数理解析研究所講究録
937 巻 1996 年 75-84

MICROLOCAL PROPERTY OF PSEUDODIFFERENTIAL OPERATORS IN CASE OF WAVE FRONT SETS DEFINED BY WAVELET TRANSFORMS

SHINYA MORITOH (森藤 紳哉)

Abstract. We define a class of wavelet transforms as a continuous and microlocal version of the Littlewood-Paley decompositions. Hörmander's wave front sets may be characterized in terms of our wavelet transforms. We prove microlocal property of pseudodifferential operators in case of our wave front sets.

INTRODUCTION

We define a class of wavelet transforms as a continuous and micro-local version of the Littlewood-Paley decompositions. Hörmander's wave front sets may be characterized in terms of our wavelet transforms. We remark that the components of our decompositions are not linearly independent but can be treated as if they were.

This paper consists of two parts. The former part is the comparison between the wave front sets defined by our wavelet transforms and Hörmander's wave front sets. The latter part is to show microlocal property of pseudodifferential operators in case of our wave front sets. First, we define our wavelet transforms as follows:

Definition 1. Suppose that the function $\psi(x)$ (called wavelet) has the following properties: $\psi(x) \in \mathcal{S}(\mathbb{R}^n)$, $\hat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $\hat{\psi}(\xi) \geqq 0$. Let Ω=supp$\hat{\psi}(\xi)$, $(0, \cdot\cdot, 0, 1)$ is the central axis of Ω, and r_ξ is any rotation which sends $\xi/|\xi|$ to $(0, \cdot\cdot, 0, 1)$. When $n = 1$, $\Omega \subset (0, \infty)$ and when $n \geqq 2$, Ω is connected, does not contain the origin 0 and $\psi(x) = \psi(rx)$ for any $r \in SO(n)$ satisfying $r(0, \cdot\cdot, 0, 1) = (0, \cdot\cdot, 0, 1)$. Then our wavelet transform is defined as follows:

for $f(t) \in \mathcal{S}'(\mathbb{R}^n)$, $(x, \xi) \in \mathbb{R}^{2n}$,

$$W_\psi f(x,\xi) = \begin{cases} \int_{\mathbb{R}} f(t)|\xi|^{1/2}\overline{\psi(\xi(t-x))}dt, & \text{if } n = 1, \\ \int_{\mathbb{R}^n} f(t)|\xi|^{n/2}\overline{\psi(|\xi| r_\xi(t-x))}dt, & \text{if } n \geqq 2. \end{cases}$$

Remark 1. $W_\psi f(x,\xi)$ is rewritten as follows:

$$\int_{\mathbb{R}^n} \hat{f}(\tau)\cdot|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)\cdot e^{i\tau x}d\tau.$$

From this, the meaning of our wavelet transforms is clear.

Remark 2. Our wavelet transforms in $\mathbb{R}^n$ are the reduced versions of those defined by R.Murenzi(See [3]).

Remark 3. The domain of a wavelet transformation is usually the L_2-space(See [1]), but can be extended to $\mathcal{S}'(\mathbb{R}^n)$,that is, the dual space of $\mathcal{S}(\mathbb{R}^n)$.

Now, we define our wave front set $WF_\psi(f)(\subset \mathbf{R}^n_x\times\mathbf{R}^n_\xi)$ of $f\in\mathcal{S}'(\mathbb{R}^n)$ as follows.

Definition 2. We say $(x_0,\xi^0)\notin WF_\psi(f)$ if there exists a neighbourhood $U(x_0)$ of x_0 and a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 such that $|W_\psi f(x,\xi)| = O(|\xi|^{-N})$ as $|\xi|$ tends to ∞ for any $N\in\mathbb{N}$ in $U(x_0)\times\Gamma(\xi^0)$.

Moreover, we define the refinement $WF^{(s)}_\psi(f)$ as follows.

Definition 3.

$$(x_0,\xi^0)\notin WF^{(s)}_\psi(f)\Leftrightarrow \iint_{U(x_0)\times\Gamma(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dx d\xi<\infty.$$

It is easy to prove that if $f\in L_2(\mathbb{R}^n)$,

$$WF_\psi(f) = \text{the closure of} \bigcup_{s\geqq 0} WF^{(s)}_\psi(f).$$

We need the following definition to state Theorem 1.

Definition 4. Let cone$\Omega=\{t\xi|\xi\in\Omega,t>0\}$. We say $(x_0,\xi^0)\notin\overline{WF}^\psi$ if $x_0\notin \text{proj}_x WF$ and $\xi^0\in\mathbb{R}^n$, or $x_0\in\text{proj}_x WF$ and r(coneΩ) does not intersect $\{\xi\in\mathbb{R}^n;(x_0,\xi)\in WF\}$ for any $r\in SO(n)$ with r(coneΩ) including ξ^0. That is to say, the set $\overline{WF}^\psi$ is the expanded set of WF only in the frequency space.

Here, the set WF is the wave front set in the sense of L.Hörmander(see [3]), and $\text{proj}_x WF$ denotes the projection of WF onto x-space.

Theorem 1. *Let $f \in L_2(\mathbb{R}^n)$, and $s \geqq 0$. When $n = 1$, $WF_{\psi}^{(s)}(f) = WF^{(s)}(f)$. When $n \geqq 2$, $WF_{\psi}^{(s)}(f) \subseteq \overline{WF^{(s)}(f)}^{\psi}$ and $WF^{(s)}(f) \subseteq \overline{WF_{\psi}^{(s)}(f)}^{\psi}$. We have the same inclusions between $WF_{\psi}(f)$ and $WF(f)$.*

The latter part of this paper is to show microlocal property of pseudodifferential operator in case of our wave front sets. First we define the operator P_{ψ} as follows:

$$P_{\psi}f(s) = C_{\psi}{}^{-1} \iint e^{i\xi\cdot(s-t)} p_{\psi}(s,\xi,t)f(t)dtd\xi,$$

where

$$p_{\psi}(s,\xi,t) = e^{-i\xi\cdot(s-t)}p(s,\xi) \int \psi_{x,\xi}(s)\overline{\psi_{x,\xi}(t)}dx, p(s,\xi) \in S^m_{1,0}$$

and C_{ψ} is defined in Proposition 1. (We abbreviate the kernel of our wavelet transform as $\psi_{x,\xi}\cdot$)

Theorem 2. *We have*

$$p_{\psi}(s,\xi,t) \in S^{-\infty}_{1,0}(\mathbb{R}^{3n}\backslash\{s=t\}), p_{\psi}(s,\xi,t) \in S^m_{1,0}$$

and that $p_{\psi}(s,\xi,t)$ converges to $p(s,\xi)$ pointwisely as Ω tends to $(0,\cdots,0,1)$.

Theorem 3. *In case $p(s,\xi)=\sum_{|k|\leqq n} s^k p_k(\xi) \in S^m_{1,0}$, we have*

$$WF_{\psi}(P_{\psi}f) \subset \bigcup_{|k|\leqq n} WF_{\psi_k}(f),$$

where $\psi_k(t) = t^k\psi(t)$. The right hand side converges to $WF(f)$ as Ω tends to $(0,\cdots,0,1)$.

Theorem 4. *In case $p_k(\xi)$ is a polynomial with respect to ξ in addition to the assumption in Theorem 3, we have*

$$WF_{\psi}(Pf) \subset \bigcup_{|l|\leqq n} \bigcup_{k\leqq l} WF_{\psi_k^l}(f),$$

where $\psi_k^l = P_l(D_t)(t^k\psi(t))$. We have the usual microlocal property as Ω tends to $(0,\cdots,0,1)$.

1. Proof of Theorem 1.

As we have already defined, the wavelet $\psi(x)$ is essentially of two parameters, since it is rotationally invariant around Θ when $n \geqq 2$. For the purpose of proving **Theorem 1**,we prepare three propositions. (Here, $\Theta=(0,\cdot\cdot,1)$)

Proposition 1(Parseval formula and inversion formula). *We have for $f,g \in L_2(\mathbb{R}^n)$,*

$$\iint W_\psi f(x,\xi)\overline{W_\psi g(x,\xi)}dxd\xi = C_\psi \int f(t)\overline{g(t)}dt.$$

Here,

$$C_\psi = (2\pi)^n \int \frac{|\hat{\psi}(\xi)|^2}{|\xi|^n}d\xi.$$

From this, we also have:

$$f(t) = C_\psi^{-1} \iint W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi (t-x))dxd\xi,$$

when $n \geqq 2$. When $n=1$, $|\xi|r_\xi(t-x)$ is replaced by $\xi(t-x)$. For $f \in S'$,this inversion formula must be regarded in the distribution sense.

Proposition 2(Locality). *If x_0 does not belong to $\operatorname{supp} f$,then there exists a neighbourhood $U(x_0)$ of x_0 such that $W_\psi f(x,\xi)$ is rapidly decreasing in ξ with respect to $x \in U(x_0)$ uniformly.*

Proposition 3(Global Sobolev property).

$$f \in H^s(\mathbb{R}^n) \Leftrightarrow \iint |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s < \infty.$$

Proof of Proposition 1. We obey the method that I.Daubechies[1] employed to prove in the case $n=1$. We have

$$\begin{aligned}
&\iint W_\psi f(x,\xi)\overline{W_\psi g(x,\xi)}dxd\xi \\
&= \iint [\int \hat{f}(\tau)\overline{|\xi|^{-\frac{n}{2}}\hat{\psi}(|\xi|^{-1}r_\xi \tau)e^{-i\tau\cdot x}}d\tau] \\
&\quad \overline{\cdot[\int \hat{g}(\tau)\overline{|\xi|^{-\frac{n}{2}}\hat{\psi}(|\xi|^{-1}r_\xi \tau)e^{-i\tau\cdot x}}d\tau]}dxd\xi \\
&= (2\pi)^n \int d\xi |\xi|^{-n} \int d\tau \hat{f}(\tau)\cdot\overline{\hat{g}(\tau)}\cdot\hat{\psi}(|\xi|^{-1}r_\xi\tau)^2. \qquad (1)
\end{aligned}$$

We exchange variables from τ into

$$\omega = |\xi|^{-1} r_\xi \tau.$$

If we denote the Haar measure on S^{n-1} by $d\theta$, we have

$$\frac{d\xi}{|\xi|^n} = \frac{d|\xi|}{|\xi|} \cdot d\theta_\xi = (-\frac{d|\omega|}{|\omega|}) \cdot (-d\theta_\omega) = \frac{d\omega}{|\omega|^n}.$$

Therefore,the right hand side of (1) is equal to

$$(2\pi)^n \int d\tau \hat{f}(\tau) \cdot \overline{\hat{g}(\tau)} \int \frac{d\omega}{|\omega|^n} \hat{\psi}(\omega)^2 = C_\psi \int f(x) \cdot \overline{g(x)} dx.$$

Proof of Proposition 2. We take n as 1, for the proof is the same in the case $n \geqq 2$. Because there exists a neighbourhood $U_1(x_0)$ of x_0 such that $f(t) \equiv 0$, it follows that, when $t \in U_1(x_0)$,

$$\begin{aligned} |W_\psi f(x,\xi)| =& |\int f(t) |\xi|^{\frac{1}{2}} \overline{\psi(\xi(t-x))} dt| \\ & \leqq \|f\|_{L_2} \cdot (\int_{U_1(x_0)^c} |\xi| |\psi(\xi(t-x))|^2 dt)^{\frac{1}{2}}. \end{aligned}$$

Because ψ belongs to $\mathcal{S}(\mathbb{R})$, there exists a neghbourhood $U_2(x_0) \Subset U_1(x_0)$ satisfying this proposition.

Proof of Proposition 3. It suffices to prove when $n \geqq 2$, and the proof is quite similar to that of the Parseval formula. We have

$$\begin{aligned} & \iint |W_\psi f(x,\xi)|^2 (1+|\xi|^2)^s dx d\xi \\ =& (2\pi)^n \int d\tau |\hat{f}(\tau)|^2 \int \frac{d\xi}{|\xi|^n} (1+|\xi|^2)^s \hat{\psi}(\frac{r_\xi}{|\xi|}\tau)^2 \\ =& (2\pi)^n \int d\tau |\hat{f}(\tau)|^2 \int \frac{d\omega}{|\omega|^n} (1+\frac{|\tau|^2}{|\omega|^2})^s \hat{\psi}(\omega)^2. \end{aligned}$$

Here, if we use the polar coordinate representation of $\omega = (r,\theta)$ and denote $\int \hat{\psi}(r,\theta)^2 d\theta$ by $S(r)$,this is equal to $\int \frac{dr}{r} S(r)(1+\frac{|\tau|^2}{r^2})^s$.

By the assumption on $\hat{\psi}(\omega)$, supp$S(r)$ is a compact set included in $(0,\infty)$. This fact makes $\int \frac{dr}{r} S(r)(1+\frac{|\tau|^2}{r^2})^s$ equivalent to $(1+|\tau|^2)^s$. After all, we get a characterization of $H^s(\mathbb{R}^n)$ by using the wavelet transform.

Now, we are prepared to state the proof of Theorem 1.

Proof of **Theorem 1**. It suffices to show when $n \geqq 2$. Moreover, by the fact that $WF_\psi(f)$=the closure of $\bigcup_{s\geqq 0} WF_\psi^{(s)}(f)$, it suffices to prove the statement for any $s \geqq 0$ fixed.

Step 1.

Suppose that $(0,\xi^0)$ does not belong to the set $\overline{WF^{(s)}(f)}^{\psi}$. Let $\Gamma(\xi^0)$ be the union of r(coneΩ) for all rotations r such that ξ^0 is included in r(coneΩ), then there exists a function $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ which is always equal to 1 near $x=0$ and satisfies $\int_{\Gamma(\xi^0)} |(\hat{\phi f})(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$. This follows from the definition of $\overline{W}^{\psi}$, the definition of Hörmander's wave front set and Heine-Borel's lemma.

What we want to say is that there exist a conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0 and a neighbourhood $U(0)$ of 0, satisfying:

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dx d\xi < \infty.$$

Here, using the inversion formula, we divide $W_\psi f(x,\xi)$ into two parts:

$$W_\psi f(x,\xi) = |\xi|^{\frac{n}{2}} \int (\phi f)(t)\cdot \overline{\psi(|\xi| r_\xi(t-x))} dt \tag{2}$$

$$+|\xi|^{\frac{n}{2}} \int ((1-\phi)f)(t)\cdot \overline{\psi(|\xi| r_\xi(t-x))} dt. \tag{3}$$

If we take a set $U(0) \Subset \{\phi(x)\equiv 1\}$,then, by the argument of propotion 2, (3) is rapidly decresing in $|\xi|$ with respect to $x \in U(0)$ uniformly. Therefore, it is easy to see that $(0,\xi_0) \notin WF_\psi^{(s)}((1-\phi)f)$. On the other hand, if we take $\tilde{\Gamma}(\xi^0)$ sufficiently small, then we obtain:

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dx d\xi$$

$$\leqq \int_{\tilde{\Gamma}(\xi^0)} d\xi \int_{\mathbf{R}_x^n} |W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dx d\xi$$

$$=(2\pi)^n \int_{\mathbf{R}_\tau^n} d\tau |(\hat{\phi f})(\tau)|^2 \int_{\tilde{\Gamma}(\xi^0)} \frac{d\xi}{|\xi|^n}(1+|\xi|^2)^s \hat{\psi}(\frac{r_\xi}{|\xi|}\tau)^2.$$

If we change variables from τ into $\omega = \frac{r_\xi}{|\xi|}\tau$ as before , ω must be in Ω. Therefore, we can see that τ stays in $\Gamma(\xi^0)$ because we took $\tilde{\Gamma}(\xi^0)$ very small. The inequality above is followed by

$$\leqq (2\pi)^n \int_{\Gamma(\xi^0)} d\tau |(\hat{\phi f})(\tau)|^2 \int_{\Omega} \frac{d\omega}{|\omega|^n}(1+\frac{|\tau|^2}{|\omega|^2})^s \hat{\psi}(\omega)^2$$

$$\leqq C \int_{\Gamma(\xi^0)} |(\hat{\phi f})(\tau)|^2 (1+|\tau|^2)^s d\tau < \infty. \text{(Here, C is a constant.)}$$

Therefore, we have that $(0,\xi^0) \notin WF_\psi^{(s)}(\phi f)$.

Step 2.

Suppose that $(0,\xi^0)$ does not belong to the set $\overline{WF_\psi^{(s)}(f)}^{\psi}$. If we take a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 as in Step 1., then there exists a neighbourhood $U(0)$ of $x=0$ and satisfies

$$\iint_{U(0)\times\Gamma(\xi^0)} |W_\psi f(x,\xi)|^2 (1+|\xi|^2)^s dx d\xi < \infty,$$

as in Step 1. Here, using the inversion formula, we divide f into two parts: $f = f_\Gamma + f_{\Gamma^c}$, where

$$f_\Gamma(t) = C_\psi^{-1} \iint_{\Gamma(\xi^0)\times\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(t-x))dx d\xi,$$

$$f_{\Gamma^c}(t) = C_\psi^{-1} \iint_{\Gamma(\xi^0)^c\times\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(t-x))dx d\xi.$$

Then,

$$\widehat{f_{\Gamma^c}}(\tau) = C_\psi^{-1} \int_{\Gamma(\xi^0)^c} \int_{\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)e^{-i\tau\cdot x} dx d\xi.$$

If we take a sufficiently small conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0, then we obtain

$$\hat{\psi}(\frac{r_\xi}{|\xi|}\tau) \equiv 0 \text{ for any} \tau \in \tilde{\Gamma}(\xi^0) \text{ and for any } \xi \in \Gamma(\xi^0)^c.$$

Therefore, it follows that $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma^c})$.

Next, we choose a function $\phi(x) \in C_0^\infty(\mathbf{R}^n)$ satisfying the condition that $\mathrm{supp}\phi(x)$ is compactly supported in $U(0)$ and that $\phi(x) \equiv 1$ in some neighbourhood $U_1(0)$ of 0. Then, we further divide $f_\Gamma(t)$ into two parts:

$$f_\Gamma = f_{\Gamma,\phi} + f_{\Gamma,1-\phi},$$

where,

$$f_{\Gamma,\phi}(t) = C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} \phi(x) \cdot W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi(t-x)) dx d\xi,$$

$$f_{\Gamma,1-\phi}(t) = C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} (1-\phi(x)) W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi(t-x)) dx d\xi.$$

Let $U_2(0) \Subset \{\phi(x) \equiv 1\}$, then we can easily see that $f_{\Gamma,1-\phi}(t)$ is C^∞ with respect to $t \in U_2(0)$, by Proposition 2, and the exchange of order of differentiation and integration. Therefore, it follows that $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma,1-\phi})$.
Lastly, we want to show that $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma,\phi})$. This is the heart of matter in proving **Theorem 1**. In fact, more strongly, we can show that $f_{\Gamma,\phi}$ globally belongs to Sobolev space $H^s(\mathbb{R}^n)$. Its Fourier transform is given by

$$\widehat{f_{\Gamma,\phi}}(\tau) = C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} \phi(x) \cdot W_\psi f(x,\xi) \cdot |\xi|^{-\frac{n}{2}} \hat{\psi}(\frac{r_\xi}{|\xi|}\tau) e^{-i\tau\cdot x} dx d\xi.$$

If we put $g(x,\xi) = \phi(x) W_\psi f(x,\xi) \cdot (1+|\xi|^2)^{\frac{s}{2}}$, then it follows from the hypothesis and from the fact that $\mathrm{supp}\phi(x)$ is included in $U(0)$ that

$$\iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} |g(x,\xi)|^2 dx d\xi < \infty.$$

If we denote the partial Fourier transform of $g(x,\xi)$ from x to τ by $\hat{g}(\tau,\xi)$, we have

$$\begin{aligned}
&\widehat{f_{\Gamma,\phi}}(\tau)(1+|\tau|^2)^{\frac{s}{2}} \\
=&C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} g(x,\xi) e^{-i\tau\cdot x} \cdot |\xi|^{-\frac{n}{2}} \hat{\psi}(\frac{r_\xi}{|\xi|}\tau)(\frac{1+|\tau|^2}{1+|\xi|^2})^{\frac{s}{2}} dx d\xi \\
=&C_\psi^{-1}(2\pi)^{\frac{n}{2}} \int_{\Gamma(\xi^0)} \hat{g}(\tau,\xi) \cdot K(\tau,\xi) d\xi.
\end{aligned}$$

Here, $K(\tau,\xi)$ is defined by $|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)(\frac{1+|\tau|^2}{1+|\xi|^2})^{\frac{s}{2}}$.
Because supp$\hat{\psi}$ is a compact set not including the origin 0, there exists a constant C such that

$$|K(\tau,\xi)| \leqq C|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau).$$

Therefore, by using the result in the proof of Proposition 1(i.e. the continuous decomposition of the unity), the integral $\int |K(\tau,\xi)|^2 d\xi$ is bounded from above. (the bound is $(2\pi)^{-n}C_\psi C^2$.)
After all, we obtain the following inequality:

$$\begin{aligned}\int |\widehat{f_{\Gamma,\phi}}(\tau)|^2(1+|\tau|^2)^s d\tau \leqq & C_\psi^{-1}C^2\int(\int_{\Gamma(\xi^0)}|\hat{g}(\tau,\xi)|^2 d\xi)d\tau \\ =& C'\int_{\Gamma(\xi^0)} d\xi \int_{\mathbf{R}^n_\tau}|\hat{g}(\tau,\xi)|^2 d\tau \\ =& C' \iint_{\Gamma(\xi^0)\times\mathbf{R}^n_x} |g(x,\xi)|^2 dx d\xi < \infty.\end{aligned}$$

(**Theorem 1**) q.e.d.

2. Proof of Theorems 2,3 and 4.

By using Proposition 2, we can show the first part of Theorem 2. If we note

$$\int \psi_{x,\xi}(s)\overline{\psi_{x,\xi}(t)}dx = \int |\hat{\psi}(\omega)|^2 e^{i(s-t)\cdot|\xi|r_\xi^{-1}\omega}d\omega,$$

we can prove the second part. The proof of Theorems 3 and 4 is based on the following calculations:

(a) If $p(s,\xi)=p(\xi)$, by using Proposition 2, Theorems 3 and 4 are clear.

(b) If for example $p(s,\xi)=s$, by rewriting $s\psi_{x,\xi}(s)$ as $\{(s-x)+x\}\psi_{x,\xi}(s)$ and using Proposition 2, Theorem 3 is clear.

(c) If for example $p(s,\xi)=s$, by using integration by parts and Proposition 2, Theorem 4 is clear.

Complete proof is obtained by the combination of (a),(b)and(c).

Acknowledgements. The auther would like to express his sincere gratitude to Prof. Hikosaburo Komatsu, Prof. Kiyoomi Kataoka, Prof. Kenji Asada and Dr. Susumu Tanabe for many valuable suggestions and encouragement. He would like to express his sincere gratitude also to Mr. Susumu Yamazaki for helpful discussions.

REFERENCES

1. Daubechies I., *Ten Lectures on Wavelets*, SIAM CBMS-61, 1992.
2. Hörmander L., *Fourier integral operators.I*, Acta.Math. **127** (1971), 79–183.
3. Murenzi R., *Wavelet transforms associated to the n-dimensional Euclidean group with dilations: signals in more than one dimension*, Wavelets (J.M.Combes, A.Grossmann and Ph. Tchamitchian, eds.), Springer-Verlag, 1989.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF TOKYO
7-3-1 HONGO BUNKYO-KU TOKYO 113, JAPAN