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NONINTEGRABILITY INDUCED BY NON-RESONANCE CHECKING

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Abstract. We discuss the non-integrability of non-homogeneous nonlinear lattices via the singularity analysis of normal variational equations of Lamé type. It is shown that quartic nonlinear lattices without cubic terms have no other additional quantities besides the Hamiltonians by checking the non-resonance condition of the monodromy for the special solutions in terms of the Jacobi elliptic function.

1. From a Nonhomogeneous Nonlinear Lattice to Lamé Equations

Even today, it is still difficult to investigate the dynamical origin of thermal behaviour without resort to numerical simulations, which gives us the basis of statistical mechanics. In this report, by using the singularity analysis, we discuss the non-integrability, one of the most important necessary condition of ergordicity for nonlinear lattices which can be considered as idealistic models for thermal behaviour. We consider the following one-dimensional lattice:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i=1}^{n+1} v(q_{i-1} - q_i), \tag{1}$$

where

$$v(X) = \frac{\mu_2}{2}X^2 + \frac{\mu_4}{4}X^4 + \cdots + \frac{\mu_{2m}}{2m}X^{2m}. \tag{2}$$

Fermi-Pasta-Ulam(FPU) lattice [4] is a special type of the systems with the potential function (2) as follows:

$$H_{FPU} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{\mu_2}{2}\sum_{i=1}^{n+1}(q_{i-1} - q_i)^2 + \frac{\mu_4}{4}\sum_{i=1}^{n+1}(q_{i-1} - q_i)^4. \tag{3}$$

If we impose the fixed boundary condition as

$$q_0 = q_{n+1} = 0, \quad n = \text{odd}, \tag{4}$$

it is easy to check that

$$\Gamma: \quad q_1 = C\phi(t), q_2 = 0, q_3 = -C\phi(t), \cdots, q_{n-1} = 0, q_n = (-1)^{\frac{n-1}{2}}C\phi(t) \tag{5}$$

is a special solution. Thus, the equation of $\phi(t)$ is equivalent to the following Hamiltonian system with one degree of freedom:

$$\ddot{\phi} + 2\mu_2\phi + 2\mu_4 C^2\phi^3 + \cdots + 2\mu_{2m}C^{2m-2}\phi^{2m-1} = 0, \tag{6}$$

where Hamiltonian is

$$H(\phi,\dot{\phi}) = \frac{1}{2}(\dot{\phi})^2 + \mu_2\phi^2 + \frac{\mu_4 C^2}{2}\phi^4 + \cdots + \frac{\mu_{2m}C^{2m-2}}{m}\phi^{2m} = \text{Const.} \tag{7}$$

Then the total energy ϵ is given by

$$\epsilon = H = H(\phi,\dot{\phi})\frac{n+1}{2}C^2 = \frac{n+1}{2}C^2(\mu_2 + \frac{1}{2}\mu_4 C^2 + \cdots \frac{1}{m}\mu_{2m}C^{2m-2}) \tag{8}$$

for the initial condition (5). In the case of the FPU lattice, we can determine C as follows:

$$C = \sqrt{\frac{\sqrt{\mu_2^2 + \frac{4\epsilon}{n+1}\mu_4} - \mu_2}{\mu_4}}. \tag{9}$$

By combining (7) with (8), the underlying equation of $\phi(t)$ can be rewritten by the differential equation of $\phi(t)$ as

$$\frac{1}{2}(\dot{\phi})^2 = \gamma_2(1-\phi^2) + \frac{\gamma_4}{2}(1-\phi^4) + \cdots + \frac{\gamma_{2m}}{m}(1-\phi^{2m}), \tag{10}$$

where

$$\gamma_{2m}(\epsilon, \{\mu_{2j}|j = 1,\cdots,m\}) \equiv \mu_{2m}C^{2m-2}. \tag{11}$$

In case of the case of the FPU lattices (3), the solution of this differential equation (10) with the condition

$$\gamma_{2m=4} \neq 0 \tag{12}$$

is given explicitly by the elliptic function

$$\phi(t) = cn(k;\alpha t), \tag{13}$$

where

$$\alpha = \sqrt{2\gamma_2 + 2\gamma_4}, \quad k = \sqrt{\frac{\gamma_4}{2\gamma_2 + 2\gamma_4}}, \tag{14}$$

$cn(k;\alpha t)$ is the Jacobi cn elliptic function, and k is the modulus of the elliptic integral. We remark that because

$$\gamma_2 + \gamma_4 = \mu_2 + C^2\mu_4 = \sqrt{\mu_2^2 + \frac{4\epsilon}{n+1}\mu_4} > 0, \tag{15}$$

holds for $\mu_4 > 0, \mu_2 \geq 0$, the modulus of the elliptic function k satisfies the following relation:

$$0 \leq k \leq \frac{1}{\sqrt{2}}. \tag{16}$$

Thus, the special solutions of the FPU lattices for $\mu_4 > 0, \mu_2 \geq 0$ have the two *fundamental periods* in the complex time plane as follows:

$$T_1(\epsilon,\mu) = \frac{2K(k)}{\alpha}, \quad T_2(\epsilon,\mu) = \frac{2K(k)+2iK'(k)}{\alpha}, \tag{17}$$

where $K(k)$ and $K'(k)$ are the complete elliptic integrals of the first kind:

$$K(k) = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}, \quad K'(k) = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-(1-k^2)v^2)}}. \tag{18}$$

Poles are located at $t = \tau$, where $\tau = \frac{2K(k)}{\alpha} + i\frac{K'(k)}{\alpha}$ (mod T_1, T_2) in the parallelogram of each period cell. Let us consider the variational equations along these special solutions. The variational equations are obtained by

$$\begin{aligned} \dot{\eta}_j = \ddot{\xi}_j &= -\sum_{k=1}^{n} \left.\frac{\partial^2 V}{\partial q_k \partial q_j}\right|_{\Gamma} \xi_k \\ &= -(\gamma_2 + 3\gamma_4\phi^2 + 5\gamma_6\phi^4 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})(2\xi_j - \xi_{j-1} - \xi_{j+1}) \text{ for } 1 \leq j \leq n, \end{aligned} \tag{19}$$

where $\xi_0 = \xi_{n+1} = \eta_0 = \eta_{n+1} = 0$ and $\xi_j = \delta q_j, \eta_j = \delta p_j \quad (1 \leq j \leq n)$.
Moreover, these linear variational equations in the form of the vector

$$\frac{d^2}{dt^2}\boldsymbol{\xi} = -(\gamma_2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2}) \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \boldsymbol{\xi} \tag{20}$$

can be decoupled as follows. After we note that the eigenvalues of the $n \times n$ symmetric matrix

$$\boldsymbol{G} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{21}$$

are obtained as $\{4\sin^2(\frac{j\pi}{2(n+1)}) | 1 \leq j \leq n\}$ by a normal orthogonal transformation $\boldsymbol{G} \to \boldsymbol{OGO}^{-1}$, the variational equations (19) are rewritten in the decoupled form:

$$\ddot{\xi}'_j(t) = -4\sin^2(\frac{j\pi}{2(n+1)})(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})\xi'_j(t) \quad (1 \leq j \leq n), \tag{22}$$

where $\boldsymbol{\xi}' = \boldsymbol{O}\boldsymbol{\xi}$. Clearly, these equations are written in the form of vector *Hill's equation*[6]

$$\frac{d^2\boldsymbol{\xi}'}{dt^2} + \boldsymbol{A}(t)\boldsymbol{\xi}' = 0, \quad \boldsymbol{A}(t+T) = \boldsymbol{A}(t), \tag{23}$$

where $T = T_1, T_2$. For $j = \frac{n+1}{2}$, we have the relation

$$\xi'_{\frac{n+1}{2}} = \sqrt{\frac{2}{n+1}}(\xi_1 - \xi_3 + \xi_5 + \cdots + (-1)^{\frac{n-1}{2}}\xi_n). \tag{24}$$

Thus, the corresponding variational equation

$$\ddot{\xi}'_{\frac{n+1}{2}} = -2(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})\xi'_{\frac{n+1}{2}}(t) \tag{25}$$

has a time-dependent integral $I(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}; t) \equiv I(\boldsymbol{\xi}, \boldsymbol{\eta}; t)$ because

$$\begin{aligned} I(\boldsymbol{\xi}, \boldsymbol{\eta}; t) &= \boldsymbol{D}H \equiv (\boldsymbol{\eta} \cdot \tfrac{\partial}{\partial \boldsymbol{p}} + \boldsymbol{\xi} \cdot \tfrac{\partial}{\partial \boldsymbol{q}})H = \boldsymbol{\eta} \cdot \boldsymbol{p} + \boldsymbol{\xi} \cdot \boldsymbol{V}_q \\ &= C\dot{\phi}(\eta_1 - \eta_3 + \eta_5 + \cdots + (-1)^{\frac{n-1}{2}}\eta_n) \\ &+ 2(C\gamma_2\phi + C\gamma_4\phi^3 + \cdots + C\gamma_{2m}\phi^{2m-1})(\xi_1 - \xi_3 + \xi_5 + \cdots + (-1)^{\frac{n-1}{2}}\xi_n), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \tfrac{1}{C}\tfrac{dI}{dt} &= \dot{\phi}(\ddot{\xi}_1 - \ddot{\xi}_3 + \cdots + (-1)^{\frac{n-1}{2}}\ddot{\xi}_n) \\ &+ 2\dot{\phi}(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})(\xi_1 - \xi_3 + \cdots + (-1)^{\frac{n-1}{2}}\xi_n) = 0. \end{aligned} \tag{27}$$

We call Eq. (25) the *tangential variational equation*. On the other hands, a $(2n-2)$-dimensional *normal variational equation*(NVE) is given by the equation of (22) with the tangential variational equation (25) removed as follows:

$$\begin{aligned} \dot{\eta}'_j &= -4\sin^2(\tfrac{j\pi}{2(n+1)})(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})\xi'_j, \\ \dot{\xi}'_j &= \eta'_j \quad \text{for } 1 \le j(\neq \tfrac{n+1}{2}) \le n. \end{aligned} \tag{28}$$

In case of the FPU lattice, the normal variational equation (28) becomes the *Lamé equation* [14]

$$\frac{d^2y}{dt^2} - (E_1 cn^2(k; \alpha t) + E_2)y = 0, \tag{29}$$

where $E_1 = 12\frac{1}{\alpha^2k^2}\sin^2(\frac{j\pi}{2(n+1)})$ and E_2 are constants.

2. Non-integrability Theorem by Picard-Vessiot Theory

Morales and Simó obtained the following theorem on the non-integrability based on the application of Picard-Vessiot theory to Ziglin's analysis[17, 18] for Hamiltonian systems with two degrees of freedom.

Theorem 1 (Morales and Simó [8],1994) *When the normal reduced variational equation is of Lamé type, if $A \equiv E_1\alpha^2k^2 \neq m(m+1), m \in \boldsymbol{N}$ and the Lamé equation satisfying this condition on A is not algebraically solvable(Brioschi-Halphen-Crawford and Baldassarri solutions), then the the initial Hamiltonian system does not have a first integral, meromorphic in a connected neighborhood of the integral curve Γ, which is functionally independent together with H.*

In case of the present analysis, A is given by the following formula:

$$A = E_1\alpha^2k^2 = 12\sin^2(\frac{j\pi}{2(n+1)}) = 6(1 - \cos(\frac{j\pi}{n+1})). \tag{30}$$

We can easily check that $\cos(\frac{j\pi}{n+1}) \notin \boldsymbol{Q}$ if and only if $j \notin \{\frac{n+1}{3}, \frac{n+1}{2}, \frac{2(n+1)}{3}\}$. When $A \notin \boldsymbol{Q}$, the above condition on the algebraic solvability of the Lamé equation is not satisfied. Thus, to check the algebraic solvability of the Lamé equations

$$\frac{d^2\xi_j}{dt^2} - (\frac{12}{\alpha^2k^2}\sin^2(\frac{j\pi}{2(n+1)})cn^2(k; \alpha t) + E_2)\xi_j = 0 \quad (j \neq \frac{n+1}{2}), \tag{31}$$

it is sufficient to examine the following two cases:

$$A = 6(1-\cos(\frac{1\pi}{3})) = 3, \quad A = 6(1-\cos(\frac{2\pi}{3})) = 9. \tag{32}$$

It is known [2] that the condition on A for the Brioschi-Halphen-Crawford solutions is given by

$$A = m(m+1), \quad m+\frac{1}{2} \in \boldsymbol{N}, \tag{33}$$

and that the condition on A for the Baldassarri solutions is given by

$$A = m(m+1), \quad m+\frac{1}{2} \in \frac{1}{3}\boldsymbol{Z} \cup \frac{1}{4}\boldsymbol{Z} \cup \frac{1}{5}\boldsymbol{Z} \setminus \boldsymbol{Z}. \tag{34}$$

However, the following relations

$$m(m+1) = 3 \to m = \frac{-1 \pm \sqrt{13}}{2} \notin \boldsymbol{Q}, \quad m(m+1) = 9 \to m = \frac{-1 \pm \sqrt{37}}{2} \notin \boldsymbol{Q} \tag{35}$$

hold, which guarantee that *all* $n-1$ Lamé equations (31) do not belong to the solvable case. In case of the systems with n degrees of freedom, we have $n-1$ Lamé equations which corresponds to $n-1$ normal variational equations.

Thus, we obtain the following theorem:

Theorem 2 ([12],1995) *The FPU lattice for $\mu_4 > 0, \mu_2 \geq 0$ does not have $n-1$ first integrals, meromorphic in a connected neighbourhood of the integral curve Γ, which are functionally independent together with H.*

3. Non-integrability Theorem Induced by Non-resonance Hypothesis

The theorem on the non-integrability using Picard-Vessiot theory in Section 2 does not depend on the total energy. In this section, it is shown that we can get stronger non-integrability results if the non-resonance condition on the monodromy group is employed.

We can consider the monodromy matrices g defined by the analytic continuation of the solution $\zeta'(t) = (\xi_1'(t), \eta_1'(t), \cdots, \hat{\xi}'_{\frac{n+1}{2}}(t), \hat{\eta}'_{\frac{n+1}{2}}(t), \cdots, \xi_n'(t), \eta_n'(t))$ of the NVE (28) along the periodic orbits in the phase curves $\Gamma(\epsilon, t)$ as follows:

$$\boldsymbol{\zeta}'(T_1) = g_1\boldsymbol{\zeta}'(0), \quad \boldsymbol{\zeta}'(T_2) = g_2\boldsymbol{\zeta}'(0). \tag{36}$$

The periods of (36) are T_1, T_2 in (17), respectively. These two fundamental periods T_1 and T_2 naturally form the parallelogram, whose associate monodromy matrices are given by $g_1g_2g_1^{-1}g_2^{-1}(\equiv g_*)$. These monodromy matrices are naturally endowed with the symplectic structure and the pairing properties of the eigenvalues, namely $\{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}, \cdots, \sigma_n, \sigma_n^{-1}\}$ [1]. In practice, the explicit calculation of the eigenvalues of the monodromy matrices is an unsuccessful business except rare cases such as Hamiltonian systems with homogeneous polynomial functions[16], *Riemann's equation*, and the *Jordan-Pochhammer equations*[9, 5, 3]. This is one of the unavoidable difficulties in performing Ziglin's analysis of general dynamical systems as well as checking the non-resonance condition[11]. However, if we restrict ourselves to the eigenvalues of the *commutator* $g_* = g_1g_2g_1^{-1}g_2^{-1}$ for the case of the FPU lattice, we can confirm that the eigenvalues of the *commutator* $g_* = g_1 g_2 g_1^{-1} g_2^{-1}$ can be computed as follows; the normal variational equation (28) is the *Lamé equation*

$$\frac{d^2y}{dt^2} - (E_1 cn^2(k;\alpha t) + E_2)y = 0, \tag{37}$$

where E_1 and E_2 are constants, and the eigenvalues σ of the commutator $g_* = g_1 g_2 g_1^{-1} g_2^{-1}$ are known [7, ?] to be determined by the indicial equation

$$\Delta^2 - \Delta - (\alpha^2 k^2)E_1 = 0, \quad \sigma = \exp(2\pi i \Delta) \tag{38}$$

with the singular point (pole) τ located at the center of the parallelogram as follows:

$$\tau = \frac{T_1 + T_2}{2} = \frac{2K(k) + iK'(k)}{\alpha}. \tag{39}$$

If we apply the indicial equation(38) to the NVE (28) of the FPU lattices, the exponents of the eigenvalues of g_* are given by

$$\Delta^2 - \Delta - 12\frac{\gamma_4}{\alpha^2 k^2}\sin^2(\frac{j\pi}{2(n+1)}) = 0. \tag{40}$$

Noting

$$\frac{\gamma_4}{\alpha^2 k^2} = 1 \tag{41}$$

from (14), we obtain the eigenvalues of the commutator as

$$\begin{aligned} \sigma_j &= \exp(2\pi i \frac{1 \pm \sqrt{1 + 48\sin^2\frac{j\pi}{2(n+1)}}}{2}) \\ &= -\exp(\pm\pi i\sqrt{25 - 24\cos(\frac{j\pi}{n+1})}) \end{aligned} \tag{42}$$

for $1 \leq j (\neq \frac{n+1}{2}) \leq n$, because there is *only one* pole singularity inside the parallelogram. For general nonlinear lattices with the reflection symmetry, we can assume the pole singularity of $\phi(t)$ at $t = \tau$ inside the parallelogram in the complex time plane:

$$\begin{aligned} &\phi(t) = C'(t-\tau)^{\beta}, \quad \beta < 0 \\ &\xi_j = C''(t-\tau)^{\nu_j}. \end{aligned} \tag{43}$$

With the redefinition of $t - \tau$ as t, and by the underlying equations of $\phi(t)$ and $\xi(t)$, (10) and (22), the condition for β is given by

$$C'\beta(\beta-1)t^{\beta-2} = -2\gamma_{2m}(C't^{\beta})^{2m-1}. \tag{44}$$

Namely, we have

$$\beta = \frac{-1}{m-1}, \quad (C')^{2m-2} = -\frac{\beta(\beta-1)}{2\gamma_{2m}}. \tag{45}$$

Then, ν_j is obtained from the formula

$$C''\nu_j(\nu_j - 1)t^{\nu_j - 2} = C''4\sin^2(\frac{j\pi}{2(n+1)})\frac{m(2m-1)}{2(m-1)^2}t^{-2}t^{\nu_j}. \tag{46}$$

Thus, we finally arrive at the indicial equation for ν_j as follows:

$$\nu_j^2 - \nu_j - 2\frac{m(2m-1)}{(m-1)^2}\sin^2(\frac{j\pi}{2(n+1)}) = 0. \tag{47}$$

Note that the phase factor $\exp(2\pi i\nu_j)$ is obtained by means of the analytic continuation along the closed loop around the singular point t^{ν_j}. Consequently the phase factors of the indicial equation (47) are given by

$$\begin{aligned} &\exp(2\pi i\nu_j) \\ &= \exp\{2\pi i\frac{1\pm\sqrt{1+8\frac{m(2m-1)}{(m-1)^2}\sin^2(\frac{j\pi}{2(n+1)})}}{2}\} \\ &= -\exp\{\pm\pi i\sqrt{1+8\frac{m(2m-1)}{(m-1)^2}\sin^2(\frac{j\pi}{2(n+1)})}\} \\ &= -\exp\{\pm\pi i\sqrt{\frac{(3k-2)^2}{(k-2)^2}-\frac{8k(k-1)}{(k-2)^2}\cos(\frac{j\pi}{(n+1)})}\}, \end{aligned} \tag{48}$$

where $k \equiv 2m$ is the degree of the potential polynomial of $v(X)$ in (2). It is easy to confirm that in the case that $k = 2m = 4$, the present formula (48) recover the eigenvalues (42) of the monodromy matrices obtained from the *Jacobi elliptic function.* To summarize, we obtain the special solutions of hyper-elliptic function for the general nonlinear lattices with the reflection symmetry and correspondingly obtain the phase factors (48) around the singularity of the solutions in the complex time plane. Furthermore, if we restrict ourselves to the FPU lattices, we can obtain the exact eigenvalues or the characteristic multipliers of the monodromy matrices $g_1g_2g_1^{-1}g_2^{-1}$ which happens to equal the phase factors (48).

If the eigenvalues $\{\sigma_1,\sigma_1^{-1},\cdots,\sigma_n,\sigma_n^{-1}\}$ of monodromy matrices do not satisfy the following relation

$$\sigma_1^{l_1}\sigma_2^{l_2}\cdots\sigma_n^{l_n} = 1 \tag{49}$$

for any set of integers $l_1,\cdots,l_n$ except the trivial case $l_1 = l_2 = \cdots = l_n$, the monodromy matrices are called *non-resonant.* It is known that the existence of a non-resonant monodromy matrix is a basic assumption in order to perform Ziglin's analysis[17]. Moreover, if there are straight line solutions whose monodromy matrices are non-resonant and if the variational equations along the straight line solutions can be diagonalized into decoupled variational equations by constant matrices, Ziglin's theorem can be generalized to Yoshida's theorem [15] for Hamiltonian systems composed by kinetic energy terms and potential energy terms.

Yoshida's theorem asserts the following in terms of the monodromy matrices: Suppose that there exists an additional complex analytic integral, and that one of the monodromy matrices g_a is non-resonant. Then it is necessary that one of the following two cases, namely,

(I) $g_b(\lambda_j)$ must preserve the eigendirection of $g_a(\lambda_j)$, i.e., $g_a(\lambda_j)$ must commute with $g_b(\lambda_j)$,

(II) $g_b(\lambda_j)$ must permute the eigendirection of $g_a(\lambda_j)$, i.e., $g_b(\lambda_j)$ is written by $\begin{bmatrix} 0 & \beta \\ -\frac{1}{\beta} & 0 \end{bmatrix}$ in the base of g_a having the eigenvalues i and $-i$

for some suffix j, at least occurs for any other monodromy matrix g_b represented by the base of g_a. By using Yoshida's theorem and the variational analysis in the former section, we obtain the following result:

Theorem 3 ([13],1995) *If the n quantities* $\{\sqrt{25-24\cos\frac{j\pi}{n+1}} | j = 1, \cdots, n\}$ *are rationally independent, then the FPU lattices (3) have no analytic first integrals besides the Hamiltonian itself for odd n* (≥ 3), $\mu_2 \geq 0, \mu_4 > 0$ *and for sufficiently small energy* $\epsilon(\approx 0)$.

We remark that the algebraic condition of the rational independence of the set $\{\sqrt{25-24\cos\frac{j\pi}{n+1}} | j = 1, \cdots n\}$ comes from the non-resonance hypothesis of the commutator $g_* = g_1 g_2 g_1^{-1} g_2^{-1}$. See [10] for checking the rational independency using an algebraic number theory on the *cyclotomic field* $\boldsymbol{Q}(\exp(\frac{\pi i}{n+1}))$ over $\boldsymbol{Q}$.

Thus, we obtain the following theorem:

Theorem 4 ([13],1995) *The FPU lattices (3) have no analytic first integrals besides the Hamiltonian itself for odd n* (≥ 3), $\mu_2 > 0$, $\mu_4 > 0$ *and for fully small energy* $\epsilon(\approx 0)$.

(**Proof of Theorem 3**)

From the non-resonance hypothesis on g_*, we can apply the Yoshida theorem[15] to the FPU lattices. According to the above argument, if we prove that at least one monodromy matrix $g_s \in \{g_1, g_2\}$ does not have the following properties

(a) $g_s(\lambda_j)$ preserves the eigendirection of $g_*(\lambda_j)$
and
(b) $g_s(\lambda_j)$ permutes the eigendirection of $g_*(\lambda_j)$(the eigenvalues of $g_s(\lambda_j)$ are $i, -i$),

at once for any suffix $j(1 \leq j(\neq n+1/2) \leq n)$ of $g_s(\lambda_j)$, then the FPU lattices [4] are concluded to have no other analytic conserved quantities besides the Hamiltonian itself, i.e., the assertion of the present theorem holds. When we take the limit $\epsilon \to 0$, the relations

$$\gamma_4 = \mu_4 C^2, \cdots, \gamma_{2m} = \mu_{2m} C^{2m-2} \to 0, \quad \alpha \to \sqrt{2\mu_2}, \quad k \to 0 \tag{50}$$

holds in Eq.(14) and

$$T_1 \to \frac{1}{\sqrt{2\gamma_2}}\pi, \quad T_2 \to \frac{1}{\sqrt{2\gamma_2}}\pi + i\infty \tag{51}$$

in Eq. (17). In the low-energy limit, the variational equations (28) approaches to the following equations:

$$\begin{aligned} \dot{\eta'}_j &= -4\sin^2(\tfrac{j\pi}{2(n+1)})\mu_2 \xi'_j, \\ \dot{\xi'}_j &= \eta'_j \quad \text{for } 1 \leq j(\neq \tfrac{n+1}{2}) \leq n. \end{aligned} \tag{52}$$

In this limit, $g_1(\lambda_j)$ tend to

$$g_1(\lambda_j) = \begin{bmatrix} \cos(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})\pi) & \frac{1}{2\sin(\frac{j\pi}{2(n+1)})\sqrt{\mu_2}}\sin(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})\pi) \\ -2\sin(\frac{j\pi}{2(n+1)})\sqrt{\mu_2}\sin(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})\pi) & \cos(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})\pi) \end{bmatrix}, \tag{53}$$

and the eigenvalues of $g_1(\lambda_j)$ tend to $\{\exp(i\pi(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})), \exp(-i\pi(\sqrt{2}\sin(\frac{j\pi}{2(n+1)}))\}$; $g_1(\lambda_j)$ for any j does not have the property of (b).

Now assume that $g_2(\lambda_j)$ for some $j(\neq \frac{n+1}{2})$ has the property of (a). Then if $g_1(\lambda_j)$ as well as $g_2(\lambda_j)$ has also the property of (a), we have

$$g_*(\lambda_j) = g_1(\lambda_j) g_2(\lambda_j) g_1^{-1}(\lambda_j) g_2^{-1}(\lambda_j) = \boldsymbol{id}, \tag{54}$$

where $\boldsymbol{id}$ denotes the 2×2 identity matrix. This relation (54) means that $g_1(\lambda_j)$ and $g_2(\lambda_j)$ commute each other and clear contradicts the non-resonance hypothesis of g_*. Consider the other case where $g_1(\lambda_j)$ has the property of (a) and $g_2(\lambda_j)$ does not have the property of (a). However, in the representation of $g_1(\lambda_j)$ and $g_2(\lambda_j)$ in the basis of $g_*(\lambda_j)$ as follows:

$$g_1(\lambda_j) = \begin{bmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{bmatrix}, \quad g_2(\lambda_j) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ (ad - bc = 1), \tag{55}$$

the relation

$$g_*(\lambda_j) = g_1(\lambda_j)g_2(\lambda_j)g_1^{-1}(\lambda_j)g_2^{-1}(\lambda_j) = \begin{bmatrix} ad - \mu^2 bc & ab(\mu^2 - 1) \\ cd(\frac{1}{\mu^2} - 1) & ad - \frac{bc}{\mu^2} \end{bmatrix} \tag{56}$$

must be satisfied. Since $g_*(\lambda_j)$ is assumed to have a diagonal representation as $g_*(\lambda_j) = \mathrm{diag}\left[\sigma_j, \sigma_j^{-1}\right]$ and from (56), we obtain

$$a = 0, \quad d = 0, \quad bc = -1, \tag{57}$$

when $g_*(\lambda_j) \neq \boldsymbol{id}$; $g_2(\lambda_j)$ must have the property of (b).

Therefore, in the basis of $g_*(\lambda_j)$, we have

$$g_1(\lambda_j) = \begin{bmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{bmatrix} \text{ and } \ g_2(\lambda_j) = \begin{bmatrix} 0 & \beta \\ -\frac{1}{\beta} & 0 \end{bmatrix}. \tag{58}$$

These relations (58) results in

$$\begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j^{-1} \end{bmatrix} = g_*(\lambda_j) = g_1(\lambda_j)g_2(\lambda_j)g_1^{-1}(\lambda_j)g_2^{-1}(\lambda_j) = g_1^2(\lambda_j), \tag{59}$$

where $\sigma_j = -\exp\{\pi i\sqrt{25 - 24\cos\frac{j\pi}{n+1}}\}$. The relation (59) causes again a contradiction with the fact that the eigenvalues of $g_1(\lambda_j)$ approach $\{\exp(i\pi(\sqrt{2}\sin(\frac{j\pi}{2(n+1)})), \exp(-i\pi(\sqrt{2}\sin(\frac{j\pi}{2(n+1)}))\}$; as in the limit $\epsilon \to 0$. The precise proof is given in Ref. ([13]). We have seen that $g_1(\lambda_j)$ for any j has neither the property of (a) or the property of (b). Now the theorem holds. (■**End of proof of Theorem 3**)

4. Summary

We have investigated the non-integrability of non-homogeneous nonlinear lattices by the different methods based on the singularity analysis of the normal variational equations of Lamé type. It is shown that the non-integrability result using Picard-Vessiot theory holds for the model with an arbitrary finite energy and non-integrability result using the non-resonance hypothesis of the normal variational equations is much stronger than the former non-integrability result, although the latter non-integrability result holds only in the low energy limit.

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