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SINGULARITIES OF THE BERGMAN KERNEL FOR A TWO-DIMENSIONAL PSEUDOCONVEX TUBE DOMAIN WITH CORNERS

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INTRODUCTION

The study on the Bergman kernel has a long history and contains enormous works. Especially, the regularity of the Bergman kernel has investigated by many people. Let $\Omega = \{z; f(z,\overline{z}) < 0\} \Subset \mathbb{C}^n$ be a strictly pseudoconvex bounded domain with C^∞ (resp. analytic) boundary: that is, f is a C^∞ (resp. analytic) function satisfying that $df \neq 0$ at $f = 0$ and that the matrix $\left(\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}\right)$ is positive definite at every point of $\partial\Omega$. We denote the Bergman kernel for Ω by $B(z,\overline{w})$. In 1974, C. Fefferman proved the following:

0.1 Theorem. ([F]) *Assume that $\Phi : \Omega \to \widetilde{\Omega}$ is a biholomorphic mapping between bounded pseudoconvex domains with C^∞ boundary. Then, Φ can be extended smoothly up to the boundary.* ∎

In order to prove this theorem, he obtained a new precise result on singularities of $B(z,\overline{z})$ near the boundary. In fact $B(z,\overline{z})$ has a form of typical asymptotic expansion appearing in the theory of pseudodifferential operators. Seeing his result, Boutet de Monvel and Sjöstrand found out the following Fourier integral representation of the Bergman kernel in 1976:

0.2 Theorem. ([B-Sj]) *Assume that $f(z,\overline{z})$ is C^∞, then $B(z,\overline{w})$ has the following asymptotic expansion:*

$$B(z,\overline{w}) \equiv \int_0^\infty e^{\sqrt{-1}tg(z,\overline{w})} b(z,\overline{w},t)\,dt \qquad \text{mod. } C^\infty\text{kernels.}$$

Here $g(z,\overline{w})$ is an almost holomorphic extension of the function $g(z,\overline{z}) = \frac{1}{\sqrt{-1}}f(z,\overline{z})$, and the amplitude $b(z,\overline{w},t)$ is an element of $S^n(\overline{\Omega}\times\overline{\Omega^a}\times\mathbb{R}_+)$ and allows an asymptotic expansion at $t=\infty$ of the form: $\sum_{k=0}^{\infty} t^{n-k}b_k(z,\overline{w})$ where Ω^a denotes the complex conjugate of Ω. ∎

Inspired by their result, Kashiwara obtained a holonomic system satisfied by the Bergman kernel when Ω has analytic boundary:

0.3 Theorem. ([Kash]) *Assume that $f(z,\overline{z})$ is analytic. Then The Bergman kernel $B(z,\overline{w})$ satisfies the following microdifferential equations near the hypersurface $\{f(z,\overline{w})=0\}$ which is the complexification of the boundary $\partial\Omega$: For any microdifferential operatorsP, Q satisfying*

$$P(z,\partial_z)Y(-f(z,\overline{w})) = Q(w,\partial_{\overline{w}})Y(-f(z,\overline{w})), \tag{0.1}$$

it follows that

$${}^tP(z,\partial_z)B(z,\overline{w}) = {}^tQ(w,\partial_{\overline{w}})B(z,\overline{w}). \tag{0.2}$$

Here ^{t} denotes the formal adjoint of operators, Y denotes the Heaviside function, and the equalities (0.2) *and* (0.3) *hold as holomorphic microfunctions.* ∎

He also showed that the Bergman kernel can be determined locally modulo functions holomorphic at the boundary. Precisely he obtained the following theorem by using the microlocal Bergman kernel (that is, a microlocalization of the Bergman kernel):

0.4 Theorem. (see also [Kash]) *Under the same condition and notation, for any $z_0\in\partial\Omega$ there exist some neighborhood $U\ni z_0$ and $a(z,\overline{w})$, $b(z,\overline{w})\in\mathcal{O}(U\times U^a)$ such that the Bergman kernel has the following form in $U\times U^a$:*

$$B(z,\overline{w}) = \frac{a(z,\overline{w})}{f(z,\overline{w})^{n+1}} + b(z,\overline{w})\log(-f(z,\overline{w})).$$

Here $\mathcal{O}(U\times U^a)$ denotes the set of holomorphic functions on $U\times U^a$. ∎

(See [Kan2] for proofs of theorems 0.3 and 0.4 and further study, and also note that if we replace the Bergman kernel and the Heaviside function with the Szegö kernel and

Dirac's δ-function respectively, similar results hold).

Hence, if $\partial\Omega$ is analytic and strictly pseudoconvex on some neighborhood of $y^1 \in \partial\Omega$, we see that the Bergman kernel has the form above at y^1. However, when a domain has non-smooth boundary, it seems that the study of singularities of the Bergman kernel is not so satisfactory. Therefore in this article, as a simple example in non-smooth boundary cases we will consider singularities of the Bergman kernel for a two-dimensional pseudoconvex tube domain $\Omega = \mathbb{R}^2 + \sqrt{-1}W$, with $W = W_1 \cap W_2$, where each W_j is strictly convex domain as follows:

$$\begin{cases} W_j = \{y \in \mathbb{R}^2; \varphi_j(y) < 0\} \text{ with an analytic function } \varphi_j \ (j = 1,2) \text{ such that} \\ (1)\partial W_1 \text{ and } \partial W_2 \text{ intersect transversally.} \\ (2) \text{ If } \varphi_j(y_0) = 0, \text{ then } d\varphi_j(y_0) \neq 0 \text{ and the Hessian matrix} \\ \quad \left(\frac{\partial^2 \varphi_j}{\partial y_k \partial y_l}(y_0)\right)_{1 \leq k,l \leq 2} \text{ is positive definite for } j = 1,2. \end{cases}$$

In this article, we interpret the Bergman kernel as a microfunction on some conormal bundle. We will denote the Bergman kernel for Ω by $B(z, \overline{w})$. In Section 1, we first recall the integral representation of the Bergman kernel for the pseudoconvex tube domain. The Bergman kernel is holomorphic except for the diagonal points $\{z = w\}$ at the boundary. (Note that this fact was already known before Fefferman's work). Hence, setting $z := x + \sqrt{-1}y$, $w := u + \sqrt{-1}v$ we study the singularity of $B(z, \overline{w})$ at $\{x = u, y + v = 0, y = y^1\}$ with $y^1 \in \partial W_1 \cap \partial W_2$ as a hyperfunction. Precisely, we set a holomorphic function $f(z, \overline{w}) := B(z + \sqrt{-1}y^1, \overline{w} - \sqrt{-1}y^1)$ and investigate singularity of $f(z, \overline{w})$ at $\{x = u = 0, y = v = 0\}$. We can see that hyperfunction $f((x,u) + \sqrt{-1}(\Gamma \times \Gamma^a)0)$ is well-defined, with $\Gamma := \{y; y + y^1 \in W\}$, $\Gamma^a := -\Gamma$. We define θ_j^1 $(j = 1,2)$ by the relation

$$\frac{-d\varphi_j(y^1)}{|d\varphi_j(y^1)|} = \omega(\theta_j^1) \quad (j = 1,2).$$

here and hereafter $\omega(\theta)$ denotes $(\cos\theta, \sin\theta)$. Without loss of generality, we may assume that

$$0 < \theta_1^1 < \theta_2^1 < 2\pi, \text{ and } 0 < \theta_2^1 - \theta_1^1 < \pi.$$

Thus we have the following:

$$\begin{aligned}
&\mathrm{SS}(f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0))\cap\{x=u\}\\
&\qquad\subset\{(x,u;\sqrt{-1}(\omega(\theta),-\omega(\theta)))\in\sqrt{-1}T^*\mathbb{R}^4;x=u,\,\theta_1^1\le\theta\le\theta_2^1\}\\
&=\{(x,u;\sqrt{-1}\sum_{j=1}^{2}(t\frac{\partial\varphi_1}{\partial y_j}(y^1)+(1-t)\frac{\partial\varphi_2}{\partial y_j}(y^1))(du_j-dx_j))\in\sqrt{-1}T^*\mathbb{R}^4;\\
&\qquad\qquad x=u,\,0\le t\le 1\}.
\end{aligned}$$

where $\mathrm{SS}(\cdot)$ denotes the singularity spectrum of a hyperfunction. Moreover we can define a function $g(z,\overline{w},\theta)$ and also see that

$$\mathrm{sp}(f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0))=\mathrm{sp}\left(\int_{\theta_1^1}^{\theta_2^1}g((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0,\theta)\,d\theta\right).$$

Here sp: $\mathcal{B}_{\mathbb{R}^4}\widetilde{\longrightarrow}\pi_*\mathcal{C}_{\mathbb{R}^4}$ denotes the spectral isomorphism from the sheaf of hyperfunctions to that of microfunctions. In Section 2 we obtain an asymptotic expansion of $f(z,\overline{w})$ above from the microlocal point of view using the result of Section 1. Under the notations above, our main theorem is the following:

Main Theorem. *There exists a sequence* $\{R_j(\theta)\}_{j=0}^{\infty}$ *such that for any* $\varepsilon>0$ *the boundary value of*

$$f(z,\overline{w})-\frac{1}{(2\pi)^2}\int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon}d\theta\int_{A_j}^{\infty}\sum_{j=0}^{\infty}R_j(\theta)\,r^{3-j}e^{\sqrt{-1}\langle z-\overline{w},\omega(\theta)\rangle r}dr \tag{0.3}$$

is micro-analytic at

$$\{(x,u;\sqrt{-1}(\omega(\theta),-\omega(\theta)))\in\sqrt{-1}T^*\mathbb{R}^4;x=u,\theta_1^1+\varepsilon<\theta<\theta_2^1-\epsilon\}. \tag{0.4}$$

Here $\{R(\theta)\}_{j=0}^{\infty}$ *satisfies following conditions:*

(1) *there exists a complex neighborhood* U *of* $]\theta_1^1,\theta_2^1[$ *such that each* $R_j(\theta)$ *is a holomorphic function of* U *with*

$$R_0(\theta)=\frac{\tan(\theta_2-\theta)\tan(\theta-\theta_1)}{\tan(\theta_2-\theta)+\tan(\theta-\theta_1)},$$

(2) *for any* $V \Subset U$ *there exist constants* $\widetilde{C}$, $\widetilde{M}$ *such that*

$$\sup_{\theta \in V} |R_j(\theta)| \leq j! \widetilde{C} \widetilde{M}^j \qquad (\forall j \geq 0).$$

Where in (0.3) $A_j := \max\{0, (j-3)A\}$ *with* A *is some positive constant depending on* $\widetilde{M}$. *In other words, on the set* (0.4) *the following equality holds as a microfunction:*

$$f((x,u) + \sqrt{-1}(\Gamma \times \Gamma^a)0) = R(D_x)\delta(x-u). \tag{0.14}$$

Here $R(D_x)$ *denotes a microdifferential operator defined by the symbol* $\sum_{j=0}^{\infty} R_j(\theta) r^{2-j}$ *where* $r\omega(\theta)$ *denotes the symbol of* $\frac{1}{\sqrt{-1}}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ *by the polar coordinates. Moreover the second term of* (0.3) *is calculated as follows:*

$$\begin{aligned}
&\frac{3}{2\pi^2} \int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} \frac{R_0(\theta)}{\langle z-\overline{w}, \omega(\theta)\rangle^4} d\theta \\
&+ \frac{1}{(2\pi)^2} \sum_{j=1}^{3} \int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} d\theta \int_0^{\infty} R_j(\theta)\, r^{3-j} e^{\sqrt{-1}\langle z-\overline{w}, \omega(\theta)\rangle r} dr \\
&+ \frac{1}{(2\pi)^2} \sum_{j=4}^{\infty} \int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} d\theta \int_{(j-3)A}^{\infty} R_j(\theta)\, r^{3-j} e^{\sqrt{-1}\langle z-\overline{w}, \omega(\theta)\rangle r} dr. \qquad \blacksquare
\end{aligned}$$

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§1. Preliminaries

In this article we will use the following notation: for $z = (z_1, \cdots, z_n) = x + \sqrt{-1}y$, $w = (w_1, \cdots, w_n) = u + \sqrt{-1}v \in \mathbb{C}^n$, $\langle z, w\rangle$ denotes $\sum_{j=1}^{n} z_j w_j$. $\omega(\theta)$ denotes $(\cos\theta, \sin\theta)$. If U is an open subset of $\mathbb{C}^n$, then we denote the Hilbert space of square

integrable and holomorphic functions on U by $\mathcal{O}_{\mathcal{L}_2}(U)$.

Let $W \subset \mathbb{R}^n$ be a bounded domain, and

$$\Omega := \mathbb{R}^n + \sqrt{-1}W = \{z \in \mathbb{C}^n\,;\, \mathrm{Im}\, z \in W\}$$

be a pseudoconvex tube domain. It is well-known that this condition is equivalent to the convexity of W. We denote the Bergman kernel for Ω (that is, the reproducing kernel function for $\mathcal{O}_{\mathcal{L}_2}(\Omega)$) by $B(z, \overline{w})$. We recall the following well-known formula:

1.1 Proposition. (cf. [Kor]) *The Bergman kernel $B(z, \overline{w})$ for Ω has the following form:*

$$B(z, \overline{w}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{\sqrt{-1}\langle z - \overline{w}, \xi\rangle}}{\int_W e^{-2\langle v, \xi\rangle} dv} d\xi. \qquad \blacksquare$$

Hereafter we assume that the dimension $n = 2$, $W = W_1 \cap W_2$, where each W_j is strictly convex domain as follows:

$$\begin{cases} W_j = \{y \in \mathbb{R}^2; \varphi_j(y) < 0\} \text{ with an analytic function } \varphi_j\ (j = 1, 2) \text{ such that} \\ (1) \partial W_1 \text{ and } \partial W_2 \text{ intersect transversally.} \\ (2) \text{ If } \varphi_j(y_0) = 0, \text{ then } d\varphi_j(y_0) \neq 0 \text{ and the Hessian matrix} \\ \quad \left(\frac{\partial^2 \varphi_j}{\partial y_k \partial y_l}(y_0)\right)_{1 \leq k, l \leq 2} \text{ is positive definite for } j = 1, 2. \end{cases}$$

As mentioned in Introduction the Bergman kernel $B(z, \overline{w})$ is holomorphic except for the diagonal points $\{z = w\}$ at the boundary and our main concern is singularities at corner points. Thus, we investigate singularities on the set

$$\{(x + \sqrt{-1}y, u + \sqrt{-1}v);\ x = u,\ y + v = 0,\ y = y_0\}.$$

Here $y_0 \in \partial W_1 \cap \partial W_2$. Set $\partial W_1 \cap \partial W_2 = \{y^1, y^2, \cdots, y^{2N}\}$ by the assumption on W, and define θ_j^l by the following relation:

$$\frac{-d\varphi_j(y^l)}{|d\varphi_j(y^l)|} = \omega(\theta_j^l) = (\cos\theta_j^l, \sin\theta_j^l)$$

(recall that $\omega(\theta) = (\cos\theta, \sin\theta)$). Without loss of generality, we may assume

$$0 < \theta_1^1 < \theta_2^1 < \theta_2^2 < \theta_1^2 < \cdots < \theta_2^{2N} < \theta_1^{2N} < 2\pi,$$

$$0 < \theta_2^{2l-1} - \theta_1^{2l-1},\ \theta_1^{2l} - \theta_2^{2l} < \pi\,(l = 1, 2, \cdots, N).$$

Thus, we can define the continuous surjection $y(*) : [0, 2\pi] \ni \theta \mapsto y(\theta) \in \partial W$ by

$$y(\theta) := \begin{cases} y^{2l-1} & (\theta_1^{2l-1} \le \theta \le \theta_2^{2l-1}) \\ y^{2l} & (\theta_2^{2l} \le \theta \le \theta_1^{2l}) \\ y \ \text{ satisfying } \frac{-d\varphi_2(y)}{|d\varphi_2(y)|} = \omega(\theta) & (\theta_2^{2l-1} < \theta < \theta_2^{2l}) \\ y \ \text{ satisfying } \frac{-d\varphi_1(y)}{|d\varphi_1(y)|} = \omega(\theta) & (\theta_1^{2l} < \theta < \theta_1^{2l+1}) \end{cases}$$

(l=1,2, ... N).

Here we set $\theta_1^{2N+1} := \theta_1^1$. Hence we have

$$\begin{aligned} B(z, \overline{w}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{\sqrt{-1}\langle z - \overline{w}, \omega(\theta)\rangle r}}{\int_W e^{-2\langle v, \omega(\theta)\rangle r} dv} r\, dr\, d\theta \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{\sqrt{-1}\langle z - \overline{w} - 2\sqrt{-1}y(\theta), \omega(\theta)\rangle r}}{\int_W e^{-2\langle v - y(\theta), \omega(\theta)\rangle r} dv} r\, dr\, d\theta. \end{aligned}$$

From now on we consider the singularity of $B(z, \overline{w})$ at $\{x = u,\ y + v = 0,\ y = y^1\}$ since singularities at $\{y^2, \cdots, y^N\}$ are similar. For any $\theta \in [0, 2\pi]$, $s \ge 0$, define

$$H(s, \theta) := \{v \in W; \langle v - s\omega(\theta) - y(\theta), s\omega(\theta)\rangle = 0\}$$

and let $dv(s, \theta)$ denote the volume element of the line $\{v \in W; \langle v - s\omega(\theta) - y(\theta), s\omega(\theta)\rangle = 0\}$. Thus by setting

$$\rho(s, \theta) := \int_{H(s,\theta)} dv(s, \theta)$$

we have

$$\begin{aligned} \int_W e^{-2\langle v - y(\theta), \omega(\theta)\rangle r} dv &= \int_0^{c_\theta} ds \int_{H(s,\theta)} e^{-2\langle v - y(\theta), \omega(\theta)\rangle r} dv(s, \theta) \\ &= \int_0^{c_\theta} e^{-2sr} \rho(s, \theta)\, ds, \end{aligned}$$

where c_θ is a constant depending on θ such that

$$\int_0^{c_\theta} ds \int_{H(s,\theta)} dv(s, \theta) = \mu(W),$$

and $\mu(W)$ denotes the Lebesgue measure of W. Note that by the assumption on W we easily see

$$\begin{cases} 0 < \inf_{\theta} c_\theta < \sup_{\theta} c_\theta < \infty, \\ 0 < \mu(W) < \infty. \end{cases}$$

Set

$$K(r,\theta) := \int_0^{c_\theta} e^{-2sr}\rho(s,\theta)\,ds.$$

Define a holomorphic function $f(z,\overline{w})$ by

$$\begin{aligned} f(z,\overline{w}) :=& B(z+\sqrt{-1}y^1, \overline{w}-\sqrt{-1}y^1) \\ =& \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{\sqrt{-1}\langle z-\overline{w}+2\sqrt{-1}(y^1-y(\theta)),\omega(\theta)\rangle r}}{K(r,\theta)} r\,dr\,d\theta. \end{aligned}$$

Then, $f(z,\overline{w})$ is holomorphic when Im z, Im $w \in \Gamma := \{y \in \mathbb{R}^2;\ y+y^1 \in W\}$, and the singularity of $B(z,\overline{w})$ at $\{x=u,\ y+v=0,\ y=y^1\}$ is equivalent to that of $f(z,\overline{w})$ at $\{x=u,\ y=v=0\}$. Hence, a hyperfunction $f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0)$ is well-defined, where $\Gamma^a := -\Gamma$ (see [Kan1], [K-K-K], and [S-K-K] for hyperfunctions and microfunctions theory). Define a continuous function $g(z,\overline{w},\theta)$ by

$$g(z,\overline{w},\theta) := \frac{1}{(2\pi)^2}\int_0^{\infty} \frac{e^{\sqrt{-1}\langle z-\overline{w}+2\sqrt{-1}(y^1-y(\theta)),\omega(\theta)\rangle r}}{K(r,\theta)} r\,dr.$$

Thus we can define a hyperfunction $g((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0,\theta)$ similarly to $f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0)$. Clearly we have

$$f(z,\overline{w}) = \int_0^{2\pi} g(z,\overline{w},\theta)\,d\theta,$$

or equivalently,

$$f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0) = \int_0^{2\pi} g((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0,\theta)\,d\theta.$$

By using the estimate of the singularity spectrum (= the analytic wave front set) for a hyperfunction we have the following:

$$\begin{aligned} &\mathrm{SS}(f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0))\cap\{x=u\} \\ &\quad\subset \{(x,u;\sqrt{-1}(\omega(\theta),-\omega(\theta)))\in\sqrt{-1}T^*\mathbb{R}^4;\ x=u,\ \theta_1^1\le\theta\le\theta_2^1\} \\ &= \{(x,u;\sqrt{-1}\sum_{j=1}^{2}(t\frac{\partial\varphi_1}{\partial y_j}(y^1)+(1-t)\frac{\partial\varphi_2}{\partial y_j}(y^1))(du_j-dx_j))\in\sqrt{-1}T^*\mathbb{R}^4; \\ &\qquad\qquad x=u,\ 0\le t\le 1\}. \end{aligned}$$

where SS(·) denotes the singularity spectrum of a hyperfunction. On the other hand, by the definition of $g(z,\overline{w},\theta)$ we can also see that

$$\mathrm{sp}(f((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0))=\mathrm{sp}\left(\int_{\theta_1^1}^{\theta_2^1} g((x,u)+\sqrt{-1}(\Gamma\times\Gamma^a)0,\theta)\,d\theta\right).$$

Here sp: $\mathcal{B}_{\mathbb{R}^4}\widetilde{\longrightarrow}\pi_*\mathcal{C}_{\mathbb{R}^4}$ denotes the spectral isomorphism from the sheaf of hyperfunctions to that of microfunctions.

§2. Asymptotic expansion

In this section, we will obtain an asymptotic expansion of $g(z,\overline{w},\theta)$ from the microlocal point of view. For $\omega(\theta)=(\cos\theta,\sin\theta)$, we set ${}^t\omega(\theta):=(-\sin\theta,\cos\theta)$. Hence we have $\langle\omega(\theta),{}^t\omega(\theta)\rangle=0$. For any $v\in H(s,\theta)$, there exists a unique real number t such that

$$v=y(\theta)+s\cdot\omega(\theta)+t\cdot{}^t\omega(\theta),$$

that is, $H(s,\theta)$ is parametrized by t. For $s\geq 0$, we have $H(s,\theta)\cap\partial W=\{w_1^s,w_2^s\}$. By the assumption on W and definition of $y(\theta)$, we can find the function $w_j^s=w_j(s,\theta)$ and we have

$$\rho(s,\theta)=|w_1(s,\theta)-w_2(s,\theta)|.$$

On the other hand, we will consider equations

$$\varphi_j(y(\theta)+s\cdot\omega(\theta)+t\cdot{}^t\omega(\theta))=0\qquad\theta\in]\theta 1^1,\theta_2^1[\,,\ s\geq 0,\ (j=1,2),$$

Thus, we can apply analytic version of the implicit function theorem by the assumption that $0<\theta_1^2-\theta_1^1<\pi$; that is, we can find a strictly positive constant δ and analytic functions $t_j=t_j(s,\theta)$ $(j=1,2,\ \theta\in]\theta 1^1,\theta_2^1[\,,\ 0\leq s\leq\delta)$ such that

$$\varphi_j(y^1+s\cdot\omega(\theta)+t_j(s,\theta)\cdot{}^t\omega(\theta))=0,$$

$$t_j(0,\theta)=0.$$

We note that there exist complex neighborhoods L and U of $\{s;\ 0\leq s\leq\delta\}$ and $]\theta 1^1,\theta_2^1[$ respectively such that t_j's are holomorphic on $L\times U$ $(j=1,2)$. The lemma below is proved by direct calculation:

2.1 Lemma. *If* $0 \leq s \leq \delta$ *and* $\theta \in]\theta 1^1, \theta_2^1[$ *, then*

$$\rho(s,\theta) = t_1(s,\theta) - t_2(s,\theta),$$
$$w_j(s,\theta) = y^1 + s \cdot \omega(\theta) + t_j(s,\theta) \cdot {}^t\omega(\theta). \qquad \blacksquare$$

By replacing δ small enough, we can obtain for $|s| \leq \delta$, $\theta \in U$ the Taylor expansion

$$(t_1 - t_2)(s,\theta) = \sum_{j=1}^{\infty} \frac{a_j(\theta)}{j!} s^j.$$

We can see by the implicit function theorem that

$$a_1(\theta) = \frac{\partial(t_1 - t_2)}{\partial s}(0,\theta) = \frac{1}{\tan(\theta - \theta_1)} + \frac{1}{\tan(\theta_2 - \theta)}.$$

Hence by shrinking L and U, we may assume that

$$\text{(2.1)} \qquad \begin{cases} a_1(\theta) \neq 0 & (\forall \theta \in U), \\ |t_j(s,\theta)|\text{'s are bounded on } s \in L,\ \theta \in V & (\forall V \Subset U\ (j = 1,2)). \end{cases}$$

Thus there exist constants C and M such that

$$\text{(2.2)} \qquad \begin{cases} \sup_{\theta \in V} |a_j(\theta)| \leq j! C M^j & (\forall j \geq 1), \\ M\delta < 1. \end{cases}$$

Thus we can prove the following lemma:

2.2 Lemma. *$K(r,\theta)$ has the following asymptotic expansion :*

$$K(r,\theta) \sim \sum_{j=1}^{\infty} \frac{a_j(\theta)}{(2r)^{j+1}} \qquad (r \longrightarrow \infty).$$

Precisely, for any $V \Subset U$, there exist positive numbers r_0 and C_V such that for every $r \geq r_0$, $\theta \in V \cap \mathbb{R}$ and $N \geq 1$ the following inequality holds:

$$\left| K(r,\theta) - \sum_{j=1}^{N} \frac{a_j(\theta)}{(2r)^{j+1}} \right| < \frac{C_V^{N+1}(N+1)!}{r^{N+2}}. \qquad \blacksquare$$

On the other hand, by virtue of (2.1) we easily see that $\sum_{j=1}^{\infty} a_j(\theta)/(2r)^{j+1}$ has formal inverse $\sum_{j=0}^{\infty} R_j(\theta) r^{2-j}$: that is, as formal series with $\mathcal{O}(U)$ coefficients the following equality holds:

$$\text{(2.3)} \qquad \left(\sum_{j=1}^{\infty} \frac{a_j(\theta)}{(2r)^{j+1}} \right)^{-1} = \sum_{j=0}^{\infty} R_j(\theta) r^{2-j}.$$

Here, $R_0(\theta) := a_1(\theta)^{-1}$ and $R_j(\theta)$'s are determined inductively by (2.3). Therefore, we can see that for any $V \Subset U$ each $R_j(\theta)$ satisfies a similar inequality to (2.2) ($j = 0, 1, 2, \dots$). Thus, we obtain the following theorem (cf. [A], [B1] and [Kat2] for symbolic calculus):

2.3 Theorem. *There exists a sequence* $\{R_j(\theta)\}_{j=0}^{\infty}$ *such that for any* $\varepsilon > 0$ *the boundary value of*

$$f(z,\overline{w}) - \frac{1}{(2\pi)^2}\int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} d\theta \int_{A_j}^{\infty} \sum_{j=0}^{\infty} R_j(\theta)\, r^{3-j} e^{\sqrt{-1}\langle z-\overline{w},\omega(\theta)\rangle r} dr \tag{2.4}$$

is micro-analytic at

$$\{(x,u\,;\sqrt{-1}(\omega(\theta),-\omega(\theta))) \in \sqrt{-1}T^*\mathbb{R}^4; x = u, \theta_1^1 + \varepsilon < \theta < \theta_2^1 - \epsilon\}. \tag{2.5}$$

Here $\{R(\theta)\}_{j=0}^{\infty}$ *satisfies following conditions:*

(1) *there exists a complex neighborhood* U *of* $]\theta_1^1, \theta_2^1[$ *such that each* $R_j(\theta)$ *is a holomorphic function of* U *with*

$$R_0(\theta) = \frac{\tan(\theta_2-\theta)\tan(\theta-\theta_1)}{\tan(\theta_2-\theta)+\tan(\theta-\theta_1)},$$

(2) *for any* $V \Subset U$ *there exist constants* $\widetilde{C}$, $\widetilde{M}$ *such that*

$$\sup_{\theta\in V} |R_j(\theta)| \leq j!\widetilde{C}\widetilde{M}^j \qquad (\forall j \geq 0).$$

Where in (2.4) $A_j := \max\{0, (j-3)A\}$ *with* A *is some positive constant depending on* $\widetilde{M}$. *In other words, on the set* (2.5) *the following equality holds as a microfunction:*

$$f((x,u) + \sqrt{-1}(\Gamma \times \Gamma^a)0) = R(D_x)\delta(x-u),$$

Here $R(D_x)$ *denotes a microdifferential operator defined by the symbol* $\sum_{j=0}^{\infty} R_j(\theta) r^{2-j}$ *where* $r\omega(\theta)$ *denotes the symbol of* $\frac{1}{\sqrt{-1}}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ *by the polar coordinates. Moreover the second term of* (2.4) *is calculated as follows:*

$$\begin{aligned} &\frac{3}{2\pi^2}\int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} \frac{R_0(\theta)}{\langle z-\overline{w},\omega(\theta)\rangle^4} d\theta \\ &+ \frac{1}{(2\pi)^2}\sum_{j=1}^{3}\int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} d\theta \int_0^{\infty} R_j(\theta)\, r^{3-j} e^{\sqrt{-1}\langle z-\overline{w},\omega(\theta)\rangle r} dr \\ &+ \frac{1}{(2\pi)^2}\sum_{j=4}^{\infty}\int_{\theta_1^1+\varepsilon}^{\theta_2^1-\varepsilon} d\theta \int_{(j-3)A}^{\infty} R_j(\theta)\, r^{3-j} e^{\sqrt{-1}\langle z-\overline{w},\omega(\theta)\rangle r} dr. \end{aligned} \tag{2.6}$$

∎

2.3 Remark. By (2.6), we can see that

$$B(z,\overline{z}) = O\left(\mathrm{dist}(y,y^1)^{-4}\right) \quad (W \ni y \longrightarrow y^1).$$

On the other hand, if $v_0 \in \partial(W_1 \cap W_2)\backslash(\partial W_1 \cap \partial W_2)$, then we can apply the proof of Kashiwara's theorem (0.4) and obtain that

$$B(z,\overline{z}) = O\left(\mathrm{dist}(y,v_0)^{-3}\right) \quad (W \ni y \longrightarrow v_0).$$

Hence we can hardly expect that similar result to theorem 0.3 holds at $(x,u\,;\sqrt{-1}\sum_{j=1}^{2}(t\frac{\partial\varphi_1}{\partial y_j}(y^1)+(1-t)\frac{\partial\varphi_2}{\partial y_j}(y^1))(du_j-dx_j))$, $(x=u,\ t=0,1)$.

References

[A] Aoki T., *Symbols and formal symbols of pseudodifferential operators*, Group Representation and Systems of Differential Equations, Advanced Studies in Pure Math. **4** (1984), 181–208.

[B1] Boutet de Monvel L., *Opérateurs pseudo-differentiels analytiques et opérateurs d'ordre infini*, Ann. Inst. Fourier, Grenoble **22**-3 (1972), 229–268.

[B2] ———, *Complément sur le noyau de Bergman*, Seminaire équations aux dérivées partielles 1985–86, Ecole Polytech. Paris.

[B-Sj] Boutet de Monvel L. and Sjöstrand J., *Sur la singularité des noyaux de Bergman et de Szegö*, Astérisque, Soc. Math. de France **34–35** (1976), 123–164.

[F] Fefferman C., *The Bergman Kernel and Biholomorphic Mappings of Pseudoconvex Domains*, Invent. Math. **26** (1974), 1–65.

[H] Hörmander L, *An Introduction to Complex Analysis in Several Variables, 3rd ed.*, North Holland, 1990.

[Kan1] Kaneko A., *Introduction to Hyperfunctions Part I(2nd ed.),II*, Univ. of Tokyo press 1990, 1982 (in Japanese) ; English translation from Kluwer, 1988.

[Kan2] ———, *Introduction to Kashiwara's Microlocal Analysis for the Bergman Kernel*, Lecture Note, Univ. of Tokyo, 1989.

[Kash] Kashiwara M., *Analyse micro-locale du noyau de Bergman*, Séminaire Goulaouic-Schwartz 1976–77, Ecole Polytech. Paris.

[K-K-K] Kashiwara M., Kawai T. and Kimura T., *Foundation of Algebraic Analysis*, Kinokuniya, 1980 (in Japanese) ; English translation from Princeton, 1986.

[K-Sc 1] Kashiwara M. and Schapira P., *Microlocal Study of Sheaves*, Astérisque, Soc. Math. de France **128** (1985).

[K-Sc 2] ———, *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, Berlin–Heidelberg–New York, 1990.

[Kat1] Kataoka K, *On the theory of Radon transformations of hyperfunctions*, J. Fac. Sci. Univ. of Tokyo Sem. IA **28** (1981), 331–412.

[Kat2] ———, *Lectures at Univ. of Tokyo*, 1990, 1992.

[KoG] Komatsu G., *Lecture at Univ. of Tokyo*, 1991.

[Kor] Korányi A., *The Bergman kernel function for tubes over convex cones*, Pacific J. Math. **12** (1962), 1355–1359.

[S-K-K] Sato M., Kawai T. and Kashiwara M., *Microfunctions and Pseudo-differential Equations*, Hyperfunctions and Pseudo-Differential Equations, Komatsu H.(Ed.), Proceedings Katata 1971, Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 265–529.

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