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A CLASSICAL APPROACH TO STUDIES ON PROPAGATION OF ANALYTIC SINGULARITIES

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1. Introduction

It is natural to consider the problems in the framework of hyperfunctions, when we study "propagation of analytic singularities." Many authors have investigated such problems from the viewpoint of "Algebraic Analysis." On the other hand, "propagation of singularities" has been investigated in the frameworks of C^∞ or Gevrey classes by applications of "Classical Analysis." In this article we attempt to study "propagation of analytic singularities," applying "Classical Analysis." There is Hörmander's book [3] for a short introduction to theory of hyperfunctions, which is not so hard for us, studying in the C^∞ category, to understand. There is also Treves' book as to analytic pseudodifferential operators, which were studied by Boutet de Monvel and Kree [1]. Combining the methods in these two books, we will apply the arguments in Kajitani and Wakabayashi [4] to the studies of "propagation of analytic singularities."

2. Function spaces

Let $\varepsilon \in \mathbf{R}$, and denote $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$. We denote

$$\widehat{\mathcal{S}}_\varepsilon := \{v(\xi) \in C^\infty(\mathbf{R}^n);\ e^{\varepsilon\langle\xi\rangle}v(\xi) \in \mathcal{S}\}.$$

we say that $v_j \to v$ in $\widehat{\mathcal{S}}_\varepsilon$ as $j \to \infty$ if $e^{\varepsilon\langle\xi\rangle}v_j(\xi) \to e^{\varepsilon\langle\xi\rangle}v(\xi)$ in $\mathcal{S}$ as $j \to \infty$. Since $\mathcal{D}$ is dense in $\widehat{\mathcal{S}}_\varepsilon$, it is obvious that the dual space $\widehat{\mathcal{S}}_\varepsilon'$ of $\widehat{\mathcal{S}}_\varepsilon$ is identified with $\{e^{\varepsilon\langle\xi\rangle}v(\xi) \in \mathcal{D}';\ v \in \mathcal{S}'\}$. For $\varepsilon \geq 0$ we can define

$$\mathcal{S}_\varepsilon := \mathcal{F}^{-1}[\widehat{\mathcal{S}}_\varepsilon]\ (= \mathcal{F}[\widehat{\mathcal{S}}_\varepsilon] = \{u \in \mathcal{S};\ e^{\varepsilon\langle\xi\rangle}\hat{u}(\xi) \in \mathcal{S}\}),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathcal{S}$ (or $\mathcal{S}'$), respectively, and $\hat{u}(\xi) = \mathcal{F}[u](\xi)$. We introduce the topology in $\mathcal{S}_\varepsilon$ so that $\mathcal{F} : \widehat{\mathcal{S}}_\varepsilon \to \mathcal{S}_\varepsilon$ is homeomorphic. Denote by $\mathcal{S}'_\varepsilon$ the dual space of $\mathcal{S}_\varepsilon$ for $\varepsilon \geq 0$. Then we can define the transposed operators ${}^t\mathcal{F}$ and ${}^t\mathcal{F}^{-1}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$ which map $\mathcal{S}'_\varepsilon$ and $\widehat{\mathcal{S}}'_\varepsilon$ onto $\widehat{\mathcal{S}}'_\varepsilon$ and $\mathcal{S}'_\varepsilon$, respectively. Since $\widehat{\mathcal{S}}_{-\varepsilon} \subset \widehat{\mathcal{S}}'_\varepsilon$ ($\subset \mathcal{D}'$) for $\varepsilon \geq 0$, we can define $\mathcal{S}_{-\varepsilon} := {}^t\mathcal{F}^{-1}[\widehat{\mathcal{S}}_{-\varepsilon}]$ for $\varepsilon \geq 0$. It is easy to see that $\mathcal{S}'_{-\varepsilon} := \mathcal{F}[\widehat{\mathcal{S}}'_{-\varepsilon}]$ is the dual space of $\mathcal{S}_{-\varepsilon}$, $\widehat{\mathcal{S}}'_{-\varepsilon} \subset \mathcal{S}' \subset \widehat{\mathcal{S}}'_\varepsilon$ and $\mathcal{S}'_{-\varepsilon} \subset \mathcal{S}' \subset \mathcal{S}'_\varepsilon$ for $\varepsilon \geq 0$, and that $\mathcal{F} = {}^t\mathcal{F}$ on $\mathcal{S}'$. So we write ${}^t\mathcal{F}$ as $\mathcal{F}$. Let K be a compact subset of $\mathbf{C}^n$, and let $\mathcal{A}'(K)$ be the space of analytic functionals carried by K. Denote $\mathcal{A}'$ ($= \mathcal{A}'(\mathbf{R}^n)$) $:= \cup_{K \subset\subset \mathbf{R}^n} \mathcal{A}'(K)$. Then we have

$$\mathcal{A}' \subset \cap_{\varepsilon<0} \mathcal{S}_\varepsilon \subset \mathcal{F} := \cap_{\varepsilon>0} \mathcal{S}'_\varepsilon .$$

Following [3], we put

$$\begin{aligned} U(x, x_{n+1}) &= (sgn\ x_{n+1}) \exp[-|x_{n+1}|\langle D\rangle] u(x)/2 \\ &\quad (= (sgn\ x_{n+1}) \mathcal{F}_\xi^{-1}[\exp[-|x_{n+1}|\langle\xi\rangle]\hat{u}(\xi)](x)/2) \quad \text{for } u \in \mathcal{F}, \end{aligned}$$

where $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, $x_{n+1} \in \mathbf{R}$ and $D = i^{-1}\partial = i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. Then we can see that

$$\begin{aligned} &U(x, x_{n+1}) \in C^\infty(\mathbf{R}^{n+1} \setminus (\mathbf{R}^n \times \{0\})), \\ &U(x, x_{n+1}) = -U(x, -x_{n+1}) \quad \text{if } x_{n+1} \neq 0, \\ &U(x, x_{n+1})\big|_{x_{n+1}>0} \in C^\infty([0, \infty); \mathcal{F}), \\ &u(x) = U(x, +0) - U(x, -0), \\ &(1 - \Delta_{x,x_{n+1}}) U(x, x_{n+1}) = 0 \quad \text{in } \mathbf{R}^{n+1} \setminus (\mathbf{R}^n \times \{0\}). \end{aligned}$$

We can define supp u for $u \in \mathcal{F}$ as follows;

$$\begin{aligned} x^0 \notin \text{supp } u \overset{def}{\Longleftrightarrow} \text{“}&U(x, x_{n+1}) \text{ can be extended} \\ &\text{in a neighborhood of } (x^0, 0) \text{ as a } C^2\text{-function.”} \end{aligned}$$

Then, for a compact subset K of $\mathbf{R}^n$ we have the following:

(i) $u \in \mathcal{A}'(K) \Longleftrightarrow$ supp $u \subset K$,

(ii) $\exists v \in \mathcal{F}$ *s.t.* supp $v \subset K$ and supp $(u - v) \subset \overline{\mathbf{R}^n \setminus K}$, where $\overline{A}$ denotes the closure of A. Moreover, supp $(v - w) \in \partial K$ if $w \in \mathcal{F}$, supp $w \subset K$ and supp $(u - w) \subset \overline{\mathbf{R}^n \setminus K}$, where ∂K denotes the boundary of K in $\mathbf{R}^n$.

For a bounded open subset Ω of $\mathbf{R}^n$ the space of hyperfunctions $B(\Omega)$ in Ω is defined by

$$B(\Omega) = \mathcal{A}'(\overline{\Omega})/\mathcal{A}'(\partial\Omega)$$

(see [3]). By the property (ii) of supp u we can define

$$\text{supp } [u] = \text{supp } u \cap \Omega,$$

where $u \in \mathcal{A}'(\overline{\Omega})$ and $[u]$ ($\in B(\Omega)$) denotes the residue class of u. We can also define the restriction $u|_\omega$ of $u \in B(\Omega)$ to an open subset ω of Ω. For any open subset Ω of $\mathbf{R}^n$ (or any real analytic manifold Ω) the space $B(\Omega)$ of hyperfunctions in Ω is defined by its sheaf property (see [3]). We note that hyperfunctions can be locally identified with elements in $\mathcal{A}'$.

3. Pseudodifferential operators in $\mathcal{S}'_\varepsilon$

Let $R_0 > 0$ and $\delta_1, \delta_2 \in \mathbf{R}$, and let $a(\xi, y, \eta)$ and $b(\xi, y, \eta)$ be symbols in $C^\infty(\mathbf{R}^n_\xi \times \mathbf{R}^n_y \times \mathbf{R}^n_\eta)$ satisfying

$$\begin{aligned} |\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma a(\xi, y, \eta)| &\leq C_{\alpha,\gamma}(B/R_0)^{|\beta|}\langle\xi\rangle^{m_1+|\beta|}\langle\eta\rangle^{m_2} \\ &\times \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle] \quad \text{if } \langle\xi\rangle \geq R_0|\beta|, \\ |\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma b(\xi, y, \eta)| &\leq C_{\alpha,\gamma}(B/R_0)^{|\beta|}\langle\xi\rangle^{m_1}\langle\eta\rangle^{m_2+|\beta|} \\ &\times \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle] \quad \text{if } \langle\eta\rangle \geq R_0|\beta|. \end{aligned}$$

Define

$$a(D_x, y, D_y)u(x) = (2\pi)^{-n}\mathcal{F}_\xi^{-1}\Big[\int e^{-iy\cdot\xi}\Big(\int e^{iy\cdot\eta}a(\xi, y, \eta)\hat{u}(\eta)d\eta\Big)dy\Big](x)$$

for $u \in \mathcal{S}_\infty := \cap_\varepsilon \mathcal{S}_\varepsilon$.

Proposition 3.1. *$a(D_x, y, D_y)$ (resp. $b(D_x, y, D_y)$) can be defined as a continuous linear operator, which maps $\mathcal{S}_{\varepsilon_2}$ to $\mathcal{S}_{\varepsilon_1}$ and $\mathcal{S}'_{-\varepsilon_2}$ to $\mathcal{S}'_{-\varepsilon_1}$, if $\varepsilon > 0$, $\varepsilon_2 > \delta_1 + \delta_2 + \varepsilon_1$, $R_0 \geq \max\{1, ene \max\{1 + \sqrt{2}, B\}(\varepsilon_2 - \varepsilon_1 - \delta_1 - \delta_2)^{-1}\}$ and $1/R_0 \geq \varepsilon'$, where $\varepsilon = \varepsilon_2 - \delta_2$ and $\varepsilon' = \varepsilon_1 + \delta_1$ (resp. $\varepsilon = -\varepsilon_1 - \delta_1$ and $\varepsilon' = \delta_2 - \varepsilon_2$). In particular, $a(D_x, y, D_y)$ maps continuously $\cup_{\varepsilon>0}\mathcal{S}_\varepsilon$ to $\cup_{\varepsilon>0}\mathcal{S}_\varepsilon$ and $b(D_x, y, D_y)$ maps continuously $\mathcal{F}$ to $\mathcal{F}$, if $\delta_1 = \delta_2 = 0$.*

We need symbol calculus for a various kind of symbol classes. We give here only a few results on symbol calculus. Let $\mathcal{U}$ be an open conic set in $T^*\mathbf{R}^n \setminus 0$. First we assume that $a(x,\xi,y,\eta)$ satisfies the following: (i)

$$\begin{aligned}
&\left|D_x^{\beta+\tilde\beta}\partial_\xi^{\alpha+\tilde\alpha}D_y^{\lambda+\tilde\lambda}\partial_\eta^{\rho+\tilde\rho}a(x,\xi,y,\eta)\right| \\
&\le C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\rho|,|\tilde\lambda|}(A_1/R_0)^{|\alpha|}(B_1/R_0)^{|\beta|}(A_2/R_0)^{|\rho|}(B_2/R_0)^{|\lambda|} \\
&\qquad \times \langle\xi\rangle^{m_1-|\tilde\alpha|+|\beta|}\langle\eta\rangle^{m_2-|\tilde\rho|+|\lambda|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle] \\
&\qquad\qquad \text{if } \langle\xi\rangle \ge R_0(|\alpha|+|\beta|) \text{ and } \langle\eta\rangle \ge R_0(|\rho|+|\lambda|).
\end{aligned}$$

(ii) $\delta \le 0$ or $\exists\varepsilon > 0$:

$$\begin{aligned}
&\left|D_x^{\beta+\tilde\beta}\partial_\xi^{\alpha+\tilde\alpha}D_y^{\lambda+\tilde\lambda}\partial_\eta^{\rho+\tilde\rho}a(x,\xi,y,\eta)\right| \\
&\le C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\rho|,|\tilde\lambda|}(A_1/R_0)^{|\alpha|}(B_1/R_0)^{|\beta|}(A_2/R_0)^{|\rho|}(B_2/R_0)^{|\lambda|} \\
&\qquad \times \langle\xi\rangle^{m_1-|\tilde\alpha|+|\beta|}\langle\eta\rangle^{m_2-|\tilde\rho|}\langle|\xi|+|\eta|\rangle^{|\lambda|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle] \\
&\qquad \text{if } |x-y|<\varepsilon,\ \langle\xi\rangle \ge R_0(|\alpha|+|\beta|),\ \langle\eta\rangle \ge R_0|\rho| \text{ and } \langle|\xi|+|\eta|\rangle \ge R_0|\lambda|.
\end{aligned}$$

(iii) $\exists\varepsilon > 0$ and $0 \le \exists\varepsilon' \le 1/2$:

$$\begin{aligned}
&\left|D_x^{\beta}\partial_\xi^{\alpha+\tilde\alpha}D_y^{\lambda}\partial_\eta^{\rho+\tilde\rho}a(x,\xi,y,\eta)\right| \\
&\le C_{|\tilde\alpha|,|\tilde\rho|}(A_1/R_0)^{|\alpha|}B_1'^{|\beta|}(A_2/R_0)^{|\rho|}B_2'^{|\lambda|}|\beta|!|\lambda|! \\
&\qquad \times \langle\xi\rangle^{m_1-|\tilde\alpha|}\langle\eta\rangle^{m_2-|\tilde\rho|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle] \\
&\qquad \text{if } (x,\eta)\in\mathcal{U},\ |x-y|<2\varepsilon,\ |\xi-\eta|<2\varepsilon'\langle\eta\rangle,\ \langle\xi\rangle \ge R_0|\alpha| \text{ and } \langle\eta\rangle \ge R_0|\rho|.
\end{aligned}$$

(iv)

$$\begin{aligned}
&\left|D_x^{\beta}\partial_\xi^{\alpha+\tilde\alpha}D_y^{\lambda+\tilde\lambda}\partial_\eta^{\rho+\tilde\rho}a(x,\xi,y,\eta)\right| \\
&\le C_{|\tilde\alpha|,|\tilde\rho|,|\tilde\lambda|}(A_1/R_0)^{|\alpha|}B_1'^{|\beta|}(A_2/R_0)^{|\rho|}(B_2/R_0)^{|\lambda|}|\beta|! \\
&\qquad \times \langle\xi\rangle^{m_1-|\tilde\alpha|}\langle\eta\rangle^{m_2-|\tilde\rho|+|\lambda|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle] \\
&\qquad\qquad \text{if } (x,\eta)\in\mathcal{U},\ \langle\xi\rangle \ge R_0|\alpha| \text{ and } \langle\eta\rangle \ge R_0(|\rho|+|\lambda|).
\end{aligned}$$

(v) $\exists\varepsilon > 0$:

$$\left|D_x^{\beta}\partial_\xi^{\alpha+\tilde\alpha}D_y^{\lambda^1+\lambda^2+\tilde\lambda}\partial_\eta^{\rho+\tilde\rho}a(x,\xi,y,\eta)\right|$$

$$\leq C_{|\tilde\alpha|,|\tilde\rho|,|\tilde\lambda|}(A_1/R_0)^{|\alpha|}B_1'^{|\beta|}(A_2/R_0)^{|\rho|}(B_2/R_0)^{|\lambda^1|+|\lambda^2|}|\beta|!$$
$$\times \langle\xi\rangle^{m_1-|\tilde\alpha|+|\lambda^1|}\langle\eta\rangle^{m_2-|\tilde\rho|+|\lambda^2|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle]$$
$$\text{if } (x,\eta)\in\mathcal{U},\ |x-y|<2\varepsilon,\ \langle\xi\rangle\geq R_0(|\alpha|+|\lambda^1|) \text{ and } \langle\eta\rangle\geq R_0(|\rho|+|\lambda^2|).$$

Formally we define

$$a(x,D_x,y,D_y)u(x)=\lim_{\nu\downarrow 0}(2\pi)^{-2n}\int\exp[-\nu|\eta|^2]e^{ix\cdot\eta}$$
$$\times\Big(\int\Big(\int e^{iy\cdot(\xi-\eta)}a(x,\eta,y,\xi)\hat u(\xi)d\xi\Big)dy\Big)d\eta \quad \text{for } u\in\mathcal{S}_\infty.$$

Lemma 3.2. (i) *$a(x,D_x,y,D_y)$ is well-defined and maps continuously $\mathcal{S}_{3\delta_1+\delta_2+1}$ to $\mathcal{S}$ if $R_0\geq R_0(A_1,B_2)$, $1/R_0>3\delta_1$ and $\delta_1>0$, where $R_0(A_1,B_2)$ is a constant depending on A_1 and B_2 and locally bounded with respect to A_1 and B_2. Moreover, $a(x,D_x,y,D_y)$ maps continuously $\mathcal{S}_{\delta_2}$ to $\mathcal{S}$ if $\delta_1\leq 0$.* (ii) *Put*

$$a(x,\xi)=\sum_{j=0}^{\infty}\phi_j^{4R_0}(\xi)a_j(x,\xi),$$
$$a_j(x,\xi)=\sum_{|\gamma|=j}\gamma!^{-1}\partial_\eta^\gamma D_y^\gamma a(x,\xi+\eta,x+y,\xi)\big|_{y=0,\eta=0},$$

where the $\phi_j^R(\xi)$ are chosen so that $\phi_j^R(\xi)=0$ if $\langle\xi\rangle\leq 2Rj$, $\phi_j^R(\xi)=1$ if $\langle\xi\rangle\geq 3Rj$, and $|\partial_\xi^{\alpha+\tilde\alpha}\phi_j^R(\xi)|\leq C_{|\tilde\alpha|}(C_0/R)^{|\alpha|}\langle\xi\rangle^{-|\tilde\alpha|}$ if $|\alpha|\leq 2j$. Then,

$$\big|a_{(\beta+\tilde\beta)}^{(\alpha+\tilde\alpha)}(x,\xi)\big|\leq C_{|\tilde\alpha|,|\tilde\beta|}(A/(2R_0))^{|\alpha|}(B/(2R_0))^{|\beta|}\langle\xi\rangle^{m_1+m_2-|\tilde\alpha|}e^{\delta\langle\xi\rangle}$$

if $\langle\xi\rangle\geq 2R_0|\alpha|$, $\langle\xi\rangle\geq 2R_0|\beta|$, $\delta=\delta_1+\delta_2+nA_1B_2/R_0^2$, $A\geq 2(A_1+A_2+C_0/4)$ and $B\geq 2(B_1+B_2)$. (iii) *There is a symbol $r(x,\xi)$ such that $a(x,D_x,y,D_y)=a(x,D)+r(x,D)$ on $\mathcal{S}_\infty$ and*

$$\big|r_{(\tilde\beta)}^{(\tilde\alpha)}(x,\xi)\big|\leq C_{|\tilde\alpha|,|\tilde\beta|}\langle\xi\rangle^{(m_1)_++m_2}\exp[-\kappa\langle\xi\rangle/R_0]$$

if $\kappa>0$, $R_0\geq R_1(A_1,B_2,1/\varepsilon,\kappa)$, $1/R_0>3\delta_1$ and $1/R_0\geq\max\{4(\delta_1)_++\delta_2,9(\delta_1)_+,$ $6\delta_1+12(\delta_1)_++12\delta_2,18(\delta_1)_+-16(\delta_2)_-\}/(12\kappa)$, where $\delta_\pm=\max\{0,\pm\delta\}$. (iv)

$$\big|a_{(\beta)}^{(\alpha+\tilde\alpha)}(x,\xi)\big|\leq C_{|\tilde\alpha|}(A/R_0)^{|\alpha|}B'^{|\beta|}|\beta|!\langle\xi\rangle^{m_1+m_2-|\tilde\alpha|}\exp[(\delta_1+\delta_2)\langle\xi\rangle]$$

if $A\geq A_1+A_2+C_0/4$, $B'\geq 2\max\{B_1',2B_2'\}$, $(x,\xi)\in\mathcal{U}$, $\langle\xi\rangle\geq 2R_0|\alpha|$ and $R_0\geq 4nA_1B_2'$, and

$$\big|r_{(\beta)}^{(\tilde\alpha)}(x,\xi)\big|\leq C_{|\tilde\alpha|,R_0}B(B_1,B_2',1/\varepsilon,R_0/\kappa,\varepsilon'R_0/\kappa)^{|\beta|}|\beta|!$$

$$\times \langle\xi\rangle^{(m_1)_+ + m_2} \exp[-\kappa\langle\xi\rangle/R_0]$$

if $(x,\xi) \in \mathcal{U}$, $\kappa > 0$, $R_0 \geq R_2(A_1, B_2, B_2', 1/\varepsilon, 1/\varepsilon', \kappa)$, $1/R_0 > 3\delta_1$ and $1/R_0 \geq \max\{2\delta_1 + 2\varepsilon'|\delta_1| + 2\delta_2, 4(\delta_1)_+ + \delta_2, 2\delta_1 + |\delta_1|, 4\delta_1 + 2|\delta_1| + 2\delta_2\}/(2\kappa)$.

Next assume that $a(\xi, x, \eta, y, \zeta)$ satisfies

$$\begin{aligned}
&\left|\partial_\xi^{\alpha+\tilde\alpha} D_x^{\beta^1+\beta^2+\tilde\beta} \partial_\eta^{\gamma+\tilde\gamma} D_y^{\lambda^1+\lambda^2+\tilde\lambda} \partial_\zeta^{\rho+\tilde\rho} a(\xi, x, \eta, y, \zeta)\right| \\
&\leq C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\gamma|,|\tilde\lambda|,|\tilde\rho|} (A_1/R_0)^{|\alpha|} (B_1/R_0)^{|\beta^1|+|\beta^2|} (A_2/R_0)^{|\gamma|} \\
&\quad \times (B_2/R_0)^{|\lambda^1|+|\lambda^2|} (A_3/R_0)^{|\rho|} \langle\xi\rangle^{m_1-|\tilde\alpha|+|\beta^1|} \langle\eta\rangle^{m_2-|\tilde\gamma|+|\beta^2|+|\lambda^1|} \\
&\quad \times \langle\zeta\rangle^{m_3-|\tilde\rho|+|\lambda^2|} \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle + \delta_3\langle\zeta\rangle]
\end{aligned}$$

if $\langle\xi\rangle \geq R_0(|\alpha| + |\beta^1|)$, $\langle\eta\rangle \geq R_0(|\gamma| + |\beta^2| + |\lambda^1|)$ and $\langle\zeta\rangle \geq R_0(|\rho| + |\lambda^2|)$, and

$$\begin{aligned}
&\left|\partial_\xi^{\alpha+\tilde\alpha} D_x^{\beta} \partial_\eta^{\gamma+\tilde\gamma} D_y^{\lambda} \partial_\zeta^{\rho+\tilde\rho} a(\xi, x, \eta, y, \zeta)\right| \\
&\leq C_{|\tilde\alpha|,|\tilde\gamma|,|\tilde\rho|} (A_1/R_0)^{|\alpha|} B_1'^{|\beta|} (A_2/R_0)^{|\gamma|} B_2'^{|\lambda|} (A_3/R_0)^{|\rho|} \\
&\quad \times |\beta|!|\lambda|! \langle\xi\rangle^{m_1-|\tilde\alpha|} \langle\eta\rangle^{m_2-|\tilde\gamma|} \langle\zeta\rangle^{m_3-|\tilde\rho|} \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle + \delta_3\langle\zeta\rangle]
\end{aligned}$$

if $(x,\zeta) \in \mathcal{U}$, $|\xi| \geq R_0/4$, $|\eta| \geq R_0/4$, $|\zeta| \geq R_0/4$, $|x-y| < 2\varepsilon$, $|\eta - \zeta| < \varepsilon'\langle\zeta\rangle$, $\langle\xi\rangle \geq R_0|\alpha|$, $\langle\eta\rangle \geq R_0|\gamma|$ and $\langle\zeta\rangle \geq R_0|\rho|$, where $\varepsilon > 0$ and $0 < \varepsilon' \leq 1$. Formally we define

$$\begin{aligned}
a(D_x, y, D_y, w, D_w)u(x) = \mathcal{F}_\eta^{-1}\Big[&\lim_{\nu\downarrow 0}(2\pi)^{-2n} \int\Big(\int\Big(\int\Big(\int \exp[-\nu|\zeta|^2] \\
&\times e^{iy\cdot(\zeta-\eta)} e^{iw\cdot(\xi-\zeta)} a(\eta, y, \zeta, w, \xi)\hat{u}(\xi)d\xi\Big)dw\Big)d\zeta\Big)dy\Big](x)
\end{aligned}$$

for $u \in \mathcal{S}_\infty$.

Lemma 3.3. (i) *$a(D_x, y, D_y, w, D_w)$ is well-defined and maps continuously $\mathcal{S}_{3(\delta_2)_+ + \delta_3 + 1}$ to $\mathcal{S}_{-\delta_1'}$ if $R_0 \geq R_3(B_2)$, $1/R_0 > 3\delta_2$ and $\delta_1' > \delta_1$.* (ii) *Put*

$$\begin{aligned}
a(\xi, y, \eta) &= \sum_{j=0}^{\infty} \phi_j^{4R_0}(\eta) a_j(\xi, y, \eta), \\
a_j(\xi, y, \eta) &= \sum_{|\gamma|=j} \gamma!^{-1} \partial_\zeta^\gamma D_w^\gamma a(\xi, y, \eta + \zeta, y + w, \eta)\big|_{w=0,\zeta=0}.
\end{aligned}$$

Then,

$$\left|\partial_\xi^{\alpha+\tilde\alpha} D_y^{\beta+\tilde\beta} \partial_\eta^{\rho+\tilde\rho} a(\xi, y, \eta)\right| \leq C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\rho|} (A_1/R_0)^{|\alpha|} (B/R_0)^{|\beta|}$$

$$\times (A/R_0)^{|\rho|}\langle\xi\rangle^{m_1-|\tilde\alpha|}\langle\eta\rangle^{m_2+m_3-|\tilde\rho|+|\beta|}\exp[\delta_1\langle\xi\rangle+\delta\langle\eta\rangle]$$

if $\langle\xi\rangle \ge R_0|\alpha|$, $\langle\eta\rangle \ge 2R_0(|\beta|+|\rho|)$, $\delta=\delta_2+\delta_3+nA_2B_2/R_0^2$, $A \ge A_2+A_3+C_0/4$ *and* $B \ge B_1+B_2$. (iii) *There is a symbol* $r(\xi,y,\eta)$ *such that* $a(D_x,y,D_y,w,D_w) = a(D_x,y,D_y)+r(D_x,y,D_y)$ *on* $\mathcal{S}_\infty$ *and*

$$\left|\partial_\xi^{\tilde\alpha}D_y^{\beta+\tilde\beta}\partial_\eta^{\tilde\rho}r(\xi,y,\eta)\right| \le C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\rho|,R_0}(B(B_1,B_2',1/\varepsilon,R_0/\kappa)/R_0)^{|\beta|}$$
$$\times \langle\xi\rangle^{m_1-|\tilde\alpha|+|\beta|}\langle\eta\rangle^{(m_2)_++m_3}\exp[\delta_1\langle\xi\rangle-\kappa\langle\eta\rangle/R_0]$$

if $\langle\xi\rangle \ge R_0|\beta|$, $\kappa>0$, $R_0 \ge R_4(A_2,B_2,B_2',1/\varepsilon,\kappa)$, $1/R_0 > 3\delta_2$ *and* $1/R_0 \ge \max\{4(\delta_2)_++\delta_3,\delta_2+|\delta_2|/2\}/\kappa$. (iv)

$$\left|\partial_\xi^{\alpha+\tilde\alpha}D_y^{\beta}\partial_\eta^{\rho+\tilde\rho}a(\xi,y,\eta)\right| \le C_{|\tilde\alpha|,|\tilde\rho|}(A_1/R_0)^{|\alpha|}B'^{|\beta|}(A/R_0)^{|\rho|}$$
$$\times |\beta|!\langle\xi\rangle^{m_1-|\tilde\alpha|}\langle\eta\rangle^{m_2+m_3-|\tilde\rho|}\exp[\delta_1\langle\xi\rangle+(\delta_2+\delta_3)\langle\eta\rangle]$$

if $(y,\eta)\in\mathcal{U}$, $\langle\xi\rangle \ge R_0|\alpha|$, $\langle\eta\rangle \ge 2R_0|\rho|$, $|\xi| \ge R_0/4$, $R_0 \ge 1$, $A \ge A_2+A_3+C_0/4$ *and* $B' \ge 2\max\{B_1',2B_2'\}$, *and*

$$\left|\partial_\xi^{\tilde\alpha}D_y^{\beta+\tilde\beta}\partial_\eta^{\tilde\rho}r(\xi,y,\eta)\right| \le C_{|\tilde\alpha|,|\tilde\beta|,|\tilde\rho|,R_0}(B(B_1,B_1',B_2',R_0/\kappa)/R_0)^{|\beta|}$$
$$\times \langle\xi\rangle^{m_1-|\tilde\alpha|+|\beta|}\langle\eta\rangle^{(m_2)_++m_3}\exp[\delta_1\langle\xi\rangle-\kappa\langle\eta\rangle/R_0]$$

if $(y,\eta)\in\mathcal{U}$, $\langle\xi\rangle \ge R_0|\beta|$, $|\xi| \ge R_0/4$, $\kappa>0$, $R_0 \ge R_5(A_2,B_2,B_2',1/\varepsilon,1/\varepsilon',\kappa)$, $1/R_0 > 3\delta_2$ *and* $1/R_0 \ge \max\{4(\delta_2)_++\delta_3,\delta_2+|\delta_2|/2\}/\kappa$.

4. Analytic wave front sets and microfunctions

There are several definitions of analytic wave front sets which are equivalent to each other. For $u\in\mathcal{A}'$ we define the FBI transform $T_\lambda u(z)$ by

$$T_\lambda u(z) = u_y(\exp[-\lambda(z-y)^2/2]),$$

where $\lambda>0$ and $z\in\mathbf{C}^n$. We say that $(x^0,\xi^0)\in(T^*\mathbf{R}^n\setminus 0)\setminus WF_A(u)$ if there are a neighborhood V of $x^0-i\xi^0/|\xi^0|$ and positive constants C and c such that

$$|T_\lambda u(z)| \le Ce^{\lambda(1/2-c)} \quad \text{for } z\in V \text{ and } \lambda>0,$$

where $u\in\mathcal{A}'$. Since $WF_A(u)$ is determined by the local properties of $u\in\mathcal{A}'$, the definition of $WF_A(u)$ can be immediately extended to functions in $\mathcal{F}$ and hyperfunctions.

Proposition 4.1. *Let $u \in \mathcal{F}$, and let $(x^0, \xi^0) \in T^*\mathbf{R}^n \setminus 0$. Then, $(x^0, \xi^0) \notin WF_A(u)$ if and only if there are $R_0 > 0$ and a family $\{g^R(\xi)\}_{R \ge R_0} \subset C^\infty(\mathbf{R}^n)$ such that $g^R(\xi) = 1$ in a fixed conic neighborhood Γ of ξ^0,*

$$|g^{R(\alpha+\beta)}(\xi)| \le C_{|\beta|}(C/R)^{|\alpha|}\langle\xi\rangle^{-|\beta|} \quad \text{if } \langle\xi\rangle \ge R|\alpha|$$

and $g^R(D)u$ is analytic near x^0 for $R \ge R_0$.

Lemma 4.2. *Let Γ be an open cone in $\mathbf{R}^n \setminus \{0\}$, and let X be an open set in $\mathbf{R}^n$. Assume that a symbol $p(x, \xi)$ satisfies*

$$\begin{aligned}
&\operatorname{supp} p(x,\xi) \cap X \times \Gamma = \emptyset, \\
&|p^{(\tilde\alpha)}_{(\beta+\tilde\beta)}(x,\xi)| \le C_{|\tilde\alpha|,|\tilde\beta|}(B/R_0)^{|\beta|}\langle\xi\rangle^{m+|\beta|} \quad \text{if } \langle\xi\rangle \ge R_0|\beta|, \\
&|p^{(\tilde\alpha)}_{(\beta)}(x,\xi)| \le C_{|\tilde\alpha|}B^{|\beta|}|\beta|!\langle\xi\rangle^m \quad \text{if } x \in X.
\end{aligned}$$

Then, $WF_A(p(x,D)u) \cap X \times \Gamma = \emptyset$ if $u \in \mathcal{F}$ and $R_0 \ge \sqrt{n}e \max\{B, 2(1+\sqrt{2})\}$.

Let $\mathcal{U}$ be an open conic subset of $T^*\mathbf{R}^n \setminus 0$, and define

$$\mathcal{C}(\mathcal{U}) = B(\mathbf{R}^n)/\{u \in B(\mathbf{R}^n);\ WF_A(u) \cap \mathcal{U} = \emptyset\}.$$

Elements of $\mathcal{C}(\mathcal{U})$ are called microfunctions on $\mathcal{U}$. Let Ω be an open conic set in $T^*\mathbf{R}^n \setminus 0$, and let $P(x,\xi) \in C^\infty(\Omega)$ satisfy

$$|P^{(\alpha)}_{(\beta)}(x,\xi)| \le C_0 A_0^{|\alpha|} B_0^{|\beta|} |\alpha|!|\beta|!\langle\xi\rangle^{m-|\alpha|} \quad \text{for } (x,\xi) \in \Omega \text{ with } |\xi| \ge R_0. \tag{4.1}$$

Assume that $X \times \gamma \subset\subset X_1 \times \gamma_1 \subset\subset \Omega$, where X and X_1 are open sets in $\mathbf{R}^n$, and γ and γ_1 are open conic sets in $\mathbf{R}^n \setminus \{0\}$. Then we can construct a symbol $\widetilde{P}(x,\xi)$ so that $\widetilde{P}(x,\xi) = P(x,\xi)$ in $X \times \gamma \cap \{|\xi| \ge R_0\}$, supp $\widetilde{P}(x,\xi) \subset\subset X_1 \times \gamma_1$, supp $\widetilde{P}(x,\xi) \subset \{|\xi| \ge R_0/2\}$ and

$$\begin{aligned}
|\widetilde{P}^{(\alpha+\tilde\alpha)}_{(\beta+\tilde\beta)}(x,\xi)| &\le C_{|\tilde\alpha|,|\tilde\beta|}(A/R_0)^{|\alpha|}(B/R_0)^{|\beta|}\langle\xi\rangle^{m-|\tilde\alpha|+|\beta|} \\
&\qquad \text{if } \langle\xi\rangle \ge R_0|\alpha| \text{ and } \langle\xi\rangle \ge R_0|\beta|, \\
|\widetilde{P}^{(\alpha+\tilde\alpha)}_{(\beta)}(x,\xi)| &\le C_{|\tilde\alpha|}(A/R_0)^{|\alpha|}B'^{|\beta|}|\beta|!\langle\xi\rangle^{m-|\tilde\alpha|} \\
&\qquad \text{if } x \in X,\ \langle\xi\rangle \ge R_0|\alpha| \text{ and } |\xi| \ge R_0,
\end{aligned}$$

$$|\widetilde{P}^{(\alpha)}_{(\beta+\tilde{\beta})}(x,\xi)| \leq C_{|\tilde{\beta}|} A'^{|\alpha|} (B/R_0)^{|\beta|} |\alpha|! \langle\xi\rangle^{m-|\alpha|+|\beta|}$$
$$\text{if } \xi \in \gamma,\ \langle\xi\rangle \geq R_0|\beta| \text{ and } |\xi| \geq R_0,$$
$$|\widetilde{P}^{(\alpha)}_{(\beta)}(x,\xi)| \leq C A'^{|\alpha|} B'^{|\beta|} |\alpha|!|\beta|! \langle\xi\rangle^{m-|\alpha|}$$
$$\text{if } x \in X,\ \xi \in \gamma \text{ and } |\xi| \geq R_0.$$

From Lemma 4.2 it follows that $\widetilde{P}(x,D)u|_X$ is uniquely determined by $P(x,D)$ and $u \in \mathcal{F}$ modulo $\{f \in \mathcal{F};\ WF_A(f) \cap X \times \gamma = \emptyset\}$ if $R_0 \geq R_0(B,B')$. We can also prove analytic pseudo-locality of $\widetilde{P}(x,D)$ in $X \times \gamma$, shrinking $X \times \gamma$ if necessary. Therefore, we can define $P(x,D) : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ by $P(x,D)u|_X = \widetilde{P}(x,D)u|_X$ in $\mathcal{A}'$ for $u \in \mathcal{F}$.

5. Propagation of singularities

Let $(x^0,\xi^0) \in T^*\mathbf{R}^n \setminus 0$, and let Ω be a conic neighborhood of (x^0,ξ^0) in $T^*\mathbf{R}^n \setminus 0$. Assume that $P(x,\xi) \in C^\infty(\Omega)$ satisfies (4.1), and that $u \in \mathcal{C}(\Omega)$ satisfies $P(x,D)u = 0$ in $\mathcal{C}(\Omega)$. Under the above assumption we study conditions which give $u = 0$ near (x^0,ξ^0) (in $\mathcal{C}(\Omega)$). Let S be a closed conic subset of $T^*\mathbf{R}^n \setminus 0$ such that $(x^0,\xi^0) \in S$, and assume that

(A) supp $u \subset S$, where supp $u = WF_A(v) \cap \Omega$ if u is the residue class of v in $B(\mathbf{R}^n)$.

We choose a real-valued function $\varphi(x,\xi)$, which is defined in Ω and positively homogeneous of degree 0, such that $\varphi(x^0,\xi^0) = 0$, $\varphi(x,\xi) > 0$ if $(x,\xi) \in S \setminus \{(x^0,\lambda\xi^0);\ \lambda > 0\}$, and

$$|\varphi^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_1 A_1^{|\alpha|} B_1^{|\beta|} |\alpha|!|\beta|! \quad \text{for } (x,\xi) \in \Omega \text{ with } |\xi| = 1.$$

Let $\psi(\xi)$ and $\lambda(\xi)$ be functions in $C^\infty(\mathbf{R}^n)$ such that $\psi(\xi) = 1$ for $|\xi| \geq 1$ and $\psi(\xi) = 0$ for $|\xi| \leq 1/2$, $C^{-1}\langle\xi\rangle \leq \lambda(\xi) \leq C\langle\xi\rangle$, $\lambda(\xi) \in S^1_{1,0}$ and

$$|\lambda^{(\alpha)}(\xi)| \leq C_2 A_2^{|\alpha|} |\alpha|! \langle\xi\rangle^{1-|\alpha|} \quad \text{for } (x,\xi) \in \Omega \text{ with } |\xi| \geq 1.$$

Put

$$\Lambda^{\delta}_{a,b,j}(x,\xi)\,(=\Lambda_j(x,\xi)) = \{a(\varphi(x,\xi) - 1/j)\lambda(\xi)^{1-\delta} + b(\varphi(x,\xi) + 1/j)\lambda(\xi)\}\psi(\xi),$$

where $a, b > 0$ and $0 \le \delta \le 1$. Let $P_{\Lambda_j}(x,\xi)$ be a symbol in $S^m_{1,0}$ satisfying

$$P_{\Lambda_j}(x,\xi) \sim \sum_{\lambda,\gamma} (\lambda!\gamma!)^{-1} \partial_\eta^\lambda D_y^\lambda \partial_\zeta^\gamma D_w^\gamma \{ P(x + w + i\Lambda_{j\xi}(x, N^j(x,y,\xi+\eta), \zeta),$$

$$N^j(x,y,\xi+\eta)) \det \frac{\partial N^j}{\partial \xi}(x,y,\xi+\eta)\}|_{y=w=0,\eta=\zeta=0}$$

in $S^m_{1,0}(\Omega)$, where $a+b \le \varepsilon(j) \ll 1$, $\Lambda_{j\xi}(x,\xi,\eta) = \int_0^1 \nabla_\xi \Lambda_j(x, \xi+\theta\eta)\, d\theta$, $\Lambda_{jx}(x,y,\xi)$ $= \int_0^1 \nabla_x \Lambda_j(x+\theta y, \xi)\, d\theta$ and $\eta = N^j(x,y,\zeta)$ is the solution of $\eta + i\Lambda_{jx}(x,y,\eta) = \zeta$. We assume that

(ME) $\exists j_0 \in \mathbf{N}$ and $\exists \chi(x,\xi) \in S^0_{1,0}$ s.t. "$\chi(x,\xi)$ is positively homogeneous of degree 0 for $|\xi| \ge 1$, $\chi(x,\xi) = 1$ near $(x^0, \xi^0/|\xi^0|)$, and $\forall j \ge j_0$, $\exists a_0 > 0$ and $\exists b_0 > 0$ s.t. $0 < \forall a \le a_0$, $0 < \forall b \le b_0$, $0 < \exists \delta_0 \le 1$, $\exists \ell_k \in \mathbf{R}$ ($1 \le k \le 4$), $\exists C > 0$ and $\exists \Psi(x,\xi) \in S^0_{1,0}$, which is positively homogeneous of degree 0 for $|\xi| \ge 1$, satisfying $\operatorname{supp} \Psi \cap S = \emptyset$ and

$$\begin{aligned}\|\langle D\rangle^{\ell_1} v\| \le & C\{\|\langle D\rangle^{\ell_2} P_{\Lambda_j}(x,D)v\| + \|\langle D\rangle^{\ell_1 - 1} v\| \\ & + \|\langle D\rangle^{\ell_3}(1-\chi(x,D))v\| + \|\langle D\rangle^{\ell_4}\Psi(x,D)v\|\}\end{aligned}$$

if $v \in C_0^\infty$ and $0 < \delta \le \delta_0$," where $\|\cdot\|$ denotes the L^2-norm.

Theorem 4.3. *Assume that* (A) *and* (ME) *are satisfied. Then,* $u = 0$ *near* (x^0, ξ^0), *i.e.,* $(x^0, \xi^0) \notin \operatorname{supp} u$.

We can prove the above theorem by the same idea as in [4], after establishing symbol calculus of analytic pseudodifferential operators and pseudodifferential operators introduced here.

Let us give some applications of Theorem 4.3. If $P(x,\xi)$ is microhyperbolic, then we can choose $\varphi(x,\xi)$ so that $P_{\Lambda_j}(x,\xi)$ is elliptic. Therefore, we can prove the results of Kashiwara and Kawai [5].

For Grušin type operators *a priori* estimates (energy estimates) are well-known (see [2]). So we can immediately prove the results on analytic hypoellipticity of Métivier [6] and Okaji [7] in the space of hyperfunctions (microfunctions) by Theorem 4.3.

Finally we consider analytic hypoellipticity of operators with double characteristics. Let $(x^0, \xi^0) = (0; 0, \cdots, 0, 1) \in T^*\mathbf{R}^n \backslash 0$, and let $P(x,\xi)$ be an analytic symbol defined in a conic neighborhood of (x^0,ξ^0) such that $P(x,\xi)=\xi_1^2+\alpha(x,\xi_2,\cdots,\xi_n)+\beta(x,\xi)$ in a conic neighborhood of (x^0,ξ^0), where $\alpha(x,\xi_2,\cdots,\xi_n)$ is positively homogeneous of degree 2, $\alpha(x,\xi_2,\cdots,\xi_n)\geq 0$ and $\beta(x,\xi)\in S^1_{1,0}$ is a classical symbol. Put

$$S=\{(x,\xi)\in T^*\mathbf{R}^n\setminus 0;\ x'=0,\ \xi'=0\},$$

where $1\leq r\leq n-1$, $x'=(x_1,\cdots,x_r)\in\mathbf{R}^r$ and $\xi'=(\xi_1,\cdots,\xi_r)\in\mathbf{R}^r$. We impose the following conditions:

(H–1) $P(x,D)$ is analytic (micro) hypoelliptic in $\Omega\setminus S$, where Ω is a conic neighborhood of (x^0,ξ^0).

(H–2) $\exists U'$: a neighborhood of $(0,0)$ in $\mathbf{R}^r\times\mathbf{R}^r$, $\exists U''$: a complex neighborhood of $(0,\xi^{0\prime\prime})$ in $\mathbf{C}^{n-r}\times(\mathbf{C}^{n-r}\setminus\{0\})$ and $\exists C>0$ s.t.

$$|\alpha(x',z'',\xi_2,\cdots,\xi_r,\zeta'')|\leq C\alpha(x',0,\xi_2,\cdots,\xi_r,\xi^{0\prime\prime})$$

if $(x',\xi')\in U'$ and $(z'',\zeta'')\in U''$, where $z''=(z_{r+1},\cdots,z_n)\in\mathbf{C}^{n-r}$ and $\zeta''=(\zeta_{r+1},\cdots,\zeta_n)\in\mathbf{C}^{n-r}$.

(H–3) $\exists\varepsilon>0$, $\exists C>0$ and $\exists q_j(x',\xi')\in S^1_{1,0}$ ($1\leq j\leq 2\ell$) s.t.

$$\begin{aligned}&(1-\varepsilon)\big(\xi_1^2+\mathrm{Re}\ \alpha(x',z'',\xi_2,\cdots,\xi_r,\zeta'')\big)+\mathrm{Re}\ sub\ \sigma(P)(x',z'',\xi',\zeta'')\\&-\sum_{j=1}^{2\ell}q_j(x',\xi')^2+\sum_{j=1}^{\ell}\{q_{2j-1},q_{2j}\}(x',\xi')\geq -C\end{aligned}$$

for $(x',\xi')\in U'$ and $(z'',\zeta'')\in U''$, where $sub\ \sigma(P)(x,\xi)=\beta^0(x,\xi)-(i/2)\sum_{j=1}^n(\partial^2p/\partial x_j\partial\xi_j)(x,\xi)$, $\beta^0(x,\xi)$ is the principal symbol of $\beta(x,\xi)$ and $p(x,\xi)=\xi_1^2+\alpha(x,\xi_2,\cdots,\xi_n)$.

Theorem 4.4. *Under the assumptions* (H–1)–(H–3) $P(x,D)$ *is analytic (micro) hypoelliptic at* (x^0,ξ^0), *i.e.,* $\exists\mathcal{U}$: *a conic neighborhood of* (x^0,ξ^0) *in* $T^*\mathbf{R}^n\setminus 0$ *s.t.* supp $u\cap\mathcal{U}$ =supp $P(x,D)u\cap\mathcal{U}$ *for* $u\in\mathcal{C}(\mathcal{U})$.

Remark. Let $n=3$, and let $P(x,\xi)=\xi_1^2+\xi_2^2+a(x_1,x_2)b(x)\xi_3^2$ be an analytic symbol. Assume that $a(x_1,x_2)\geq 0$, $a(x_1,x_2)\neq 0$ for $(x_1,x_2)\neq(0,0)$ and $b(x)>0$. Then, it follows from Theorem 4.4 that $P(x,D)$ is analytic hypoelliptic.

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