. A. Ocneanu perceived a new combinatorial structure of an irreducible inclusion of AFD factors $N \subset M$ of type II_1 with finite depth and finite Jones index and axiomatized it as a paragroup. The classification of subfactors is one of the challenging problem in the theory of operator algebras. In a certain class of subfactors, namely, irreducible inclusions of AFD II_1 factors $N \subset M$ with finite Jones index and finite depth, paragroups give the complete classification machinery. This completeness is due to a theorem of S. Popa called generating property. He also studied the necessary and sufficient condition for subfactor to be approximated by certain series of finite dimensional C^*-algebras ([P1], [P2]). Paragroup theory as the combinatorial aspect of classification of subfactors looks like IRF models in exactly solvable lattice models. In fact, we have a complex valued function called *connection* in the axioms of paragroups, which is quite similar to Boltzmann weight which appeared in IRF models. Moreover, we have the same type of A-D-E classification for subfactors with index less than 4 as that of modular invariant partition functions.

A paragroup is constructed from two finite graphs and the above stated complex valued function. In the axioms of the paragroup theory, the most important one is the flatness condition. If we drop off this condition, the paragroup is no longer a complete invariant for the above class of subfactors. The flatness condition has some aspects. From one viewpoint, the flatness is an indispensable property for the completeness of the invariant as already stated. From another, it makes the "parallel transport " on the graphs well-defined. This property is an analogue of that of differential geometry and gives the name "flatness " reasonable. It is also regarded as a representation of the "associativity " of a paragroup. A paragroup is a generalization of a finite group and its unitary representations with non-degenerate paring. In the case of subfactor arising from the inclusion by finite group crossed product with an outer action, the space of a group and that of the left regular representation is given by graphs and a non-degenerate paring is given by the connection. The associativity of the group is exactly the flatness condition. If the depth of graphs is two, it is also true that associativity is equivalent to the flatness condition or equivalent to say the flatness condition is equivalent to the pentagonal relation in monoidal tensor category (see below and [Sa1] for proofs).

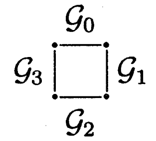
This paper is organised as follows. Part I discusses that the flatness condition is equivalent to "associativity" of a paragroup under some assumptions. Part II discusses a relation between two subfactors arising from a non-degenerate commuting square with period two. This gives an answer to a problem raised by V. F. R. Jones at workshop in Aarhus, June, 1995. See [Sa2] for proofs.

Part I. Fourier transform for paragroups and its application to the depth two case

2. Ocneanu's paragroup theory

We will review Ocneanu's paragroup theory to fix some notations. The present exposition is rather restricted. For general paragroups and more details, we refer readers to [O1], [O3] and [K].

First, we have a graph G consisting of four finite bipartite graphs $\mathcal{G}_0 = \mathcal{G}_3 = \mathcal{G}$, $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{H}$ as in the following figure.



Suppose $\mathcal{G}_j$ and $\mathcal{G}_{j+1}$ has common vertices V_{j+1} and $\mathcal{G}_j$'s have the common Perron-Frobenius eigenvalue β and the common Perron-Frobenius eigenvector μ for $j \in \mathbf{Z}/4\mathbf{Z}$. We fix a vertex in V_0 (resp. V_2) and call it $*_{\mathcal{G}}$ (resp. $*_{\mathcal{H}}$) and normalize μ so that $\mu(*) = 1$. We call four edges, one from each graph, with common vertices, a *cell*. We set one more assumption on the graphs (**Initialization axiom**) as follows.

There exists the only one vertex connected to $*_{\mathcal{G}}$ (resp. $*_{\mathcal{H}}$) in V_1 (resp. V_3) and it is the only vertex connected to $*_{\mathcal{H}}$ (resp. $*_{\mathcal{G}}$) in V_1 (resp. V_3).

We assume there exists an assignment, called a *connection* of a complex number to each cell and denote it W and we use the graphical notation for the value of W

on a cell as in the following figure.

$$\begin{array}{ccc} a & \xrightarrow{\sigma_1} & b \\ \sigma_3 \downarrow & & \downarrow \sigma_2 \\ c & \xrightarrow[\sigma_4]{} & d \end{array}$$

We set some assumptions on W as follows (**Unitarity axiom**).

$$\sum_{c,\sigma_3,\sigma_4} \begin{array}{ccc} a & \xrightarrow{\sigma_1} & b \\ \sigma_3 \downarrow & & \downarrow \sigma_2 \\ c & \xrightarrow[\sigma_4]{} & d \end{array} \; \overline{\begin{array}{ccc} a & \xrightarrow{\sigma_1'} & b' \\ \sigma_3 \downarrow & & \downarrow \sigma_2' \\ c & \xrightarrow[\sigma_4]{} & d \end{array}} = \delta_{b,b'}\delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'},$$

$$\sum_{b,\sigma_1,\sigma_2} \begin{array}{ccc} a & \xrightarrow{\sigma_1} & b \\ \sigma_3 \downarrow & & \downarrow \sigma_2 \\ c & \xrightarrow[\sigma_4]{} & d \end{array} \; \overline{\begin{array}{ccc} a & \xrightarrow{\sigma_1} & b \\ \sigma_3' \downarrow & & \downarrow \sigma_2 \\ c' & \xrightarrow[\sigma_4']{} & d \end{array}} = \delta_{c,c'}\delta_{\sigma_3,\sigma_3'}\delta_{\sigma_4,\sigma_4'}.$$

We can construct the nested graphs by reflecting each graphs vertically and horizontally. We use the notation $\tilde{}$ to mean the reflected graphs and edges. We assume that W on the nested graphs satisfies the following identity (**Renormalization rule axiom**).

$$\begin{array}{ccc} c & \xrightarrow{\sigma_4} & d \\ \tilde{\sigma_3} \downarrow & & \downarrow \tilde{\sigma_2} \\ a & \xrightarrow[\sigma_1]{} & b \end{array} = \begin{array}{ccc} b & \xrightarrow{\tilde{\sigma_1}} & a \\ \sigma_2 \downarrow & & \downarrow \sigma_3 \\ d & \xrightarrow[\tilde{\sigma_4}]{} & c \end{array} = \sqrt{\frac{\mu(a)\mu(d)}{\mu(b)\mu(c)}} \; \overline{\begin{array}{ccc} a & \xrightarrow{\sigma_1} & b \\ \sigma_3 \downarrow & & \downarrow \sigma_2 \\ c & \xrightarrow[\sigma_4]{} & d \end{array}}.$$

If a connection W satisfies the above two conditions, we call it a *bi-unitary connection.*

For an oriented edge σ, we denote the starting point, the end point and the length by $s(\sigma)$, $r(\sigma)$ and $|\sigma|$ respectively. We define an oriented *path* σ on $\mathcal{G}_0$ by a succession of edges. We take a pair of paths (ξ_+, ξ_-) called a *string* which has the starting point $*$ and the same end point with length n.

First we construct an algebra $A_{0,n}$ from above data.

Definition 2.1. *We define $A_{0,n}$ as follows. As a* **C***-vector space, a basis for $A_{0,n}$ is given by strings with length n. An algebra structure is defined as follows. A product structure is given by $(\xi_+, \xi_-) \cdot (\eta_+, \eta_-) = \delta_{\xi_-,\eta_+}(\xi_+, \eta_-)$. A star-structure is given by $(\xi_+, \xi_-)^* = (\xi_-, \xi_+)$. Thus $A_{0,n}$ is a finite dimensional C^*-algebra.*

We can embed $A_{0,n}$ into $A_{0,n+1}$ canonically. Moreover there exists the unique normalized trace which is compatible with such embedding. Using this trace, we can construct an AFD II_1 factor $A_{0,\infty} = \overline{\cup_{n=1}^{\infty} A_{0,n}}^{\text{weak}}$.

We can construct finite dimensional C*-algebras $A_{k,n}$ on nested graphs in a similar way. Although we have many ways to reach at (k,n) component, the identification of the different basis is given by the connection. We call these $A_{k,n}$ *string algebras.*

$$\begin{array}{ccccccc}
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Thus we can construct increasing sequences of AFD II_1 factors $A_{k,\infty} = \overline{\cup_{n=0}^{\infty} A_{k,n}}^{\text{weak}}$ as well as $A_{0,n}$. We call this construction of AFD II_1 factors a *string algebra construction.* We can also construct the string algebras $A_{-1,k}$ by identifying the edge connected to $*_{\mathcal{G}}$ and the edge connected to $*_{\mathcal{H}}$ by Initialization axiom. We have the following theorem.

Theorem 2.2 **([O3]).** *The inclusion $A_{0,\infty} \subset A_{1,\infty}$ is irreducible and Jones index for this inclusion is given by β^2. The increasing sequence of AFD II_1 factors*

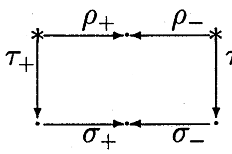
$$A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset A_{3,\infty} \subset A_{4,\infty} \cdots$$

give the basic construction for $A_{0,\infty} \subset A_{1,\infty}$. Moreover we can estimate the higher relative commutants for $A_{0,\infty} \subset A_{1,\infty}$ by $A_{0,\infty}' \cap A_{1,\infty} \subset A_{k,0}$. Also we have the estimation for $A_{-1,\infty}' \cap A_{k,\infty}$ by $A_{-1,\infty}' \cap A_{k,\infty} \subset A_{k,1}$.

Now we describe the most important axiom (**Flatness axiom**).

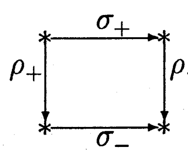
Definition 2.3 **([K], Theorem 2.1).** *We say that a bi-unitary connection is flat if it satisfies the following equivalent condition. Here $*$ means either $*_{\mathcal{G}}$ or $*_{\mathcal{H}}$.*

(1) *Any two elements $x \in A_{k,0}$ (in the vertical string algebra) and $y \in A_{0,l}$ (in the horizontal string algebra) commute.*
(2) *For each horizontal string $\rho = (\rho_+, \rho_-) \in A_{0,k}$, we get the identity as in the following figure, where $C_{\rho,\sigma} \in \mathbf{C}$ depends only on ρ and $\sigma = (\sigma_+, \sigma_-)$.*



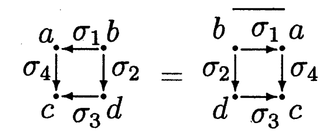
$$= \delta_{\tau_+,\tau_-} C_{\rho,\sigma}.$$

(3) *For any horizontal paths σ_+, σ_- and vertical paths ρ_+, ρ_- with all the sources and ranges equal to $*$, we get the identity in the following figure.*



$$= \delta_{\sigma_+,\sigma_-} \delta_{\rho_+,\rho_-}.$$

We explain the figures used above. First, reversed arrows mean the following figure for denoting the connections.



Next, the box-like figures mean the following. We make all the possible fillings of cells for above diagrams. Such a choice is called a *configuration.* We multiply the connection values of all the cells in a configuration and sum them over all the configurations. This is the value assigned to the above two box-like diagrams, and we mean this value by the diagram.

Definition 2.4. *We call a bi-unitary flat connection (G, W) a paragroup.*

Theorem 2.2 and the following theorem give the reason why the paragroup theory is important.

Theorem 2.5 (Popa's generating property [P1], [P2]**).** *Let $N \subset M$ be an irreducible inclusion of AFD II_1 factors with finite index and finite depth. Then we have the anti-isomorphism as follows.*

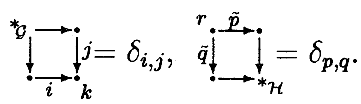
$$\left(\overline{\bigcup_{k=1}^{\infty} M_1{}' \cap M_k}^{\text{weak}} \subset \overline{\bigcup_{k=1}^{\infty} M' \cap M_k}^{\text{weak}} \right) \cong (N \subset M).$$

Thus we can say

Theorem 2.6. *A paragroup gives a complete invariant for irreducible inclusion of AFD II_1 factors with finite index and finite depth.*

3. Fourier transform for paragroups

We freely use the notations in [K] and set the paragroup (G, W) with the graph $\mathcal{G}_0 = \mathcal{G}_3 = \mathcal{G}$, $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{H}$ and denote the Perron-Frobenius eigenvalues for these graphs β. Moreover we note that we may change the connections by gauge choice, which means a choice of an appropriate unitary operator if necessary. So we choose the connection as follows.



The meaning of the Kronecker δ on the right hand side is as follows. There is only one vertex on $\mathcal{G}$ that is connected to $*_{\mathcal{G}}$ (resp. $*_{\mathcal{H}}$). For any k (resp. r), the number of edges connecting such a vertex on $\mathcal{G}$ and k (resp. r) on $\mathcal{G}_1$ (resp. $\mathcal{G}_0$) and that on $\mathcal{G}_2$ (resp. $\mathcal{G}_3$) are the same. By identifying these pairs of edges and denoting the above vertex simply by "$\cdot$" (without any label), we can impose the above formula.

A. Ocneanu defined Fourier transform for the paragroup first. We define Fourier transform following [O2] and [O3].

Definition 3.1. *We define the linear map $\mathcal{F} : A_{0,2} \longrightarrow A_{1,1}$ by*

$$\mathcal{F}(x) := \beta^3 E_{A_{1,1}}(x e_0 e_1), \ x \in A_{0,2}, \tag{3.1}$$

and call this linear map $\mathcal{F}$ Fourier transform for the paragroup.
We define the linear map $\hat{\mathcal{F}} : A_{1,1} \longrightarrow A_{0,2}$ as well by

$$\hat{\mathcal{F}}(x) := \beta^3 E_{A_{0,2}}(x e_1 e_0), \ x \in A_{1,1}, \tag{3.2}$$

and call this $\hat{\mathcal{F}}$ inverse Fourier transform for the paragroup. Here E, e_0 and e_1 mean the conditional expectation, Jones projections, respectively ([K], [O3]*).*

Here we set some notations. We choose a system of matrix units $\{e^k_{ij}\}_{i,j=1,\dots,n_k}$ (k's are odd vertices in the graph $\mathcal{G}_2$) in $A_{1,1}$ and also choose a system of matrix units $\{\lambda^r_{pq}\}_{p,q=1,\dots,n_r}$ (r's are odd vertices in the graph $\tilde{\mathcal{G}}_0$) in $A_{0,2}$ for simplicity. Moreover we use the notation $n_k = \mu(k)$ and $n_r = \mu(r)$.

Using the connection, Fourier transform and inverse Fourier transform are written as follows.

Proposition 3.2.

$$\mathcal{F}(\lambda^r_{pq}) = \sum_{i,j,k} \left(\frac{n_r}{n_k}\right)^{\frac{1}{2}} \overline{\begin{array}{ccc} & p & r \\ i & \square & \tilde{q} \\ k & & \tilde{j} \end{array}} \; e^k_{ij}, \tag{3.3}$$

$$\hat{\mathcal{F}}(e^k_{ij}) = \sum_{p,q,r} \left(\frac{n_k}{n_r}\right)^{\frac{1}{2}} \begin{array}{ccc} & p & r \\ i & \square & \tilde{q} \\ k & & \tilde{j} \end{array} \; \lambda^r_{pq}. \tag{3.4}$$

Proposition 3.3. *Fourier transform $\mathcal{F}$ is invertible and its inverse is inverse Fourier transform $\hat{\mathcal{F}}$.*

Lemma 3.4. *Fourier transform and inverse Fourier transform preserve inner products arising from* tr*. That is, we have the following identities.*

$$\text{(i)} \quad (\mathcal{F}(x), \mathcal{F}(y)) = (x, y), \quad x, y \in A_{0,2}.$$

$$\text{(ii)} \quad (\hat{\mathcal{F}}(x), \hat{\mathcal{F}}(y)) = (x, y), \quad x, y \in A_{1,1}.$$

By the above two Propositions, we can get the following Theorem.

Theorem 3.5. *The linear maps $\mathcal{F}$ and $\hat{\mathcal{F}}$ are unitary.*

Next, we introduce another product called the *convolution product* in $A_{0,2}$ and $A_{1,1}$.

Definition 3.6. *We define the following new product in* $A_{1,1}$.

$$x * y := \mathcal{F}(\hat{\mathcal{F}}(x)\hat{\mathcal{F}}(y)), \quad x, y \in A_{1,1}. \tag{3.5}$$

We also define the following new product in $A_{0,2}$.

$$x \hat{*} y := \hat{\mathcal{F}}(\mathcal{F}(x)\mathcal{F}(y)), \quad x, y \in A_{0,2}. \tag{3.6}$$

Because $\mathcal{F}$ and $\hat{\mathcal{F}}$ are inverses of each other, we get the following identity.

$$\hat{\mathcal{F}}(x * y) = \hat{\mathcal{F}}(x)\hat{\mathcal{F}}(y), \quad x, y \in A_{1,1}, \tag{3.7}$$

$$\mathcal{F}(x \hat{*} y) = \mathcal{F}(x)\mathcal{F}(y), \quad x, y \in A_{0,2}. \tag{3.8}$$

Furthermore, we get the following Proposition by definition.

Proposition 3.7. *The convolution products* $*$ *and* $\hat{*}$ *are associative.*

Definition 3.8. *We define another star structure* $\#$ *in* $A_{1,1}$ *as follows.*

$$x^{\#} = \mathcal{F}(\hat{\mathcal{F}}(x)^*), \quad x \in A_{1,1}. \tag{3.9}$$

We define another star structure $\hat{\#}$ *in* $A_{0,2}$ *as follows.*

$$x^{\hat{\#}} = \hat{\mathcal{F}}(\mathcal{F}(x)^*), \quad x \in A_{0,2}. \tag{3.10}$$

Thus we have the following property by definition.

$$\mathcal{F}(x^{\#}) = \hat{\mathcal{F}}(x)^*, \tag{3.11}$$

$$\mathcal{F}(x^{\hat{\#}}) = \mathcal{F}(x)^*. \tag{3.12}$$

Proposition 3.9. *The two algebras* ($A_{1,1}$, *the convolution product* $*$, *the star structure* $\#$) *and* ($A_{0,2}$, *the convolution* $\hat{*}$, *the star structure* $\hat{\#}$) *are finite dimensional C*-algebras.*

So we can decompose these C*-algebras as the direct sums of full matrix algebras as follows.

$$A_{1,1} = \bigoplus_{r=1}^{s} M_{n_r}(\mathbf{C}).$$

We define a system of matrix units $\{\hat{\lambda}^r_{pq}\}_{p,q=1,\ldots,n_r}$ $(r = 1, \ldots, s)$ by $\hat{\lambda}^r_{pq} = \mathcal{F}(\lambda^r_{pq})$.

$$A_{0,2} = \bigoplus_{k=1}^{l} M_{n_k}(\mathbf{C}).$$

We define a system of matrix units $\{\hat{e}^k_{ij}\}_{i,j=1,\dots,n_k}$ $(k = 1, \dots, l)$ by $\hat{e}^k_{ij} = \hat{\mathcal{F}}(e^k_{ij})$.

Thus we have two finite dimensional C*-algebra structures in each of $A_{0,2}$ and $A_{1,1}$. We will see in the next section these two algebra structures give a Kac algebra structure on each algebra.

4. The relation between the flatness condition and the pentagonal identities for the depth two case

We apply all the results of the previous section to the depth two case and prove the main theorem in this section. We note that we omit the label of the vertex v because v is the only odd vertex in the depth two graph.

We need some preparations for proving the main theorem. We adopt the notation $n = \beta^2$.

Definition 4.1. *Define linear functionals $\hat{\varphi}$, φ as follows.*

$$\hat{\varphi}(x) = \beta\ \mathrm{tr}(x),\ \ x \in A_{0,2}, \tag{4.1}$$

$$\varphi(x) = \beta\ \mathrm{tr}(x),\ \ x \in A_{1,1}. \tag{4.2}$$

Definition 4.2. *We define the linear maps $\hat{\Gamma}$ from $A_{0,2}$ to $A_{0,2} \otimes A_{0,2}$ and Γ from $A_{1,1}$ to $A_{1,1} \otimes A_{1,1}$ as follows.*

$$(\hat{\varphi} \otimes \hat{\varphi}) \left(\hat{\Gamma}(x)(a \otimes b)\right) = \hat{\varphi}\left(x\left(a \hat{*} b\right)\right),\ \ x, a, b \in A_{0,2}, \tag{4.3}$$

$$(\varphi \otimes \varphi)\left(\Gamma(x)(a \otimes b)\right) = \varphi\left(x\left(a * b\right)\right),\ \ x, a, b \in A_{1,1}. \tag{4.4}$$

Definition 4.3. *We define the linear maps W and $\hat{W}$ by the following equations.*

$$\hat{W}(x \otimes y) := \hat{\Gamma}(y)(x \otimes 1),\ \ x, y \in A_{0,2}. \tag{4.5}$$

$$W(x \otimes y) := \Gamma(y)(x \otimes 1),\ \ x, y \in A_{1,1}. \tag{4.6}$$

Lemma 4.4. *The linear maps $\hat{W}$ and W are unitary operators.*

Definition 4.5 ([B-S])**.** *Let $\mathcal{H}$ be a Hilbert space.*
A unitary operator V on $\mathcal{H} \otimes \mathcal{H}$ is called a multiplicative unitary if V satisfies the following identity.

$$V_{23}V_{12} = V_{12}V_{13}V_{23}. \tag{4.7}$$

Theorem 4.6. *The following conditions for a bi-unitary connection are equivalent.*

(i) *The bi-unitary connection is flat for $*_{\mathcal{G}}$ (resp. $*_{\mathcal{H}}$).*

(ii) *The unitary operator $\hat{W}^*$ (resp. W^*) defined in Definition 4.3 is a multiplicative unitary.*

We apply the construction of Hopf C*-algebras by Baaj-Skandalis to above multiplicative unitaries. As we know that a finite dimensional Hopf C*-algebra is a finite dimensional Kac algebra and vice versa, we get the following corollary.

Corollary 4.7. *In the case of a paragroup with depth two, we have Kac algebra structures in $A_{1,1}$, $A_{0,2}$.*

5. Duality between Kac algebra $A_{0,2}$ and Kac algebra $A_{1,1}$

In the previous section, we have constructed Kac algebra structures on $A_{1,1}$ and $A_{0,2}$. Those algebras have two algebra structures, string algebras and convolution algebras. The products of C*-algebra structures in $A_{1,1}$ and $A_{0,2}$ are closely related. The two products in $A_{0,2}$ are given by exchanging the two products in $A_{1,1}$. So the corresponding Kac algebras have a very simple relation, which is called *duality* in Kac algebra theory. For applications, we will describe this duality between $A_{0,2}$ and $A_{1,1}$ by comparing the formulae of the fundamental unitaries W and $\hat{W}$. For our purpose, we shall compute formulae of W and $\hat{W}$. We need some preparations.

In this section, we fix a paragroup with depth 2.

Proposition 5.1. *We have the following relation between fundamental unitaries W and $\hat{W}$.*

$$\hat{W} = \Sigma W^* \Sigma .$$

Here Σ means the flip map on $\mathcal{M} \otimes \mathcal{M}$.

So the Kac algebra constructed on $A_{0,2}$ and that on $A_{1,1}$ are dual to each other.

6. A realization of a paragroup with depth two

We realize a paragroup with depth two from an initial Kac algebra and describe the subfactor arising from the paragroup.

Suppose $\mathbf{K}= (\mathcal{M}, \Gamma, \kappa, \varphi)$ is a finite dimensional Kac algebra. We denote the left regular representation for this Kac algebra by π and identify the original Kac algebra and the represented algebra. Because this algebra is a finite dimensional C*-algebra, we may think that $\mathcal{M}$ and the Haar measure φ are of the following form.

$$\mathcal{M} = \bigoplus_{k=1}^{l} M_{n_k}(\mathbf{C}), \quad \varphi = \frac{1}{n}\sum_{k=1}^{l} n_k \mathrm{Tr}_{M_{n_k}(\mathbf{C})}.$$

Here $n = \dim\mathcal{M}$.

We can construct the dual Kac algebra $\hat{\mathbf{K}} = (\hat{\mathcal{M}}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$ from the initial Kac algebra $\mathbf{K}$ ([E-S]). We denote the left regular representation by λ and identify the original dual Kac algebra and the represented one. As above, we may think that $\hat{\mathcal{M}}$ and the Haar measure $\hat{\varphi}$ are of the following form.

$$\hat{\mathcal{M}} = \bigoplus_{r=1}^{s} M_{n_r}(\mathbf{C}), \quad \hat{\varphi} = \sum_{r=1}^{s} n_r \mathrm{Tr}_{M_{n_r}(\mathbf{C})}.$$

Note that we have the *Plancherel formula* for inner products arising from Haar measures ([K-P]). That is,

$$\varphi(ab^*) = \hat{\varphi}(a * b^{\#}). \tag{6.1}$$

We shall construct a string algebra from these data. At first, we set the two graphs, one from the Kac algebra and the other from the dual Kac algebra. More precisely, the graph $\mathcal{G}$ (resp. $\mathcal{H}$) has the unique vertex which is connected to the vertices corresponding to the direct summands of the multi-matrix algebra $\mathcal{M}$ (resp. $\hat{\mathcal{M}}$) by n_k (resp. n_r) edges. We fix a vertex corresponding to one dimensional representation for π (resp. λ) as a special vertex $*_{\mathcal{G}}$ (resp. $*_{\mathcal{H}}$).

Next, we introduce a bi-unitary connection to above two graphs. Connect these graphs as in the first figure in section 2.

Lemma 6.1. *We have the following formulae.*

$$\lambda^r_{pq} = \sum_{i,j,k} \frac{n}{n_k} \varphi(e^k_{ji} \lambda^r_{pq}) e^k_{ij}, \tag{6.2}$$

$$e^k_{ij} = \sum_{p,q,r} \frac{1}{n_r} \overline{\varphi(\lambda^r_{pq} e^k_{ji})} \lambda^r_{pq}. \tag{6.3}$$

Proposition 6.2. *We can define a bi-unitary connection on the above two graphs by the following formula.*

$$\begin{array}{ccc} k & \tilde{i} & \\ \bullet & \longrightarrow & \bullet \\ \tilde{j} \downarrow & & \downarrow p \\ \bullet & \longrightarrow & \bullet \\ & q & r \end{array} = \frac{n}{n_k n_r} \varphi(e^k_{ij} \lambda^r_{pq}). \tag{6.4}$$

Here e^k_{ij} (resp. λ^r_{pq}) means the system of matrix units corresponding to the decomposition of $\mathcal{M}$ (resp. $\hat{\mathcal{M}}$).

As we described in section 2, we can construct a string algebra with $A_{1,1}$ (resp. $A_{0,2}$) as $\mathcal{M}$ (resp. $\hat{\mathcal{M}}$). Thus we have Kac algebra structures in the string algebras $A_{1,1}$ and $A_{0,2}$. By Theorem 4.6, we can show that the above bi-unitary connection is a flat connection. So we have a paragroup for two graphs $\mathcal{G}$ and $\mathcal{H}$. Thus we get a subfactor from this paragroup and denote it $N \subset M$.

On the other hand, Sekine ([Se]) computed the connection for a subfactor $P \subset P \times_\alpha \mathbf{K}$ concretely. This connection is equal to the connection given in Proposition 6.2. Thus by Theorem 2.5 in section 2, a subfactor constructed above $N \subset M$ and a subfactor $P \subset P \times_\alpha \mathbf{K}$ are anti-isomorphic.

Theorem 6.3. *Assume we have a finite dimensional Kac algebra $\boldsymbol{K}$. We can construct a subfactor from the Kac algebra $\boldsymbol{K}$ and this subfactor is anti-isomorphic to a subfactor $P \subset P \times_\alpha \boldsymbol{K}$, where P is a AFD II_1 factor and α is an outer action of $\boldsymbol{K}$.*

Remark 6.4. *We can represent $N \subset M$ as Kac algebra crossed product subfactor by describing an outer action of $\boldsymbol{K}$ on M concretely (*[Da]*).*

Part II. An answer to a question raised by V. F. R. Jones

V. F. R. Jones raised the following question in his talk at Aarhus in June, 1995.

Question (V.F.R. Jones)

Suppose that we have a non-degenerate commuting square as follows.

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11} \end{array}$$

Here R_{00}, R_{01}, R_{10} and R_{11} are finite dimensional C^*-algebras such that R_{11} has a faithful trace and satisfy the following relation for the conditional expectations.

$$E_{R_{01}} \cdot E_{R_{10}} = E_{R_{00}}$$

Iterating the basic constructions for the above commuting square, we get the following series of commuting squares of period 2.

$$\begin{array}{ccccccccc} R_{00} & \subset & R_{01} & \subset & R_{02} & \subset & \cdots & \subset & R_{0\infty} \\ \cap & & \cap & & \cap & & & & \cap \\ R_{10} & \subset & R_{11} & \subset & R_{12} & \subset & \cdots & \subset & R_{1\infty} \\ \cap & & \cap & & \cap & & & & \cap \\ R_{20} & \subset & R_{21} & \subset & R_{22} & \subset & \cdots & \subset & R_{2\infty} \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ R_{\infty 0} & \subset & R_{\infty 1} & \subset & R_{\infty 2} & \subset & \cdots & & \end{array}$$

Then is the finite depth condition for $R_{0\infty} \subset R_{1\infty}$ related to that for $R_{\infty 0} \subset R_{\infty 1}$?

We will answer to this question by the method of paragroups and techniques associated with flatness. More precisely, we will prove a new theorem related to flatness and will answer to the question as its application. Also, we will give a new example of a principal graph as another application. It is natural to think that there might be a simple relation between Jones indices of the two subfactors. But we cannot expect such a relation by the following reason. Consider a series of canonical commuting squares $\{R_{k,l}\}_{k,l\geq 0}$ for some subfactor. square for $N \subset M$. For positive integers n, m, we have another series of commuting squares $\{R_{kn,lm}\}_{k,l\geq 0}$. We have two subfactors $P \subset P_1$ and $Q \subset Q_1$ arising from this series of commuting squares as above and have a relation $[P_1 : P]^n = [Q_1 : Q]^m$ in this case. Furthermore, one might expect that one of the index may be a rational power of the other in general, but this is not the case either. Indeed, a famous example in [G-H-J] gives an index $3 + \sqrt{3}$ and the other index for this commuting square is $4\cos^2(\pi/12) = 2 + \sqrt{3}$, which is not a rational power of $3 + \sqrt{3}$. So we do not have any simple inequality or equality for the two indices. We will prove that the global indices in the sense of A. Ocneanu give right numerical relation between the two subfactors. We note that the notion of the global index is first introduced by A. Ocneanu in [O1] as Jones index of the asymptotic inclusion subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ for a subfactor $N \subset M$ with finite index and finite depth.

7. An answer to the question

We freely use the notations in [O3] and [K].

We can restate the problem in the manner of the string algebra. Suppose that we have the string algebras $\{A_{k,l}\}_{k,l\geq 0}$ with starting vertex $*$ constructed from the four finite bipartite graphs and a biunitary connection W on the four graphs. Note that we have the starting vertex to make $A_{k,l}$ be the commuting square if k and l are large enough. This is a difference of the description between commuting square approach and string algebra one. We have AFD II_1 factors $A_{0,\infty}$ and $A_{1,\infty}$. Also we have AFD II_1 factors $A_{\infty,0}$ and $A_{\infty,1}$. Then is the finite depth condition for the inclusion $A_{0,\infty} \subset A_{1,\infty}$ related to that for $A_{\infty,0} \subset A_{\infty,1}$?

The following result is useful for later arguments.

Proposition 7.1 ([E-K], **Proposition 3.1**). *Suppose that $(A_{k,l})$ be string algebras as in the question in introduction. Then the following diagram is a series of commuting squares of period two.*

$$\begin{array}{ccc} A'_{0,\infty} \cap A_{k,\infty} & \subset & A_{k,0} \\ \cap & & \cap \\ A'_{0,\infty} \cap A_{k+1,\infty} & \subset & A_{k+1,0} \end{array} \tag{7.5}$$

By Proposition 7.1, we have a set of four Bratteli diagrams and the biunitary connection on the four graphs arising from the commuting square (7.5). We denote this biunitary connection by W', hereafter. We remark that the biunitary connection W' is defined on the finite graphs if the inclusion $A_{0,\infty} \subset A_{1,\infty}$ is of finite depth and that the two horizontal graphs are not necessarily connected even in the case the inclusion is of finite depth. We note that the following diagram is also a series of commuting squares of period two. So we have the biunitary connection $W' \cdot W$

and refer this biunitary connection as the composite connection of the two biunitary connections W' and W. We also remark that the two horizontal graphs arising from the commuting square 7.6 is connected if the inclusion is of finite depth.

$$\begin{array}{ccc} A'_{0,\infty} \cap A_{k,\infty} & \subset & A_{k,1} \\ \cap & & \cap \\ A'_{0,\infty} \cap A_{k+1,\infty} & \subset & A_{k+1,1} \end{array} \tag{7.6}$$

We have the following theorem under the above notations.

Theorem 7.2. *Suppose that the subfactor $A_{0,\infty} \subset A_{1,\infty}$ is of finite depth. The composite connection $W' \cdot W$ gives a flat connection.*

Corollary 7.3 (An answer to the question). *If either $A_{0,\infty} \subset A_{1,\infty}$ or $A_{\infty,0} \subset A_{\infty,1}$ is of finite depth, then so is the other.*

As we have already seen in introduction, it does not seem that we have a simple relation between the Jones indices of the two subfactors. In the rest of the paper, we show that the natural numerical relation between the two subfactors is given by the *global indices.*

Definition 7.4. *The global index for the inclusion of AFD II_1 factors $N \subset M$ is defined by*

$$\sum_{{}_M X_M : irreducible} (dim\ {}_M X_M)^2$$

and we denote it by $[[M : N]]$.

By definition, we easily know the value of the global index $[[M : N]]$ is infinity if the inclusion is of infinite depth.

The following lemma is essentially contained in Proposition 2.4 in [Y].

Lemma 7.5. *Suppose that we have an inclusion $N \subset M$ with finite index and that there exists an intermediate subfactor P. Then we have* $[[M : P]] \leq [[M : N]]$

Corollary 7.6. *The global indices for $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$ are equal.*

Corollary 7.7. *If the horizontal graphs associated with the commuting square in Proposition 7.1 are connected, then the connection arising from this commuting square is also flat.*

References

[B1] D. Bisch, A note on intermediate subfactors, *Pacific. J. Math.* **163** (1994), 201–216.

[B2] D. Bisch, Bimodules, higher relative commutants and the fusion algebra associated to a subfactor, preprint, (1995).

[B-S] S. Baaj, G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croises de C*-algèbres *Ann. Sci. Ec. Norm. Sup.* **26** (1993), 425–488.

[Da] M.-C. David, Paragroupe d'Adrian Ocneanu et algèbre de Kac, to appear in *Pac. J. Math.*

[E] M. Enock, Produit croisé d'une algèbre de von Neumann par une algèbre de Kac, *J. Funct. Anal.* **26** (1977), 16–47

[E-K] D. E. Evans & Y. Kawahigashi, The E_7 commuting square produce D_{10} as principal graph, *Publ. RIMS, Kyoto Univ.* **30** (1994), 151–166

[E-N] M. Enock & R. Nest, Irreducible inclusions of factors and multiplicative unitaries, preprint March 1994.

[E-S] M. Enock & M.Schwarz, "Kac algebras and duality", Springer Verlag (1992).

[G-H-J] F. Goodman, P. de la Harpe, & V. F. R. Jones, "Coxeter graphs and towers of algebras", MSRI publications 14, Springer, (1989).

[J] V.F.R. Jones, Index of subfactors, *Invent. Math.* **72** (1983), 1–15.

[K] Y. Kawahigashi, On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors, *J. Funct. Anal.* **127** (1995), 63–107.

[K-P] G.I. Kac & V.G. Paljutkin, Finite ring groups, *Trans. Moscow Math. Soc.* (1966), 251–294.

[O1] A. Ocneanu, Quantized group, string algebras and Galois theory for algebras, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, (1988), pp. 119–172.

[O2] A. Ocneanu, "Graph geometry, quantized groups and nonamenable subfactors", Lake Tahoe Lectures, June–July, (1989).

[O3] A. Ocneanu, "Quantum symmetry, differential geometry of finite graphs and classification of subfactors", University of Tokyo Seminary Notes **45**, (Notes recorded by Y. Kawahigashi), (1991).

[P1] S. Popa, Classification of subfactors: reduction to commuting squares, *Invent. Math.* **101** (1990), 19–43.

[P2] S. Popa, Classification of amenable subfactors of type II, *Acta Math.*, **172** (1994), 163–255.

[Sa1] N. Sato, Fourier transform for paragroups and its application to the depth two case (1995), preprint.

[Sa2] N. Sato, A relation between two subfactors arising from a non-degenerate commuting square— An answer to the problem raised by V. F. R. Jones— (1995), preprint.

[Se] T. Sekine, Connections associated with finite dimensional Kac algebra actions (1994), preprint.

[Sz] W. Szymanski, Finite index subfactors and Hopf algebra crossed products, *Proc. Amer. Math. Soc.* Vol **120** No.2 (1994), 519–528.

[Y] S. Yamagami, A note on Ocneanu's approach to Jones' index theory, *Internat. J. Math.* 4 (1993), 859–871

[Ya] T. Yamanouchi, Construction of an outer action on a finite-dimensional Kac algebra on the AFD factor of type II_1 , *Internat. J. Math.* 4 (1993), 1007–1045.

Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153, JAPAN

E-mail address: nobuya@ms.u-tokyo.ac.jp