Gevrey singularities for nonlinear wave equations

大阪大学 加藤圭一
(OSAKA UNIVERSITY: KEICHI KATO)

1. Introduction

We consider the following semilinear wave equations,

(1) \[ \square u = f(u) \quad \text{in} \quad \Omega \subset \mathbb{R}_t \times \mathbb{R}^2_x, \]

where \( u \) is a real valued function, \( \square = \partial^2/\partial t^2 - \Delta \) with \( \Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2 \), \( \Omega \) is a bounded domain which contains the origin and \( f(u) \) is a polynomial of \( u \) with \( f(0) = 0 \).

We study the interaction of Gevrey singularities for this equation. We assume that solutions that we study here are all in \( H^s(\Omega) \) with \( s > 3/2 \) where \( H^s(\Omega) \) is a Sobolev space of order \( s \) in \( \Omega \). In 1982, J. Rauch and M. Reed [4] have made an example in which three singularities produce new singularities. In 1984, J. M. Bony [2] and R. Melrose and N. Ritter [3] have had a general result for \( C^\infty \)-singularity independently. We put \( \Sigma_j = \{(t, x) \in \mathbb{R}^3; t = 4^j \cdot x\} \) (\( j = 1, 2, 3 \)) with \( \omega_j \in S^1 \). Their result for the equation (1) is as follows.

**Theorem 1.1 (J. M. Bony [2], R. Melrose and N. Ritter [3]).** If \( u \) is conormal with respect to \( \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \) in \( \Omega_- = \Omega \cup \{t < 0\} \), then the solution \( u \) is \( C^\infty \) in \( K \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{t^2 = |x|^2\}) \) where \( K \) is a domain of determine with respect to \( \Omega_- \).

In this talk, we shall make the Gevrey version of the above result.

**Definition 1.1 (Gevrey conormal distribution).** For \( s > 3/2, \sigma \leq 1 \), we call that \( u \in H^s(\Sigma, G^\sigma) \), if and only if for any compact set \( K \subset \Omega \) and for any vector fields \( V_1, \ldots, V_l \) with analytic coefficients and any integer \( l \) which are tangent to \( \Sigma \), there exist constants \( C, A > 0 \) such that

(2) \[ \|V_1^{\alpha_1} \cdots V_l^{\alpha_l} u\|_{H^s(K)} \leq CA^{\|\alpha\|}(\|\alpha\|)^{\sigma} \]

for any integers \( \alpha_1, \ldots, \alpha_l \).

**Theorem 1.2.** Suppose that \( u \) is in \( H^s(\Omega) \) for some \( s > 5/2 \), \( u \) satisfies the equation (1) and \( u \in H^s(\Sigma_1, G^\sigma; \Omega_-) \). Then we have

(3) \[ u \in H^s(\Sigma_1, G^\sigma; K), \]
where \( \Omega_- = \Omega \cap \{(t, x); t < 0\} \), \( K \) is the domain of determine with respect to \( \Omega_- \).

**Theorem 1.3.** Suppose that \( u \) is in \( H^s(\Omega) \) for some \( s > 5/2 \), \( u \) satisfies the equation (1) and \( u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; \Omega_-) \). Then we have

\[
u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; K),
\]

where \( \Omega_- = \Omega \cap \{(t, x); t < 0\} \), \( K \) is the domain of determine with respect to \( \Omega_- \).

**Theorem 1.4 (Main result).** Suppose that \( u \in H^s(\Omega) (s > 5/2) \), \( u \) satisfies the equation (1) and

\[
u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(\sigma)}; \Omega_-).
\]

Then \( u \) is a Gevrey class function of order \( \sigma \) in \( K \setminus \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+ \), where \( \Gamma_+ = \{t^2 = |x|^2, t > 0\} \), \( \Omega_- = \Omega \cap \{t < 0\} \) and \( K \) is a domain of determine with respect to \( \Omega_- \).

**Corollary 1.1.** Suppose that \( u \in H^s(\Omega) (s > 5/2) \), \( u \) satisfies the equation (1) and

\[
u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(1)}; \Omega_-).
\]

Then \( u \) is real analytic in \( K \setminus \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+ \), where \( \Gamma_+ = \{t^2 = |x|^2, t > 0\} \), \( \Omega_- = \Omega \cap \{t < 0\} \) and \( K \) is a domain of determine with respect to \( \Omega_- \).

## 2. Preliminaries

Let \( K \) be a relatively compact set in \( R^3 = R_t \times R_x^2 \) such that each subset \( K \cap \{(t, x); s \leq t \leq T\} \) is a domain of determine with respect to \( K \cap \{(t, x); t \leq s\} \) for \( S \leq s \leq T \). For \( m > 5/2 \) and \( f \in H^m(K) \), we put

\[
E_m(t)[f] = \|f(t)\|_{H^{m-1/2}(K(t))} + \|\partial_t f(t)\|_{H^{m-3/2}(K(t))}
\]

with \( K(s) = K \cap \{(t, x); t = s\} \).

**Proposition 2.1 (Energy estimate).** For \( f \in H^m(K) \), we have

\[
E_m(t_2)[f] \leq E_m(t_1)[f] + C(T) \int_{t_1}^{t_2} \|\Box f\|_{H^{m-3/2}(K(s))} \, ds
\]

for \( S \leq t_1 < t_2 \leq T \).

**Proposition 2.2.** For \( u, v \in H^m(K) \), we have

\[
E_m(t)[uv] \leq C(n)E_m[u]E_m[v].
\]
Let $Q$ be an analytic vector field on $K$. We define a quantity $\|f(t)\|_{G^s_{A}(Q;E_m)}$ by

\begin{equation}
(10) \quad \|f(t)\|_{G^s_{A}(Q;E_m)} = \sum_{l=0}^{\infty} \frac{A^l}{l!^s} E_m(t)[P^l]
\end{equation}

and we put $\|f(t)\|_{X(Q)} = \|f\|_{G^s_{A}(Q;E_m)}$ and $\|f\|_{Y_{A}([t_1,t_2];Q)} \|f\|_{Y([t_1,t_2];Q)} = \sup_{t_1 \leq t \leq t_2} \|f(t)\|_{X(Q)}$ for abbreviation.

**Proposition 2.3.** For $c \in \mathbb{R}$,
\begin{equation}
(11) \quad \|f\|_{G^s_{A}(Q+C;E_m)} \leq e^{\epsilon A} \|f\|_{X(Q)}
\end{equation}

**Proposition 2.4.**
\[\|uv\|_{X(Q)} \leq C(n)\|u\|_{X(Q)}\|v\|_{X(Q)}\]

For $\Sigma_1$ and $\Sigma_2$, we put $\tilde{\omega}_j = (1, -\omega_j)$, $\tilde{\omega}_j^* = (1, \omega)$, $\nabla = (\partial_t, \partial_{x_1}, \partial_{x_2})$ and put
\begin{align}
X_1 &= (\tilde{\omega}_1 \times \tilde{\omega}_2) \cdot \nabla \\
X_2 &= (t - \omega_2 \cdot x)\tilde{\omega}_1^* \cdot \nabla \\
X_3 &= \tilde{\omega}_2^* \cdot \nabla \\
X_4 &= (t - \omega_1 \cdot x)\tilde{\omega}_2^* \cdot \nabla
\end{align}

**Proposition 2.5.** We have
\begin{align}
[X_j, X_k] &= 0 \quad \text{for \quad } 1 \leq j, k \leq 4, \\
[\Box, X_1] &= [\Box, X_3] = 0, \\
[\Box, X_2] &= [\Box, X_3] = C_1 \Box + C_2 X_1^2
\end{align}
for some $C_1$ and $C_2$.

**Proposition 2.6.** (1) $X_1, X_2$ and $X_3$ are all tangent to $\Sigma_1$.

(2) $X_1, X_2$ and $X_4$ are all tangent to $\Sigma_1 \cup \Sigma_2$.

**Proposition 2.7.** (1) $X_1, X_2$ and $X_3$ are linearly independent in $\Sigma_1^c$.

(2) $X_1, X_2$ and $X_4$ are linearly independent in $(\Sigma_1 \cup \Sigma_2)^c$.

3. LEMMAS

In this section, we prepare several lemmas which are used to prove the theorems. We put $P = t \partial_t + x \cdot \partial_x$. Let $K'$ be a relatively compact open set in $K$ satisfying the same condition $K$ of the section 2. We consider the following linearized equation, 
\[ \begin{cases} \Box v = F(w), \\
 v = u(-\epsilon, x) \quad \partial_t v = \partial_t u \quad \text{for \quad } t = -\epsilon, \end{cases} \]
where we take $\epsilon$ is so small that $K'(-\epsilon)$ determines $K' \cap \{-\epsilon < t < T\}$. Let $S$ denote the mapping that corresponds $w$ to $v$. We put $u_0 = S[0]$ and $u_n = Su_{n-1}$. Since $u_0$
is a solution to the homogenous linear wave equation, there exists a constant $A$ such that $\|u_0\|_{Y_A([-\epsilon,T];P)} < \infty$. We put $B_0 = \max(\|u_0\|_{Y([-\epsilon,T];P)}, 2\|u(t)\|_{X(P)})$.

**Lemma 3.1.** If $u$ satisfies the assumption of Theorem 1.2 or 1.3 or 1.4, we have

$$ (19) \quad \|u\|_{Y([t_1,t_2];P)} \leq \|u(t_1)\|_{X(P)}, $$

for $-\epsilon \leq t_1 < t_2 \leq T$ with $t_2 - t_1 < 1/(2C(T)F(C(n))G(B_0))$.

**Proof.** Using Propositions 2.1, 2.3 and 2.4, we have the lemma. \square

**Lemma 3.2 (the Energy estimate).** If $u$ satisfies the assumption of Theorem 1.2 or 1.3 or 1.4, we have

$$ (20) \quad \sup_{0 \leq t \leq T} \|u(t)\|_{X(P)} \leq \|\phi\|_{C^s_t}^{1}[\|\nabla;E_m\|]. $$

**Proof.** Using the lemma 3.1 several times, we have the lemma. \square

4. **Proof of Theorem 1.1 and 1.2**

First we prove Theorem 1.1. From Propositions 2.6 and 2.7 it suffices to show that for every compact set $K' \subset K$ there exist constants $C_1$ and $A_1$ such that

$$ (21) \quad \|X^{\alpha_1}X^{\alpha_2}X^{\alpha_3}u\|_{H^m(K')} \leq C_1A_1^{\alpha_1}|\alpha|^\sigma, $$

for all non negative integers $\alpha_1, \alpha_2$ and $\alpha_3$ with $|\alpha| = \alpha_1 + \alpha_2\alpha_3$. We can prove the above by the same argument as in the proof of Lemma 3.2.

5. **Regularity in the Interior of the Cone**

Let $\sigma$ be a real number greater than or equal to 1. We put $P = t\partial_t + x \cdot \partial_x$. The following lemma is a key lemma to prove Theorem 1.4.

**Lemma 5.1 (Key lemma).** Suppose that

$$ (22) \quad \|P^l u\|_{H^l(K)} \leq C_1A_1^l(l!)^\sigma \quad \text{for } \forall l \in N \cup \{0\} $$

and $u$ satisfies the equation (1). Then $u$ is a Gevrey class function of order $\sigma$ in $\Gamma_+$, where $\Gamma_+ = \{(t,x) \in \mathbb{R}^2; t^2 > |x|^2, t > 0\}$.

**Proof.** For simplicity, we prove only the case $f(u) = u^m$. Let $B \subset \Gamma_+$ be a relatively compact ball. It suffices to show that $u$ is a Gevrey class function of order $\sigma$ in each $B \subset \Gamma_+$. We put $M = \Box^2 + P^4$.

Let $\chi(x)$ be a $C^\infty$ function in $B$ such that $0 < \chi \leq 1$ in $B$ and $\chi(x) = \text{dist}(x, \partial B)$ near $\partial B$. We put $\psi(x) = \chi(x)^N$ and we take $N$ sufficiently large that $\|\partial^\beta(\psi u)\|_B \leq c\|M\psi u\|_B$ for $|\beta| \leq 4$.

We show that

$$ (23) \quad \|\psi^{\beta_0}\Box^\alpha P^l u\| \leq C_2A_2^{\beta_0 + l}(l!)^\sigma $$
for some $C_2 > 0$ and $A_2 > 0$ for all $\alpha \geq 0$ and all $l \geq 0$. We show (23) by induction
with respect to $\alpha$.

When $|\alpha| = 0$, (23) is nothing but the assumption (22). We assume that (23) is
valid until $|\alpha| = m$.

First we prove the case $0 \leq m \leq 3$. For $|\alpha| = m + 1$, we have

$$
||\psi^{[\alpha]} \partial^{\alpha} P^l u||_B \leq ||\partial^{[\alpha]} \psi^{[\alpha]} P^l u||_B + \|[\psi^{[\alpha]}, \partial^{\alpha}] P^l u||_B.
$$

The second term of the right hand side is estimated by

$$
\sum_{\alpha' < \alpha} C_3 ||\psi^{[\alpha']} \partial^{\alpha'} P^l u|| \leq C_4 C_2 A_2^{m+1} ((m + l)!)^\sigma
$$

if we take $A_2 \geq 2C_4$. Since $|\alpha| = m + 1 \leq 4$, the first term is estimated by

$$
||M \psi^{[\alpha]} P^l u||_B \leq ||\psi^{[\alpha]} M P^l u||_B + \|[M, \psi^{[\alpha]}] P^l u||_B.
$$

The second term of the right hand side of the above inequality can be estimated by

$$
C_5 \sum_{|\alpha'| \leq 3} ||\psi^{[\alpha']} \partial^{\alpha'} P^l u||_B \leq \frac{1}{4} C_2 A_2^{m+1} ((m + l)!)^\sigma
$$

if we take $A_2$ sufficiently large. The first term is estimated by

$$
||\psi^{[\alpha]} \partial^2 P^l u||_B + ||\psi^{[\alpha]} P^{l+4} u||_B.
$$

The second term of the right hand side of the above can estimated by

$$
C_6 ||P^{l+4} u||_B \leq C_6 C_2 A_2^{l+4} ((l + 4)!)^\sigma
$$

$$
\leq \frac{1}{8} C_2 A_2^{l+m+1} ((l + m + 1)!)^\sigma,
$$

if we take $C_2$ and $A_2$ sufficiently large. The first term of the right hand side of the
above is estimated by

$$
||\psi^{[\alpha]} (P + 4)^l \Box u||_B = ||\psi^{[\alpha]} (P + 4)^l (u^m)||_B
$$

$$
\leq m ||\psi^{[\alpha]} (P + 4)^l u^{2m-1}|| + \left( \frac{m}{2} \right)^2 \sum_{j=0}^2 ||\psi^{[\alpha]} (P + 4)^l u^{m-2}(\partial_j u)^2||_B.
$$

The second term of the right hand side of the above is estmated by

$$
\frac{m(m-1)}{2} \sum_{j=0}^2 \sum_{a_1 + \cdots + a_m = \alpha} \frac{\alpha!}{\alpha_1! \cdots \alpha_m!} \sum_{l_1 + \cdots + l_m = l} \frac{l!}{l_1! \cdots l_m!} ||\psi^{[\alpha]} (P + 4)^l u||_B \times
$$

$$
||\psi^{[\alpha]} P^l u||_B \cdots ||\psi^{[\alpha]} P^{l-2} u||_B \psi^{[\alpha]} P^{l-1} \partial_j u||_B ||\psi^{[\alpha]} P^{l-2} \partial_j u||_B.
$$
This can be estimated by $\frac{1}{16}C_2^{i+m+1}((l+m+1)!)^\sigma$ if we take $C_2$ and $A_2$ sufficiently large. The first term can be also estimated by $\frac{1}{16}C_2^{i+m+1}((L+m+1)!)^\sigma$.

Next we prove the case $m \geq 4$. For $|\alpha| = m-3$ and $|\beta| = 4$, we have

$$\|\psi^{m+1}\partial^{\alpha+\beta}P^lu\|_B \leq \|\partial^{\beta}\psi^{m+1}\partial^\alpha P^lu\|_B + \|[[\psi^{m+1}, \partial^\beta]\partial^\alpha P^lu]\|_B.$$  

The second term of the right hand side of the above can be estimated by

$$C_7 \sum_{\beta' < \beta} \|\psi^{m-3+|\beta'|}\partial^{\beta'}P^lu\|_B \leq \|C_2^{m+i}((m+l+1)!)^\sigma \leq \frac{1}{2}C_2^{m+i}((m+l+1)!)^\sigma$$

if we take $A_2 \geq 2C_7$. Since $|\beta| = 4$, the first term is estimated by

$$\|M\psi^{m+1}\partial^\alpha P^lu\|_B \leq \|\psi^{m+1}M\partial^\alpha P^lu\|_B + \|[M, \psi^{m+1}]\partial^\alpha P^lu\|_B.$$  

Using the same argument as in the case $m \leq 3$, we can estimate the right hand side of the above by $\frac{1}{4}C_2^{m+i}((m+l+1)!)^\sigma$. But we note that we do not change $C_2$ at each step of induction in the case $m \geq 4$ not as in the case $m \leq 3$.

6. PROOF OF MAIN RESULT

We devide $K \setminus \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$ into 4 parts, $\bigcup_{i=1}^{4} \mathcal{O}_i$ with

$$\mathcal{O}_1 = \{(t, x) \in \mathbb{R}^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x < 0, t - \omega_3 \cdot x < 0\} \cup \cdots$$

$$\mathcal{O}_2 = \{(t, x) \in \mathbb{R}^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x < 0\} \cup \cdots$$

$$\mathcal{O}_3 = \{(t, x) \in \mathbb{R}^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x > 0, t^2 - |x|^2 < 0\}$$

$$\mathcal{O}_4 = \{(t, x) \in \mathbb{R}^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x > 0, t^2 - |x|^2 > 0\}.$$  

For $x$ in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$, the backward light cone $\Gamma_x^-$ from $x$ does not contain the origin. So we can prove that $u$ is in $G^{(\sigma)}$ in this area by the same argument as in the proof of Theorems 1.2 and 1.3.

To prove that $u$ is in $G^{(\sigma)}$ in $\mathcal{O}_4$, we use the operator $P = t\partial_t + x \cdot \partial_x$. Using this operator, M. Beals[1] has given another proof of the theorem 1.1 of Bony and Melrose–Ritter. Note that for all relatively compact open set $L \subset \Omega_-$, there exist constants $C, A_1 > 0$ such that

$$\|P^k u\|_{H^s(L)} \leq CA_1^k(k!)^\sigma \text{ for all } k,$$
from the assumptions of Theorem 1.2. Since $[\square, P] = 2\square$, 
(40) \[ \square(Pu) = P\square u + [\square, P]u = (P + 2)f(u). \]
So we have 
(41) \[ \square(P^k u) = (P + 2)^k f(u). \]
Using the energy inequality 3.2, we have for all relatively compact open set $L \subset K$, there exist constants $C, A > 0$ such that 
(42) \[ \|P^k u\|_{H^s(L)} \leq CA^k (k!)^s \quad \text{for } \forall k. \]
From Lemma 5.1, we have that $u$ is in $G^{(\omega)}$ in $\mathcal{O}_4$.

REFERENCES


DEPARTMENT OF MATH., OSAKA UNIVERSITY, TOYONAKA, OSAKA 560 JAPAN

e-mail address:kei@math.sci.osaka-u.ac.jp