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On the monotonicity of topological entropy for bimodal real cubic maps

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1 The parameter space for bimodal real cubic maps

A real cubic maps f from the real line $\mathbf{R}$ to itself is called *bimodal* if it has two real critical points distinct each other. This map can be normalized by the real affine conjugation as one of the following forms:

$$\begin{aligned} f_{a,b}(x) &:= x^3 - 3a^2x + b \ (a > 0, b \geq 0) \\ &:= -x^3 + 3a^2x + b \ (a < 0, b \leq 0) \end{aligned}$$

Therefore the space $P := P^+ \sqcup P^-$

$$\begin{aligned} P^+ &:= \{(a,b) \in \mathbf{R}^2 | a > 0, b \geq 0\} \\ P^- &:= \{(a,b) \in \mathbf{R}^2 | a < 0, b \leq 0\} \end{aligned}$$

can be considered as the parameter space for bimodal real cubic maps. In this paper we identify a cubic map $f_{a,b}$ with a point $(a,b) \in P$ and only consider a map $f_{a,b}$ for $(a,b) \in P^+$ for the sake of simplicity.

We decompose the parameter space P^+ into two complementary subsets with qualitatively different dynamical behavior.

Definition 1.1 *We define the connectedness locus C^+ and the escape locus E^+ by*

$$\begin{aligned} C^+ &:= \{(a,b) \in P^+ | f^n_{a,b}(\pm a) \, (n \in \mathbf{N}) \, \textit{is bounded}\} \\ E^+ &:= P^+ \setminus C^+ \end{aligned}$$

Remark 1.1 *([M])*
The shape of these subsets are in Figure 1. The boundary ∂C^+ consists of the parts of real algebraic curves S_1 and S_2.

$$S_1 := \{(a,b) \in P^+ | \, b = 2(a^2 + \frac{1}{3})^{\frac{3}{2}} \, , \, 0 < a \leq \frac{1}{3}\}$$
$$S_2 := \{(a,b) \in P^+ | \, b = 2a(1-a^2) \, , \, \frac{1}{3} \leq a \leq 1\}$$

S_1 consists of $f_{a,b}$ which have the nutral fixed point, on the other hand S_2 consists of $f_{a,b}$ whose critical value $f_{a,b}(-a)$ is a fixed point of $f_{a,b}$.

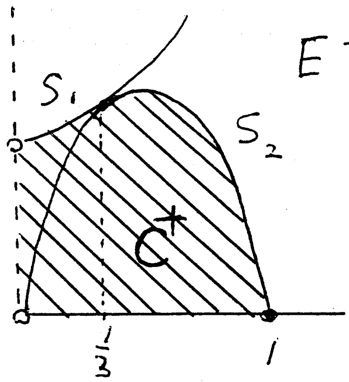


Figure 1

2 Monotonicity of the topological entropy on the escape locus

Definition 2.1 *For $f \in P^+$, the n-th lap number $l(f^n)$ is the number of the maximal intervals on which f^n the n-fold coposite of f is monotone. We define the topological entropy $h(f)$ of f by*

$$h(f) := \lim_{n\to\infty} \frac{1}{n} \log l(f^n).$$

Claim 2.1 *([M-T] Lemma 12.3)*
The function h on the parameter space

$$\begin{aligned} h : P^+ &\to \mathbf{R} \\ f &\mapsto h(f) \end{aligned}$$

is continuous.

Analogues to the monotonicity of the topological entropy for the real quadratic family $Q_c(x) = x^2 + c (c \in \mathbf{R})$, Milnor conjectured that the level set of the above function is connected and in [D-G-M-T] he considered this problem on the connectedness locus C^+ in detail.

Our main result is about the escape locus E^+.

Theorem 2.1 *On the escape locus E^+, the sets of cubic maps whose topological entropy are constant are connected (in fact they are simply connected).*

This result is a consequences of Claim 2.1 and the following claims.

Claim 2.2 *Topological entropy is monotone along ∂C^+.*

Claim 2.3 *There exists the homeomorphism T*

$$\begin{array}{rcl} T : \mathbf{R}_+ \times (0,1] & \rightarrow & E^+ \\ (s,u) & \mapsto & T(s,u) \end{array}$$

such that for $u \in (0,1]$ fixed, any real cubic maps in $T(\mathbf{R}_+, u)$ are quasi symmetric conjugate to each other and $T(s,u)$ goes to infinity if $s \rightarrow \infty$ and $T(s,u)$ goes to ∂C^+ if $s \rightarrow 0$.

Claim 2.2 is an analoguous result to the monotonicity for the quadratic family and because we can prove this by using the similar methods (namely the kneading theory and combinatorial rigidity of post critically finite rational maps), we omit the detail in this paper. After reviewing the work of Branner and Hubbard about the dynamical structure of the parameter space of complex cubic maps ([B] and [B-H]), we prove Claim 2.3 in the final section.

3 Review of the result of Branner and Hubbard

3.1 Parameter space for complex cubic maps

After complex affine cojugation, every complex cubic map $f : \mathbf{C} \to \mathbf{C}$ can be written as

$$f_{a,b}(z) = z^3 - 3a^2 z + b \ (a, b \in \mathbf{C})$$

We should remark that $\{\pm a\}$ is the critical set of $f_{a,b}$. Therefore we can take $\mathbf{C}^2$ as the parameter space $P(3)$ of complex cubic maps.

We decompose $P(3)$ into two complementary subsets *the connectedness locus* $C(3)$ and *the escape locus* $E(3)$. The connectedness locus $C(3)$ consists of cubic maps whose filled-in Julia set K_f is connected and the escape locus $E(3)$ is the complement of $C(3)$.

3.2 Escape rate to infinity

For $f \in P(3)$ define the function

$$g_f : \mathbf{C} \to \mathbf{R}_+ \cup \{0\}$$

by

$$g_f(z) := \lim_{n\to\infty} \frac{1}{d^n} \log_+(|f^n(z)|)$$

where $\log_+(|z|) := max\{0, log(|z|)\}$.

g_f is the Green function of the filled-in Julia set K_f which measures the escape rate to infinity.

We set

$$G : P(3) \to \mathbf{R}_+ \cup \{0\}$$

by

$$G(f) := max\{g_f(-a), g_f(a)\}.$$

Then G is continuous, $C(3) = G^{-1}(0)$ and for sufficiently large $r > 0$, we can show that $G^{-1}(r)$ is homeomorphic to the three dimensional sphere S^3.

3.3 Stretching rays

The map $l_s : \mathbf{C} \setminus \overline{D} \to \mathbf{C} \setminus \overline{D}$ ($s \in \mathbf{R}_+$) (where $\overline{D}$ is the closed disk) given by

$$l_s(z) := \frac{z}{|z|} \cdot |z|^s$$

is a q.c.diffeomorphism commuting with $f_0(z) = z^3$. Every $f \in P(3)$ is conjugate to f_0 on

$$U_f := \{z \in \mathbf{C} | g_f(z) > G(f)\}$$

by the analytic isomorphism φ_f satisfying

$$\frac{\varphi_f(z)}{z} \to 1 \ as \ z \to \infty.$$

Let σ_s denote the f-invariant almost complex structure on $\mathbf{C}$ satisfying

$$\sigma_s = \begin{cases} (l_s \circ \varphi_f)^*(\sigma_0) & on\, U_f \\ \sigma_0 & on\, K_f \end{cases}$$

where σ_0 denotes the standerd complex structure. Then the Measurable Riemann Mapping Theorem tells that there exists an analytic isomorphism

$$F_s : (\mathbf{C},\, \sigma_s) \to (\mathbf{C},\, \sigma_0).$$

We can uniquely choose F_s satisfying $f_s := F_s \circ f_0 \circ F_s^{-1}$ a monic, centered and $l_s \circ \varphi_f \circ F_s^{-1}$ tangent to the identity at ∞.

We call

$$R(f) := \{f_s | s \in \mathbf{R}_+\}$$

the stretching ray through f. Since $G(f_s) = sG(f)$, the stretching ray intersects $G^{-1}(r)$ in the exactly one point for any $r \in \mathbf{R}_+$.

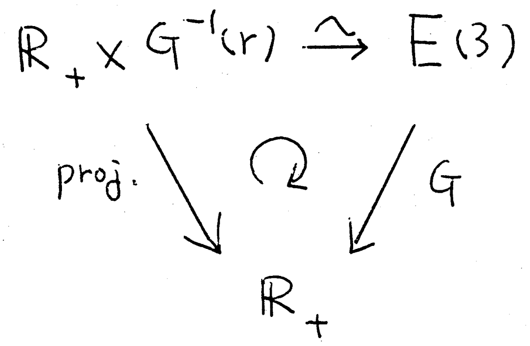
3.4 Fibration

One of the main result of [B-H] is

Theorem 3.1 *([B-H] Theorem 11.1)*
For any $r \in \mathbf{R}_+$, the map

$$\begin{array}{rcl} \mathbf{R}_+ \times G^{-1}(r) & \to & E(3) \\ (s, f) & \mapsto & f_s \end{array}$$

is a homeomorphism and makes the next diagram commutative



As a collorary $G^{-1}(r)$ for any $r \in \mathbf{R}_+$ is homeomorphic to S^3.

4 The proof of Claim 2.3

We consider the real locus $P(3) \cap \mathbf{R}^2$ of $P(3)$. Then $P(3) \cap \mathbf{R}^2$ consists of

$$f_{a,b} := z^3 - 3a^2 z + b \ (a, b \in \mathbf{R}).$$

The restriction of G to $E(3) \cap \mathbf{R}^2$ shows

Lemma 4.1 *For sufficiently large $r > 0$,*

$$G^{-1}(r) \cap \mathbf{R}^2 \simeq S^1$$

and

$$G^{-1}(r) \cap E^+ \simeq (0, 1].$$

Because l_s and real cubic map $f_{a,b}$ commute with the complex conjugation

Lemma 4.2 *The stretching ray $R(f)$ through $f \in E(3) \cap \mathbf{R}^2$ is contained in $E(3) \cap \mathbf{R}^2$. In particular for $f \in E^+$ the streching ray $R(f)$ is contained in E^+.*

Therefore above lemmmas with Theorem 3.1 show the following isomorphisms: for any $r \in \mathbf{R}_+$

$$\begin{aligned} \mathbf{R}_+ \times (G^{-1}(r) \cap \mathbf{R}^2) &\simeq E(3) \cap \mathbf{R}^2 \\ \mathbf{R}_+ \times (G^{-1}(r) \cap E^+) &\simeq E^+ \end{aligned}$$

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