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BRANCHING OF SINGULARITIES FOR SOME THIRD ORDER MICROHYPERBOLIC OPERATORS MULTIPLY CHARACTERISTIC AT $x_1 = 0$

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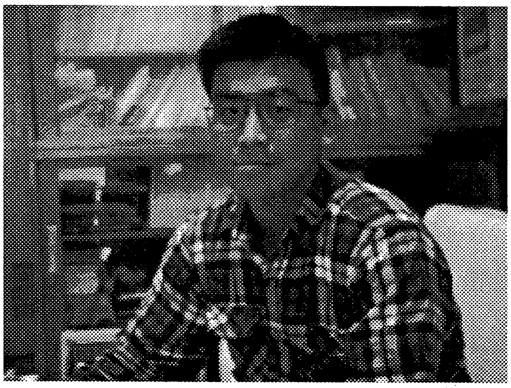

§1 INTRODUCTION

In this brief report, we consider the branching of the support of microfunction solutions to a microhyperbolic equation of third order, triply characteristic over the initial surface $x_1 = 0$. Branching of singularities has been studied by many authors. Alinhac and Taniguchi-Tozaki made researches into second order hyperbolic operators in the C^∞-category. Hanges and Oaku treated operators of the form $x_1 D_1 -$ (lower), in the C^∞- and C^ω-categories respectively. Amano-Nakamura studied an operator of arbitrary order and reduced the problem of branching of C^∞-singularities to that of Stokes phenomena.

§2 RESULTS

Let

$$P(x,D) = D_1^3 - x_1^2 D_n^2 D_1 + 2(a-b) D_n D_1 + \{2(a+b)-3\} x_1 D_n^2 + \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2, x', D') x_1^{l+1} D_1^l$$

be a microdifferential operator defined near a point p in $\{(x, i\xi) \in iT^*\mathbb{R}^n; x_1 = 0, \xi_n > 0\}$. Here we assume that ord$\alpha_{-l} \leq -l-1$ and that α_{-l} is an polynomial in $t = \frac{1}{2}x_1^2$ and x_n.

Here we write $x = (x_1, x_2, \ldots, x_n) = (x_1, x'), D_j = \frac{\partial}{\partial x_j}(1 \leq j \leq n)$,

a and b are constants such that

$$a \notin \mathbb{Z}, a+b \notin \frac{1}{2} + \mathbb{Z}, b \notin \mathbb{Z}.$$

$\sigma(P)$, the principal symbol of P, has the factorization

$$\sigma(P) = (\xi_1 - x_1\xi_n)\xi_1(\xi_1 + x_1\xi_n).$$

Hence P is microhyperbolic and triply characteristic at $x_1 = 0$. In $x_1 \neq 0$, it is simply characteristic and we can apply a propagation theorem of SKK. That is, the support of a solution to P is a union of bicharacteristic strips, each of which is parametrized by x_1. Let $b_j^{\pm}$ be the half bicharacteristic strip in $\pm x_1 > 0$ issuing from p, contained in $\xi_1 = x_1\xi_n, 0, -x_1\xi_n$ for $j = 1, 2, 3$ respectively. A microfunction solution u to P, defined in the intersection of a neighborhood of p and $\{(x; i\xi dx); x_1 > 0\}$, is said to be *j-pure* if $u = 0$ on b_k^+ $(k \neq j)$ and $u \neq 0$ on b_j. Remark that according to the general theory due to Kashiwara- Kawai on microhyperbolic operators, we have the isomorphism

$$(\Gamma_{\{x_1>0\}}\mathcal{C}_M^P)_p \simeq \mathcal{C}_{M,p}^P$$

where $M = \mathbb{R}_x^n$ and $\mathcal{C}_M^P$ is the solution sheaf. It means unique extendability of solutions across $x_1 = 0$.

Now we pose the following problem.

PROBLEM 1 (branching of singularities).

What is the support of the extension of a j-pure solution?

Let $N = \{x; x_1 = 0\} \subset \mathbb{R}^n$, ρ be the pull-back $N \times_M iT^*M \to iT^*N$ and $p' = \rho(p)$. We have the boundary value (iso)morphism

$$b.v. : (\Gamma_{\{x_1>0\}}\mathcal{C}_M^P)_p \xrightarrow{\sim} \overset{3}{\oplus}\, \mathcal{C}_{N,p'}$$

$$u \mapsto (D_1^k u(+0, x'))_{k=1,2,3}.$$

Our next problem is

PROBLEM 2 (boundary value problem with purity). *Let $f(x')$ be an element of $\mathcal{C}_{N,p'}$. Consider*

$$(*)\begin{cases} Pu = 0 \quad in \quad x_1 > 0 \\ u \ is\ j\text{-}pure \\ u(+0, x') = f(x') \end{cases}$$

We give answers to the two problems above. First we have

ANSWER TO PROBLEM 2. *(*) is uniquely solvable for a generic (a, b). More precisely, there is a holomorphic function G in $\{(a,b) \in \mathbb{C}^2; a \notin \mathbb{Z}, a+b \notin \frac{1}{2} + \mathbb{Z}, b \notin \mathbb{Z}\}$, not vanishing identically, such that (*) is uniquely solvable if $G(a, b) \neq 0$.*

REMARK

G can be written explicitly in terms of integrals which resemble that defining Beta function.

Next, we have

ANSWER TO PROBLEM 1(GENERIC CASE).

For a generic (a, b), we have the following: If a solution u is pure, its extension across $x_1 = 0$ has three branches: $u \neq 0$ on each $b_j^-(j = 1, 2, 3)$.

If $\alpha_{-l} = 0$ for all l, we can consider the case $(a, b) \in \mathbb{N} \times \mathbb{N}, \mathbb{N} = \{1, 2, 3, ...\}$.

This non-generic case is interesting in that we encounter a different kind of branching phenomenon.

ANSWER TO PROBLEM 1 (NON-GENERIC CASE).

Under the condition above, we have: (1) If a solution u is 1-pure, then $u \neq 0$ on b_1^-, $u \neq 0$ on b_2^- and $u = 0$ on b_3^-.

(2) If u is 2-pure, $u = 0$ on $b_1^- \cup b_3^-$ and $u \neq 0$ on b_2^-.

(3) If u is 3-pure, $u = 0$ on b_1^-, $u \neq 0$ on b_2^- and $u \neq 0$ on b_3^-.

Roughly speaking, it means that b_2^- is "priviledged". Under other conditions, it happens that another half-bicharacteristic strip is priviledged in a similar sense. In fact, we can prove that if $b \in \mathbb{N}$ and $a + b \in \frac{3}{2} - \mathbb{N}$, then b_1^- is priviledged and that if $a \in \mathbb{N}$ and $a + b \in \frac{3}{2} - \mathbb{N}$, then b_3^- is.

§3 SKETCH OF THE PROOF

The problems are solved by constructing a "j-pure fundamental solution". That is, we construct a morphism

$$E_j : \mathcal{C}_{N,p'} \to (\Gamma_{\{x_1>0\}}\mathcal{C}_M^P)_p$$

$$f(x') \mapsto (E_j f)(x)$$

such that $E_j f$ is a j-pure solution. Once we know the boundary values

$$D_1^k(E_j f)(+0, x') \quad (k = 0, 1, 2, j = 1, 2, 3),$$

we immediately obtain the results in the previous section. We perform the change of variables $t = \frac{1}{2}x_1^2$ for $x_1^3 P$, and apply the quantized Legendre transform with respect to (t, x'). Then the operator to be considered is

$$Q(\zeta, x', \partial_\zeta, D_{x'}) = J(\zeta, \partial_\zeta) + J'(\zeta, x', \partial_\zeta, D'_x)$$

where

$$J(\zeta, \partial_\zeta) = (\zeta^3 + \zeta)\partial_\zeta^3 + \{\frac{15}{2}\zeta^2 - i(a-b)\zeta + a + b + \frac{3}{2}\}\partial_\zeta^2 + \{12\zeta - 2i(a-b)\}\partial_\zeta + 3$$

$$J' = \sum_{m=2}^{\text{finite}} \sum_{j=0}^{m-2} \alpha_{m,j}(x', D')\zeta^j \partial_\zeta^m \in \mathcal{E}(-1)$$

$$\text{ord}\alpha_{m,j} \leq -m - 1.$$

J is an ordinary differential operator of Fuchs type called Jordan-Pochhammer hypergeometric operator. Its solutions have Euler integral representation. A suitably chosen path gives us a solution with a desired singularity. Such a solution roughly corresponds to a j-pure solution. Of course we have to deal with the perturbation J'. This is performed by successive approximation. We construct a microdifferential operator $\tilde{E}_j(\zeta, x', D')$ of order 0 such that :

$\tilde{E}_j$ is defined in (a neighborhood in $\mathbb{C}_\zeta$ of $\{\zeta; \mathrm{Re}\zeta \geq 0, \zeta \neq a_j\}$) $\times$ (a conic neighborhood in iT^*N of p'). Here $a_j = i, 0, -i$ if $j = 1, 2, 3$ respectively.

$$(\tilde{E}_j f)(\zeta, x') \in \mathcal{CO}_+^\infty \quad \text{for all } f \in \mathcal{C}_{N,p'}$$

$$\tilde{E}_j \in \zeta^{-1}\mathcal{E}(0) + \zeta^{-\frac{3}{2}}\mathcal{E}(0) \quad \text{at} \quad \zeta = \infty$$

$$Q(\zeta, x', \partial_\zeta, D'_x)\tilde{E}_j(\zeta, x', D') = 0$$

Here $\partial_\zeta = [D_\zeta, \bullet]$. Obviously, for any $f(x') \in \mathcal{C}_{N,p'}$, we have

$$Q(\zeta, x', D_\zeta, D_{x'},)[\tilde{E}_j(\zeta, x', D')f(x')] = 0$$

According to [Kat], $\tilde{E}_j f$ defines a j-pure solution, which is the definition of $(E_j f)(x)$. Its boundary values are calculated from the expansion coefficients of Q at $\zeta = \infty$.

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