<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>ハイパーブolicity of Localizations</td>
</tr>
<tr>
<td>著者</td>
<td>Nishitani, Tatsuo</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (1996), 937: 85-90</td>
</tr>
<tr>
<td>発行年月</td>
<td>1996-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60046">http://hdl.handle.net/2433/60046</a></td>
</tr>
<tr>
<td>形式</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>出版社</td>
<td>KYOTO UNIVERSITY</td>
</tr>
</tbody>
</table>

このページは、研究資料として使用されています。
Hyperbolicity of Localizations

阪大理 西谷 達雄 (Tatsuo Nishitani)

1. INTRODUCTION

Let \( P(x, D) \) be a differential operator of order \( m \) in an open set \( \Omega \subset \mathbb{R}^{n+1} \) with coordinates \( x = (x_0, x') = (x_0, x_1, \ldots, x_n) \), hence a sum of differential polynomials \( P_j(x, D) \) of order \( j \) \((j \leq m)\) with symbols \( P_j(x, \xi) \). In [7] Ivrii-Petkov has proved a necessary condition for the Cauchy problem to \( P(x, D) \) is correctly posed which asserts that \( P_{m-j}(z) \) must vanish of order \( r - 2j \) at \( z \) if \( P_m(z) \) vanishes of order \( r \) at \( z \) with \( z = (x, \xi) \in T^*\Omega \setminus 0 \). This enables us to define the localization \( P_{z_0}(z) \) at a multiple characteristic \( z_0 \) \((\text{of } P_m(z))\), which is a polynomial on \( T_{z_0}(T^*\Omega) \), following Helffer [4].

In this note we show that \( P_{z_0}(z) \) is hyperbolic, that is verifies Gårding’s condition if the Cauchy problem to \( P(x, D) \) is correctly posed. The proof is based on the arguments of Svensson [9] and Nishitani [8]. Since \( P_{z_0}(z) \) is hyperbolic, following Atiyah-Bott-Gårding [1], one can define the localizations \( P_{(z_0, z_1, \ldots, z_s)}(z) \) successively as the localization of \( P_{(z_0, z_1, \ldots, z_{s-1})}(z) \) at \( z_s \) which are hyperbolic polynomials on \( T_{z_0}(T^*\Omega) \cong \cdots \cong T_{z_s}(T^*\Omega) \) (see also Hörmander [5, II]). It may occur the case that the lineality \( \Lambda_{(z_0, z_1, \ldots, z_s)}(P_m) \) of \( P_{m}(z_0, z_1, \ldots, z_s)(z) \) (see (2.8) below) is an involutive subspace with respect to the canonical symplectic structure on \( T_{z_0}(T^*\Omega) \). In this case we prove that for the Cauchy problem to be correctly posed it is necessary that

\[
P_{(z_0, z_1, \ldots, z_s)}(z) = P_{m}(z_0, z_1, \ldots, z_s)(z),
\]

that is, no lower order terms of \( P_{(z_0, \ldots, z_s)}(z) \) occur. This argument was also used in Bernardi-Bove-Nishitani [2] with \( s = 1 \).

2. LOCALIZATION IS HYPERBOLIC

We denote by \( L_{z_0}^{m,r} \) the set of pseudodifferential operators \( P \) near \( z_0 \) with symbol \( P(x, \xi) \) verifying

\[
P(x, \xi) \sim \sum_{j=0}^{\infty} P_{m-j}(x, \xi)
\]

in every homogeneous symplectic coordinates around \( z_0 \) where \( P_{m-j}(x, \xi) \) are positively homogeneous of degree \( m - j \) in \( \xi \) and vanish of order at least \( r - 2j \) and \( P_n(x, \xi) \) vanishes exactly to the order \( r \) at \( z_0 \). Note that we may replace in the definition “every” by “some”.

**Lemma 2.1** (Helffer [4]). Let \( P \in L_{z_0}^{m,r} \). Then

\[
Q(x, \xi) = \exp \left\{ \frac{i}{2} \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j \partial \xi_j} \right\} P(x, \xi)
\]

is invariantly defined in \( L_{z_0}^{m,r}/L_{z_0}^{m,r+1} \): Let \( \chi \) be a homogeneous symplectic coordinates around \( z_0 \) and let \( F \) be a Fourier integral operator associated with \( \chi \) and \( \hat{P} = FPF^{-1} \).
Then we have
\[ \hat{Q}(\chi(x, \xi)) = Q(x, \xi) \]
in \( L_{z_0}^{m,r} / L_{z_0}^{m,r+1} \) where \( \hat{Q} \) is associated with \( \hat{P} \) by (2.1).

**Definition 2.1.** We define the localization \( P_{z_0}(x, \xi) \) of \( P \in L_{x_0, \xi}^{m,r} \) at \( z_0 = (x_0, \xi_0) \) as the lowest order term of the Taylor expansion of
\[ \mu^{2m} Q(x_0 + \mu x, \mu^{-2} \xi_0 + \mu^{-1} \xi) \]
as \( \mu \to 0 \) which is invariantly defined as a polynomial on \( T_{z_0}(T^*\Omega) \): If \( y \) are local coordinates around the origin and \( \hat{P}(y, \eta) \) is the full symbol of \( P \) for the coordinates \( (y, \eta dy) \) then we have
\[ \hat{P}_{w_0}(y'(x_0), y'(x_0)^{-1} \xi + (y\xi_0)'(x_0)x) = P_{z_0}(x, \xi), \]
where \( w_0 = (y(x_0), y'(x_0)^{-1} \xi_0) \).

Writing \( Q(x, \xi) \) as the sum of homogeneous parts \( Q_{m-j}(x, \xi) \), it is clear that
\[ P_{z_0}(x, \xi) = \sum_{r-2j \geq 0} Q_{m-j, z_0}(x, \xi), \quad (2.2) \]
\[ Q_{m-j, z_0}(z) = P_{m-j, z_0}(z) + \sum_{i < j, |\alpha| = j-i} c_{\alpha} P_{m-j, z_0}^{(\alpha)}(z) \]
with some constants \( c_{\alpha} \) where \( Q_{m-j, z_0}(x, \xi) \) and \( P_{m-j, z_0}(x, \xi) \) are defined by
\[ P_{m-j, z_0}(z) = \lim_{\mu \to 0} \mu^{-(r-2j)} P_{m-j}(z_0 + \mu z). \]

Let \( P(x, D) = \sum_{j=0}^{m} P_j(x, D) \) be a differential operator of order \( m \) on \( \Omega \) containing the origin where \( P_j(x, D) \) is the homogeneous part of degree \( j \) with symbol \( P_j(x, \xi) \). Assume that the plane \( x_0 = 0 \) is non characteristic and we are concerned with the Cauchy problem with respect to \( x_0 = \text{const.} \). Let \( z_0 \in T^*\Omega \setminus 0 \) be a characteristic of \( P \) of order \( r \);
\[ d^j P_m(z_0) = 0 \quad \text{for} \quad j < r, \quad d^r P_m(z_0) \neq 0. \]

By the necessary condition of Ivrii-Petkov [7] stated in Introduction we conclude that \( P \in L_{x_0, \xi}^{m,r} \) provided that the Cauchy problem for \( P \) is correctly posed. Then we have from Lemma 2.1 that

**Proposition 2.2** (cf. Ivrii and Petkov [7]). Assume that the Cauchy problem for \( P(x, D) \) is correctly posed near the origin and let \( z_0 \in T^*\Omega \setminus 0 \) be a multiple characteristic of \( P \). Then the localization \( P_{z_0}(z) \) is an invariantly defined polynomial on \( T_{z_0}(T^*\Omega) \).

Let us denote by \( \tilde{P}_{z_0}(x, \xi) \) the lowest order term of the Taylor expansion of
\[ \mu^{2m} P(x_0 + \mu x, \mu^{-2} \xi_0 + \mu^{-1} \xi) \]
as \( \mu \to 0 \). Note that \( \tilde{P}_{z_0}(x, \xi) \) is not coordinates free but we have
Lemma 2.3. The following two conditions are equivalent.
(i) $\tilde{P}_{z_0}(z)$ is hyperbolic with respect to $\theta = (0, e_0)$,
(ii) $P_{z_0}(z)$ is hyperbolic with respect to $\theta$.

Proof. Recall that $\tilde{P}_{z_0}(z) = \sum_{r-2j \geq 0} P_{m-j,z_0}(z)$. Since $\tilde{P}_{z_0}(z)$ is hyperbolic if and only if $P_{m-j,z_0}(z)$ are weaker than $P_{m,z_0}(z) = Q_{m,z_0}(z)$ (see Hörmander [5, II], Svensson [9]) the proof is immediate by (2.2).

Now our aim is to prove

Theorem 2.4. Assume that the Cauchy problem for $P(x,D)$ is correctly posed near the origin and let $z_0 \in T^*\Omega\setminus 0$ be a multiple characteristic of $P_m$. Then the localization $P_{z_0}(z)$ is a hyperbolic polynomial with respect to $\theta = (0, e_0)$.

Let $z_0$ be a characteristic of order $r_0$ of $P_m(z)$ so that $P_{z_0}(z)$ is a polynomial of degree $r_0$. We denote by $P_{(z_0,z_1)}(z)$ the localization of $P_{z_0}(z)$ at $z_1$, that is the first coefficient of $\mu^{r_0}P_{z_0}(\mu^{-1}z_1 + z)$ that does not vanish identically in $z$:

$$\mu^{r_0}P_{z_0}(\mu^{-1}z_1 + z) = \mu^{r_1}(P_{(z_0,z_1)}(z) + O(\mu)), \mu \to 0$$

(see Hörmander [5, II] and Atiyah-Bott-Gårding [1]). We call $r_1$ the order of $z_1$. From Lemma 3.4.2 in Atiyah-Bott-Gårding [1] it follows that $P_{(z_0,z_1)}(z)$ is again hyperbolic with respect to $\theta$. Furthermore $z_1$ is a characteristic of $P_{m,z_0}$ of order $r_1$ and $P_{(z_0,z_1)}(z)$ is the principal part of $P_{(z_0,z_1)}(z)$. On the other hand Corollary 12.4.9 in Hörmander [5, II] shows that

$$d^\nu Q_{m-j,z_0}(z_1) = 0, \nu < r_1 - 2j$$

where $d^\nu Q(z)$ denotes the $\nu$-th differential of $Q$ with respect to $z$. Since $Q_{m-j,z_0}(z)$ are homogeneous of degree $r_0 - 2j$ it is clear that

$$P_{(z_0,z_1)}(z) = \sum_{r_1-2j \geq 0} Q_{m-2j(z_0,z_1)}(z)$$

where

$$Q_{m-j(z_0,z_1)}(z) = \lim_{\mu \to 0} \mu^{-(r_1-2j)}Q_{m-j,z_0}(z_1 + \mu z)$$

which is homogeneous of degree $r_1 - 2j$ in $z$. Repeating the same arguments we get

Lemma 2.5. Let $P_{(z_0,..,z_k)}(z)$ be the localization of $P_{(z_0,..,z_{k-1})}(z)$ at $z_k$ of which order is $r_k \geq 2$;

$$P_{(z_0,..,z_k)}(z) = (P_{(z_0,..,z_{k-1})})_{z_k}(z).$$

Then we have for every $j$ with $r_k - 2j > 0$

$$d^\nu Q_{m-j(z_0,..,z_{k-1})}(z_k) = 0, \nu < r_k - 2j$$

and hence

$$Q_{m-j(z_0,..,z_k)}(z) = \lim_{\mu \to 0} \mu^{-(r_k-2j)}Q_{m-j(z_0,..,z_{k-1})}(z_k + \mu z)$$

exists. Moreover $P_{(z_0,..,z_k)}(z)$ is equal to

$$\sum_{r_k-2j \geq 0} Q_{m-j(z_0,..,z_k)}(z)$$

and hyperbolic with respect to $\theta$. 
Corollary 2.6. Let $z_k$ be a characteristic of $P_{m(z_0,\ldots,z_{k-1})}(z)$ of order $r_k \geq 2$. Then we have

\begin{equation}
\tag{2.3}
d^n P_{m-j(z_0,\ldots,z_{k-1})}(z_k) = 0, \quad \nu < r_k - 2j
\end{equation}

and then

\begin{equation}
\tag{2.4}
P_{m-j(z_0,\ldots,z_k)}(z) = \lim_{\mu \to 0} \mu^{-(r_k-2j)} P_{m-j(z_0,\ldots,z_{k-1})}(z_k + \mu z)
\end{equation}

exists.

\textit{Proof.} Assume that (2.3) and

\begin{equation}
\tag{2.5}
Q_{m-j(z_0,\ldots,z_{k-1})}(z) = P_{m-j(z_0,\ldots,z_{k-1})}(z) + \sum_{i<j,|\alpha|=j-i} c_{\alpha} P_{m-i(z_0,\ldots,z_{k-1})}(\alpha)(z)
\end{equation}

hold with $k = p$ where $c_{\alpha}$ are constants. Then it is easy to see that (2.5) with $k = p + 1$ holds. Thus (2.3) with $k = p + 1$ follows from Lemma 2.5. By induction on $k$ we get the desired conclusion.

Here we give another formula which defines $P_{(z_0,\ldots,z_s)}(z)$ directly. Let $0 < \mu_0 < \mu_1 < \ldots < \mu_s$ be a sequence of parameters with

\begin{equation}
\tag{2.6}
\mu_j = O(\mu_{j+1}^{m+1}) \quad \text{as} \quad \mu_{j+1} \to 0.
\end{equation}

Then we have

\begin{equation}
\tag{2.7}
(\mu_0 \cdots \mu_s)^{2m} Q(x_0 + \mu_0 x_1 + \cdots + \mu_0 \cdots \mu_{s-1} x_s + \mu_0 \cdots \mu_s x, \mu_0 \cdots \mu_s)^{-2} (\xi_0 + \mu_0 \xi_1 + \cdots + \mu_0 \cdots \mu_{s-1} \xi_s + \mu_0 \cdots \mu_s \xi) = \mu_0^{r_0} \cdots \mu_s^{r_s} (P_{(z_0,\ldots,z_s)}(z) + O(\mu_s))
\end{equation}

where $z_j = (x_j, \xi_j)$ and $r_j$ is the order of $z_j$.

Let $\Lambda_{(z_0,\ldots,z_s)}(P_m)$ be the lineality of $P_{m(z_0,\ldots,z_s)}$ which is a linear subspace defined by

\begin{equation}
\tag{2.8}
\{ z|P_{m(z_0,\ldots,z_s)}(w + tz) = P_{m(z_0,\ldots,z_s)}(w), \forall t \in \mathbb{R}, \forall w \in T_{z_0}(T^*\Omega) \}
\end{equation}

and let $\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$. For $S \subset T_{z_0}(T^*\Omega)$ we denote by $S^\sigma$ the annihilator of $S$ with respect to $\sigma$:

$$S^\sigma = \{ z \in T_{z_0}(T^*\Omega)|\sigma(z, w) = 0, \forall w \in S \}.$$
Theorem 2.7. Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and

$$\Lambda_{(z_0, \ldots, z_s)}(P_m)^\sigma \subset \Lambda_{(z_0, \ldots, z_s)}(P_m).$$

Then we have

$$P_{(z_0, \ldots, z_s)}(z) = P_m(z_0, \ldots, z_s)(z),$$

that is, no lower order terms occur in $P_{(z_0, \ldots, z_s)}(z)$.

Example 2.1. Let

$$P(x, \xi) = (\xi_0^2 - x_1^2 \xi_n^2 - \xi_1^2)(\xi_0^2 - x_1^2 \xi_n^2 - 2\xi_1^2) + p_2(\xi_0, x_1, \xi_1)\xi_n$$

where $p_2$ is a homogeneous polynomial of degree 2. With $z_0 = (0, e_n)$ it is clear that

$$P_{4, z_0} = (\xi_0^2 - x_1^2 \xi_n^2 - \xi_1^2)(\xi_0^2 - x_1^2 \xi_n^2 - 2\xi_1^2), \quad Q_{3, z_0} = 6ix_1 \xi_1 + p_2(\xi_0, x_1, \xi_1).$$

Let $z_1$ be $\xi_0 = x_1 = a$, $a \in \mathbb{R}$, $\xi_1 = 0$ so that

$$P_{4(z_0, z)} = 4a^2(\xi_0 - x_1)^2, \quad Q_{3(z_0, z_1)} = p_2(a, a, 0).$$

Since $\Lambda_{(z_0, z_1)}(P_4)^\sigma \subset \Lambda_{(z_0, z_1)}(P_4)$ it follows from Theorem 2.7 that $p_2(a, a, 0) = 0$. Similarly choosing $z_1$ to be $\xi_0 = a$, $x_1 = -a$, $\xi_1 = 0$ we get $p_2(a, -a, 0) = 0$. Thus

$$p_2(\xi_0, x_1, \xi_1) = c(\xi_0^2 - x_1^2) + \xi_1 p_1(\xi_0, x_1, \xi_1)$$

where $p_1$ is linear. Finally one can write

$$P(x, \xi) = (\xi_0^2 - x_1^2 \xi_n^2 - \xi_1^2 + c\xi_n)(\xi_0^2 - x_1^2 \xi_n^2 - 2\xi_1^2) + \xi_1 L(\xi_0, x_1, \xi_n, \xi_1)\xi_n$$

with a linear function $L$.

Example 2.2. Let

$$P(x, \xi) = (\xi_0 - x_0 \xi_n)^2(\xi_0 + x_0 \xi_n) + \alpha(\xi_0 - x_0 \xi_n)\xi_n + \beta(\xi_0 + x_0 \xi_n)\xi_n$$

where $\alpha, \beta \in \mathbb{C}$. With $z_0 = (0, e_n)$ we have

$$P_{3, z_0} = (\xi_0 - x_0)^2(\xi_0 + x_0), \quad Q_{2, z_0} = \alpha(\xi_0 - x_0) + (\beta - i)(\xi_0 + x_0).$$

Taking $z_1$ to be $\xi_0 = 1$, $x_0 = 1$ it follows that

$$P_{3(z_0, z_1)} = 2(\xi_0 - x_0)^2, \quad Q_{2(z_0, z_1)} = 2(\beta - i).$$

Since $\Lambda_{(z_0, z_1)}(P_3)^\sigma \subset \Lambda_{(z_0, z_1)}(P_3)$ we have $\beta = i$ by Theorem 2.7. Set

$$p_1(x, \xi) = \xi_0 - x_0 \xi_n, \quad p_2(x, \xi) = (\xi_0 - x_0 \xi_n)(\xi_0 + x_0 \xi_n) + (\alpha + i)\xi_n$$

then $\beta = i$ implies that

$$P(x, D) = p_1^w(x, D)p_2^w(x, D)$$

where $p_j^w(x, D)$ are Weyl realizations of $p_j(x, \xi)$, see Hörmander [5, III].
REFERENCES