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HAUSDORFF DIMENSION OF CANTOR SET OF INFINITELY RENORMALIZABLE DISK-DIFFEOMORPHISMS

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ABSTRACT. The notion of infinitely renormalizable diffeomorphisms is given. It is discussed that the regularity of such diffeomorphisms is closely related to Hausdorff dimension of certain Cantor sets, and checked moreover that no such diffeomorphism with C^3 is able to construct under our definition.

In two-dimensional dynamics we consider some questions for the dynamics of infinitely renormalizable diffeomorphisms, which is studied in [B-G-L-T], inspired by Denjoy's theorem and Falconer's Book[F].

A construction of an infinitely renormalizable diffeomorphism is found in [B-F] and [F-Y]. By making use of the construction they gave answers for a problem of whether there exist Kupka-Smale diffeomorphisms of the sphere with neither sinks or sources, rasied by Smale[S]. For $C^{1+\epsilon}$-infinitely renormalizable diffeomorphisms the dynamics of the Cantor set founded by them was characterized in [B-G-L-T]. It was proved [S-W] that a homeomorphism of a Cantor set with some conditions is topologically conjugate with the restriction to a Cantor set of an example constructed in [B-F].

Before describing our result taken aim at this paper for infinitely renormalizable diffeomorphisms we define an orientation preserving diffeomorphism which is called *infinitely renormalizable*.

Let D denote the unit disk centered at the origin of $\mathbb{R}^2$ and ℓ be an arbitraly integer more than one. Take an infinite sequence $\{p_i^n | n \geq 1,\ 1 \leq i \leq \ell^n\}$ of points belonging to D.

First we define a sequence $\{D_i^n | n \geq 1,\ 1 \leq i \leq \ell^n\}$ of subdisks of D and a sequence $\{r_n | n \geq 1\}$ satisfying the conditions:

(D1) $0 < r_1 < 1/2$ and $0 < r_{n+1} < r_n/2$ for $n \geq 1$,
(D2) for fixed $n \geq 1$ and all i with $1 \leq i \leq \ell^n$, D_i^n is a disk centered at p_i^n with radius r_n,
(D3) $D_i^n \cap D_j^n = \varnothing$ for $i \neq j$ and $n \geq 1$,
(D4) $\bigcup_{j=0}^{\ell-1} D_{i+j\cdot\ell^n}^{n+1} \subset D_i^n$ for $n \geq 1$ and $1 \leq i \leq \ell^n$.

Next we define a sequence $\{f_n\}$ of diffeomorphisms satisfying the conditions: for every $n \geq 1$

(F1) $f_n : D \to D$ is a C^∞-diffeomorphism,

(F2) $D_i^n = f_n^{i-1}(D_1^n)$ for $1 \leq i \leq \ell^n$ and $f_n^{\ell^n}(D_1^n) = D_1^n$,
(F3) f_{n+1} and f_n agree on the complement of $\bigcup_{i=1}^{\ell^n} D_i^n$, i.e.

$$f_{n+1}|_{(\bigcup_{i=1}^{\ell^n} D_i^n)^c} = f_n|_{(\bigcup_{i=1}^{\ell^n} D_i^n)^c}$$

(here E^c denotes the complement of E),
(F4) For $1 \leq i \leq \ell^n$ $f_n|_{D_i^n}$ is a composition of a rotation and translation such that $f_n(p_{\ell^n}^n) = p_1^n$ and $f_n(p_i^n) = p_{i+1}^n$ for $1 \leq i \leq \ell^n - 1$.

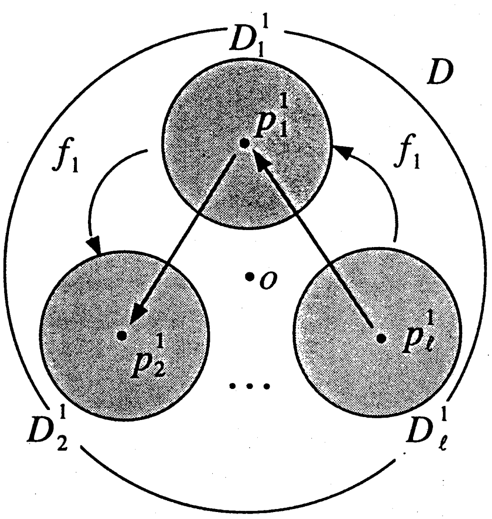

Figure 1.

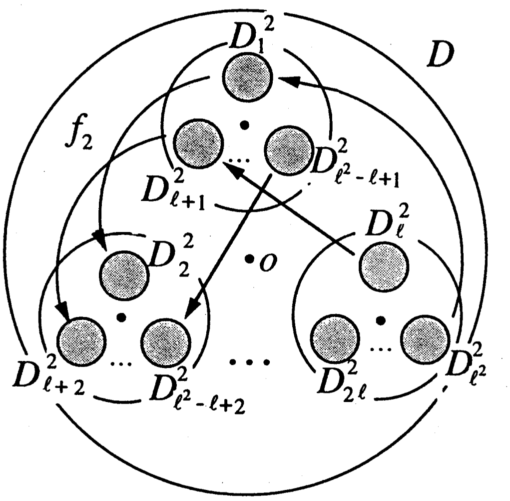

Figure 2.

Under the above notations we need the following assumption to obtain our desired diffeomorphism.

(A) There exists a constant $C > 0$ such that for $n \geq 1$

$$\sup \left\{ \max \left\{ \|D_x f_{n+1} - I\|, \ \|D_x f_{n+1}^{-1} - I\| \right\} \middle| x \in \bigcup_{i=1}^{\ell^n} D_i^n \right\} \leq C/\ell^n.$$

The assumption (A) implies that $\{f_n\}$ is a C^1-Cauchy sequence. Thus we have a limit $f : D \to D$ which is called an *infinitely renormalizable diffeomorphism*.

Obviously $K = \bigcap_{n \geq 1} \bigcup_{i=1}^{\ell^n} D_i^n$ is a f-invariant Cantor set in D. Our results which will be made precise later depend heavily on the properties of the set K.

Remark 1. The topological entropy of $f|_K$ is zero, i.e. $h(f|_K) = 0$.

Indeed, for $m > 0$ denote as $r_m(\varepsilon, E)$ the smallest cardinarity of the finite subset $\{y_1, \cdots, y_k\} \subset E$ satisfying that for $x \in E$ there is y_i $(1 \leq i \leq k)$ such that

$$\max\{|f^j(x) - f^j(y_i)| : 0 \leq j \leq m-1\} \leq \varepsilon.$$

By (D1) - (D4), for $\varepsilon > 0$ we can choose $N > 0$ such that $|D_i^N| = 2r_N < \varepsilon$ for $1 \le i \le \ell^N$. Thus

$$r_m(\varepsilon, K) \le r_m(\varepsilon, \bigcup_{i=1}^{\ell^N} D_i^N) \le \ell^N$$

for $m > 0$. Since the topological entropy of $f|_K$ is given by

$$h(f|_K) = \lim_{\varepsilon \to 0} \overline{\lim}_{m\to\infty}(1/m)\log r_m(\varepsilon, K),$$

we have $h(f|_K) \le \lim_{\varepsilon\to 0} \overline{\lim}_{m\to\infty}(1/m)\log \ell^N = 0$. □

Remark 2. $f|_K$ is minimal, i.e. for any $x \in K$ the orbit $O(x) = \{f^n(x) : n \ge 0\}$ is dense in K.

Indeed, since $r_n \to 0$ $(n \to \infty)$, for $\varepsilon > 0$ there is $N > 0$ such that $2r_N < \varepsilon$. Since $K \subset \bigcup_{i=1}^{\ell^N} D_i^N$, for two points $x, y \in K$ we have $x \in D_{i_1}^N$ and $y \in D_{i_2}^N$ for some i_1 and i_2. Thus we have $|f^n(x) - y| < \varepsilon$ since $f^n(D_{i_1}^N) = D_{i_2}^N$ for some n (by (F2)). This implies the minimality of $f|_K$. □

For our definition we remark that the choice of subdisks D_i^n to n-th stage is ruled by ℓ^n number. Now we can describe one of our results as follows.

Theorem A. *Let $f : D \to D$ be an infinitely renormalizable diffeomorphism and K be the Cantor set constructed as above.*

If f is of $C^{1+\varepsilon}$ then $\varepsilon \le dim_H(K)$ where $dim_H(K)$ denotes the Hausdorff dimension of K. Moreover if f is of $C^{2+\varepsilon}$ then $1 + \varepsilon \le dim_H(K)$.

The converse of Theorem A will be proved for infinitely renormalizable diffeomorphisms constructed without the assumption (A). We shall describe it later on. The following lemma plays an important role to show Theorem A.

Lemma. *Let $\{D_i^n | n \ge 1,\ 1 \le i \le \ell^n\}$ be the subdisks satisfying (D1)-(D4) and $f : D \to D$ be the infinitely renormalizable diffeomorphism. Then there exist a constant $C_1 > 0$ and a sequence of points x_n in $\bigcup_{i=1}^{\ell^n} D_i^n$ satisfying*

$$||D_{x_n} f - I|| \ge C_1/\ell^n \quad (n \ge 1).$$

If we establish Lemma, then Theorem A is concluded as follows.

Proof of Theorem A. Let C_1 and $\{x_n\}$ be as in Lemma. Then, for $n > 0$ there is $1 \le i_n \le \ell^n$ such that $x_n \in D_{i_n}^n$. For $n > 0$ we can take $q_n \in \partial D_{i_n}^n$. We remark that $D_{q_n}^2 f = 0$ by (F4) and $D_{q_n} f = \mathrm{id}$ by (A). From the mean value theorem we have that $||D_{x_n} f - D_{q_n} f|| \le ||D_{y_n}^2 f|| \cdot |x_n - q_n|$ for some $y_n \in D_{i_n}^n$. Thus,

$$||D_{y_n}^2 f|| \ge ||D_{x_n} f - D_{q_n} f||/|x_n - q_n| \ge C_1/2\ell^n r_n. \qquad (*)$$

Since f is of $C^{2+\varepsilon}$, there is a constant $\overline{C} > 0$ such that $||D_x^2 f - D_y^2 f|| \le \overline{C}|x - y|^\varepsilon$ for $x, y \in D$, and so

$$\overline{C}(2r_n)^\varepsilon \ge \overline{C}|y_n - q_n|^\varepsilon \ge ||D_{y_n}^2 f - D_{q_n}^2 f|| \ge C_1/(2\ell^n r_n)$$

from which we have

$$r_n \geq (C_1/\overline{C}2^{1+\varepsilon}\ell^n)^{1/(1+\varepsilon)}.$$

For $n \geq 1$ and i with $1 \leq i \leq \ell^n$ let $\tilde{D}_i^n$ be a disk centered at p_i^n (which is the center of D_i^n) with radius $(C_1/\overline{C}2^{1+\varepsilon}\ell^n)^{1/(1+\varepsilon)}$. Then $K' = \bigcap_{n\geq 1}(\bigcup_{i=1}^{\ell^n} \tilde{D}_i^n)$ has the Hausdorff dimension which is calculated as

$$\dim_H(K') = -\log \ell / \log(1/\ell)^{1/(1+\varepsilon)} = 1 + \varepsilon.$$

(for the details see Remark 6 described later on) and therefore

$$\dim_H(K) \geq \dim_H(K') = 1 + \varepsilon.$$

□

Remark 3. Under our definition, no infinitely renormalizable diffeomorphism with the C^3 is able to constructed.

Indeed, suppose f is of C^3. Let $C_1, \{x_n\}$ be as in Lemma and let $\{i_n\}, \{y_n\}, \{q_n\}$ be as in the proof of Theorem A. Then we have (∗). Use the mean value theorem. Then, $||D^2_{y_n} f - D^2_{q_n} f|| \leq ||D^3_{z_n} f|| \cdot |y_n - q_n|$ for some $z_n \in D^n_{i_n}$, from which

$$||D^3_{z_n} f|| \geq ||D^2_{y_n} f - D^2_{q_n} f|| / |y_n - q_n| \geq ||D^2_{y_n} f||/2r_n \geq C_1/(4\ell^n r_n^2).$$

Take a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\lim_{j\to\infty} z_{n_j} = z \in K$. Since $D^3_z f = 0$, we have $\lim_{j\to\infty} ||D^3_{z_{n_j}} f|| = 0$, and thus

$$0 = \lim_{j\to\infty} ||D^3_{z_{n_j}} f|| \geq \lim_{j\to\infty} C_1/(4\ell^{n_j} r_{n_j}^2).$$

Since $\ell^{n_{j_0}} r_{n_{j_0}}^2 > 1$ for some $j_0 > 0$, we have $r_{n_{j_0}} > 1/\sqrt{\ell^{n_{j_0}}}$. Let λ denote Lebesgue measure of $\mathbb{R}^2$. Then we have

$$\pi = \lambda(D) > \lambda\left(\cup_k D_k^{n_{j_0}}\right) = \ell^{n_{j_0}} \pi r_{n_{j_0}}^2 > \pi,$$

thus contradicting. Remark 3 was proved. □

A C^∞-Kupka-Smale diffeomorphism of the sphere with neither sinks or sources was constructed in [G-S-T]. The main step of it is to obtain an embedding of the 2-disk without using the technique of [B-F] and [F-Y]. Thus the method of [G-S-T] is justified by Remark 3.

Remark 4. If a C^1-diffeomorphism $f : D \to D$ constructed by Bowen-Franks is of $C^{1+\varepsilon}$, then $\varepsilon \leq \dim_H(K)$.

Indeed, the construction of an infinitely renormalizable diffeomorphism done in [B-F] does not require the assumption (A). Therefore Theorem A concludes Remark 4. □

Proof of Lemma. Since $f^{\ell^n}(D_1^n) = f_n^{\ell^n}(D_1^n) = D_1^n$ for $n \geq 1$ (by (F2)), we can find $\overline{x}_n \in D_1^n$ such that

$$||D_{\overline{x}_n} f^{\ell^n} - I|| \geq 1/\ell.$$

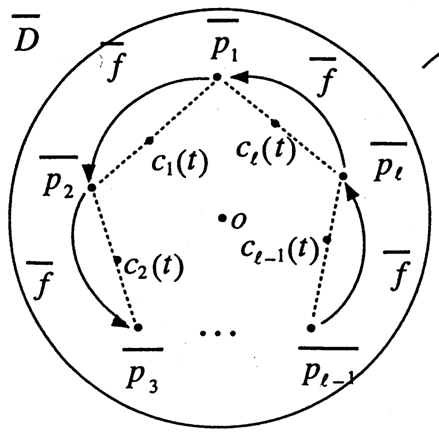

Figure 3.

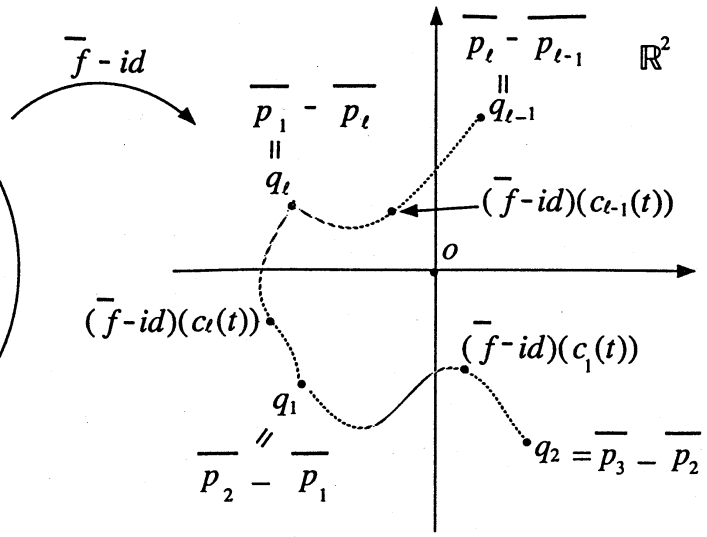

Figure 4.

Indeed, from Brouwer's theorem it follows that $f^{\ell^{n+1}}(\overline{p}^n) = \overline{p}^n$ for some $\overline{p}^n \in D_1^{n+1}$. Remark that $\overline{p}^n$ is a periodic point with period ℓ of $f^{\ell^n}|_{D_1^n}$.

For simplicity we fix $n \geq 1$ and write $\overline{D} = D_1^n$, $\overline{f} = f^{\ell^n}|_{D_1^n}$, $\overline{p}_1 = \overline{p}^n$ and $\overline{p}_i = \overline{f}^{i-1}(\overline{p}_1)$ $(2 \leq i \leq \ell)$.

Put $q_i = \overline{p}_{(i+1)\mathrm{mod}\ell} - \overline{p}_i$ for $1 \leq i \leq \ell$. Then each of q_i is non-zero and can be calculated as

$$|q_i| = \int_0^1 |\dot{c}_i(t)| dt \quad (1 \leq i \leq \ell)$$

where $c_i(t) = (1-t)\overline{p}_i + t\overline{p}_{(i+1)\mathrm{mod}\ell}$. Since $q_i = \overline{f}(\overline{p}_i) - \overline{p}_i = \overline{p}_{(i+1)\mathrm{mod}\ell} - \overline{p}_i$ for $1 \leq i \leq \ell$, obviously

$$\sum_{i=1}^{\ell} q_i = (\overline{p}_2 - \overline{p}_1) + (\overline{p}_3 - \overline{p}_2) + \cdots + (\overline{p}_1 - \overline{p}_\ell) = (0,0).$$

Remark that $(\overline{f} - \mathrm{id})(c_i(0)) = q_i$ and $(\overline{f} - \mathrm{id})(c_i(1)) = q_{(i+1)\mathrm{mod}\ell}$. Then we have that for $1 \leq i \leq \ell$

$$\begin{aligned} |q_i - q_{(i+1)\mathrm{mod}\ell}| &\leq \int_0^1 |(D_{c_i(t)}\overline{f} - I)\dot{c}_i(t)| dt \\ &\leq \int_0^1 ||D_{c_i(t)}\overline{f} - I|| \cdot |\dot{c}_i(t)| dt \\ &= \sup_{t \in [0,1]} ||D_{c_i(t)}\overline{f} - I|| \cdot |q_i|, \end{aligned}$$

from which

$$\sup_{t \in [0,1]} ||D_{c_i(t)}\overline{f} - I|| \geq |q_i - q_{(i+1)\mathrm{mod}\ell}| / |q_i|.$$

Thus, to obtain the conclusion it suffices to show that there is $1 \le i \le \ell$ satisfying

$$\frac{|q_i - q_{(i+1)\mathrm{mod}\ell}|}{|q_i|} \ge 1/\ell. \tag{†}$$

To do so if (†) is false, and put $|q_{i_0}| = \max\{|q_i| \, |1 \le i \le \ell\}$, then we have that for $1 \le i \le \ell$

$$\begin{aligned}|q_i - q_{i_0}| &\le |q_1 - q_2| + \cdots + |q_\ell - q_1| \\ &< |q_1|/\ell + \cdots + |q_\ell|/\ell \\ &\le |q_{i_0}|.\end{aligned}$$

Let $\ell \cdot q_{i_0}$ denotes the sum of ℓ time of q_{i_0} (i.e. $\ell \cdot q_{i_0} = q_{i_0} + \cdots + q_{i_0}$). Since $\sum_{i=1}^{\ell} q_i = (0,0)$, we have

$$|\ell \cdot q_{i_0}| = \left|\sum_{i=1}^{\ell} q_i - \ell \cdot q_{i_0}\right| \le \sum_{i=1}^{\ell} |q_i - q_{i_0}| < |\ell \cdot q_{i_0}|,$$

thus contradicting.

Therefore, for $n \ge 1$ there is $\overline{x}_n \in D_1^n$ such that $||D_{\overline{x}_n} f^{\ell^n} - I|| \ge 1/\ell$.

We now are a position to show the lemma. Since $||D_y f|| \le 1 + C/\ell^n$ for $y \in \bigcup_{i=1}^{\ell^n} D_i^n$ (by the assumption (A)), by the choice of $\bar{x}_n \in D_1^n$ we have that for $n \ge 1$

$$\begin{aligned}1/\ell &\le ||D_{\bar{x}_n} f^{\ell^n} - I|| \le \sum_{i=0}^{\ell^n-1} ||D_{\bar{x}_n} f^{i+1} - D_{\bar{x}_n} f^i|| \\ &\le \sum_{i=0}^{\ell^n-1} ||D_{f^i(\bar{x}_n)} f - I|| \cdot ||D_{\bar{x}_n} f^i|| \\ &\le \sum_{i=0}^{\ell^n-1} ||D_{f^i(\bar{x}_n)} f - I|| \cdot \left\{\prod_{j=0}^{i-1} ||D_{f^j(\bar{x}_n)} f||\right\} \\ &\le \sum_{i=0}^{\ell^n-1} ||D_{f^i(\bar{x}_n)} f - I|| \cdot (1 + C/\ell^n)^i \\ &\qquad\qquad (\text{since } ||D_y f|| \le 1 + C/\ell^n \text{ for } y \in \bigcup_{i=1}^{\ell^n} D_i^n) \\ &\le e^c \sum_{i=0}^{\ell^n-1} ||D_{f^i(\bar{x}_n)} f - I||,\end{aligned}$$

from which we can find $0 \le i_1 \le \ell^n - 1$ such that

$$||D_{f^{i_1}(\bar{x}_n)} f - I|| \ge \frac{1}{e^c \ell^{n+1}}.$$

Putting $x_n = f^{i_1}(\bar{x}_n)$ and $C_1 = (e^c \ell)^{-1}$, we have the conclusion of the lemma. □

For a question of whether the converse of Theorem A is true, we can give an answer for infinitely renormalizable diffeomorphisms constructed concretely as follows.

Let D and ℓ be as before and take a finite sequence $\{p_i \mid 1 \leq i \leq \ell\}$ of points belonging to D. Define a sequence $\{D_i \mid 1 \leq i \leq \ell\}$ of subdisks included in D such that

(D'1) for i, D_i is a disk centered at p_i with radius $0 < r < 1/2$,
(D'2) $D_i \cap D_j = \varnothing$ for $i \neq j$,
(D'3) $\bigcup_{i=1}^{\ell} D_i \subset D$.

We consider an C^∞-isotopy $h_t : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

(H1) $h_0(x) = x$ for $x \in \mathbb{R}^2$ and $h_t(D) = D$,$h_t(D^c) = D^c$ for $t \in [0,1]$,
(H2) $h_1(D_i) = D_{i+1}$ for $1 \leq i \leq \ell - 1$ and $h_1(D_\ell) = D_1$,
(H3) for a fixed $\alpha > 0$ and $t \in [0,1]$, $h_t|_{D^c}$ is a rotation of the angle $t\alpha$ which is centered at the origin of $\mathbb{R}^2$,
(H4) fix $1 \leq i \leq \ell$, and for $t \in [0,1]$, $h_t|_{D_i}$ is a composition of a translation and a rotation of the angle $t\alpha/\ell$ which is centered at p_i.

By using the isotopy $h_t : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ we can define a C^∞ diffeomorphism $g(i,N) : D \to D$ by

$$g(i,N) = h_{i/N} \circ h_{(i-1)/N}{}^{-1} \quad (N \in \mathbb{N}, 1 \leq i \leq N).$$

Then it follows that

$$h_1 = g(N,N) \circ g(N-1,N) \circ \cdots g(1,N).$$

From now on we construct a sequence $\{D_i^n \mid n \geq 1,\ 1 \leq i \leq \ell^n\}$ of subdisks of D satisfying (D1)-(D4) and a sequence $\{f_n\}$ of C^∞-diffeomorphisms of D satisfying (F1)-(F4). These constructions are inductively done as follows.

First put $f_1 = h_1$. Obviously $f_1 : D \to D$ is a C^∞-diffeomorphism. We write $D_i^1 = D_i$ and $p_i^1 = p_i$ for $1 \leq i \leq \ell$. Then

$$D_i^1 = f_1^{i-1}(D_1^1)\ (1 \leq i \leq \ell), \quad f_1^\ell(D_1^1) = D_1^1.$$

By (H2) and (H4) we have that $f_1(p_i^1) = p_{i+1}^1$ $(1 \leq i \leq \ell - 1)$ and $f_1^\ell(p_1^1) = p_1^1$, which satisfy conditions (F2) and (F4).

Next put

$$f_2(x) = \begin{cases} f_1(x) & (x \notin \bigcup_{i=1}^{\ell} D_i^1) \\ r \cdot g(i,\ell)\left((x - p_i^1)/r\right) + p^1_{(i+1)\mathrm{mod}\ell} & (x \in D_i^1) \end{cases}$$

where

$$(i+1)\mathrm{mod}\ell = \begin{cases} i+1 & \text{for } 1 \leq i \leq \ell - 1 \\ 1 & \text{for } i = \ell. \end{cases}$$

Define a map $\beta_2 : D \to D_1^1$ by $\beta_2(x) = r \cdot x + p_1^1$. Then $D_1^2 = \beta_2(D_1^1)$ is a disk with radius r^2. Write

$$D_i^2 = f_2^{i-1}(D_1^2)\ (2 \leq i \leq \ell^2)$$

and denote as p_i^2 the center of D_i^2 for $1 \le i \le \ell^2$.

Obviously, $f_2 : D \to D$ is a C^∞-diffeomorphism and for $x \in D_i^1$ $(1 \le i \le \ell)$

$$\begin{aligned} f_2^\ell(x) &= r \cdot g(\ell,\ell) \circ g(\ell-1,\ell) \circ \cdots \circ g(1,\ell)\left\{(x-p_i^1)/r\right\} + p_i^1 \\ &= r \cdot h_1\left((x-p_i^1)/r\right) + p_i^1. \end{aligned}$$

Thus the sequence $\{D_i^2 \mid 1 \le i \le \ell^2\}$ satisfies the conditions (D1)-(D4), and the diffeomorphism $f_2 : D \to D$ satisfies the conditions (F1)-(F4).

Continuing this process we obtain a sequence $\{f_n \mid n \ge 1\}$ of C^∞-diffeomorphisms of D satisfying

$$f_{n+1}(x) = \begin{cases} f_n(x) & (x \notin \bigcup_{i=1}^{\ell^n} D_i^n) \\ r^n \cdot g(i,\ell^n)\left((x-p_i^n)/r^n\right) + p^n_{(i+1)\mathrm{mod}\ell^n} & (x \in D_i^n) \end{cases}$$

where

$$(i+1)\mathrm{mod}\ell^n = \begin{cases} i+1 & (1 \le i \le \ell^n - 1) \\ 1 & (i = \ell^n), \end{cases}$$

and a sequence $\{D_i^n \mid n \ge 1,\ 1 \le i \le \ell^n\}$ of subdisks centered at p_i^n with radius r^n satisfying

$$\begin{aligned} D_1^{n+1} &= \beta_{n+1}(D_1^n) \\ D_i^{n+1} &= f_{n+1}^{i-1}(D_1^{n+1}) \quad (2 \le i \le \ell^{n+1}) \end{aligned}$$

where $\beta_{n+1} : D_1^{n-1} \to D_1^n$ is defined by $\beta_{n+1}(x) = r \cdot (x - p_1^{n-1}) + p_1^n$.

The sequence $\{D_i^n \mid n \ge 1,\ 1 \le i \le \ell^n\}$ satisfies (D1)-(D4), and the sequence $\{f_n\}$ satisfies (F1)-(F4).

Remark 5. We can check that the sequence $\{f_n\}$ constructed concretely as above satisfies (A).

Indeed, by Lemma 3 of [F-Y], there is $C' > 0$ such that for $N \in \mathbb{N}$ and $x \in D$

$$\|D_x g(i,N) - I\| \le C'/N.$$

Thus, for $x \in D_i^n$ $(1 \le i \le \ell^n,\ n \ge 1)$

$$\|D_x f_{n+1} - I\| = \|D_{\{(x-p_i^n)/r^n\}} g(i,\ell^n) - I\| \le C'/\ell^n.$$

By the same way we can find $C'' > 0$ such that for $N \in \mathbb{N}$ and $x \in D$

$$\|D_x g(i,N)^{-1} - I\| \le C''/N,$$

and for $x \in D_i^n$ $(1 \le i \le \ell^n,\ n \ge 1)$

$$\|D_x f_{n+1}^{-1} - I\| \le C''/\ell^n,$$

which shows (A). $\square$

Therefore we see that the class of infinitely renormalizable diffeomorphisms constructed under the first definition contains that of diffeomorphisms done under the second definition.

By Remark 5 the sequence $\{f_n\}$ is a C^1-Cauchy sequence, and so its limit $f : D \to D$ is an infinite renormalizable diffeomorphism constructed by using isotopys.

The set $K_1 = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\ell^n} D_i^n$ is a Cantor set and f-invariant. For the diffeomorphism we have the following second result of this paper.

Theorem B. *Let $f : D \to D$ be an infinitely renormalizable diffeomorphism constructed by using isotopys and K_1 be the Cantor set related to f.*

(1) *Suppose that $0 < dim_H(K_1) \leq 1$. Then f is of $C^{1+\varepsilon}$ if and only if $\varepsilon \leq dim_H(K_1)$,*

(2) *Suppose that $1 < dim_H(K_1)$. Then f is of C^2, and moreover f is of $C^{2+\varepsilon}$ if and only if $\varepsilon + 1 \leq dim_H(K_1)$.*

Remark 6. The Hausdorff dimension of K_1 is $-\log \ell / \log r$.

For the proof put $s = -\log \ell / \log r$. Then Hausdorff measure of K_1 is calculated as

$$\begin{aligned}\mathcal{H}^s(K_1) &= \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} |U_i \cap K_1|^s : \{U_i\} \text{ is a } \delta\text{-cover of } D \right\} \right) \\ &\leq \overline{\lim}_{n\to\infty} \sum_{i=1}^{\ell^n} |K_1 \cap D_i^n|^s \leq 2^s \overline{\lim}_{n\to\infty} (\ell \cdot r^s)^n = 2^s.\end{aligned}$$

Thus we have $\dim_H(K_1) \leq s$. To see $\dim_H(K_1) \geq s$ define a sequence space $I = \prod_{i=1}^{\infty} \{1, \cdots, \ell\}$, and let $\{1, \cdots, \ell\}$ have the discrete topology. Obviously I is a compact metric space under the product topology. For $k \geq 1$ denote as

$$I_{i_1,\cdots,i_k} = \{(i_1, \cdots, i_k, q_{k+1}, \cdots) : 1 \leq q_j \leq \ell,\ j \geq k+1\}$$

a subset of I with initial terms $(i_1, \cdots, i_k)$, and define a set function μ of I by

$$\mu(I_{i_1,\cdots,i_k}) = r^{ks}.$$

Because of

$$\mu(I_{i_1,\cdots,i_k}) = r^{ks} = r^{ks}(\ell \cdot r^s) = \sum_{j=1}^{\ell} (r^{k+1})^s = \sum_{j=1}^{\ell} \mu(I_{i_1,\cdots,i_k,j})$$

shows that μ is a Borel probability measure of I.

For $k \geq 1$ we write $J_k = \{(i_1, \cdots, i_k) |\ 1 \leq i_j \leq \ell,\ 1 \leq j \leq k\}$ and for convenience

$$D_{i_1,i_2,\cdots,i_k} = D^k_{i_1+\ell\cdot i_2+\cdots+\ell^{k-1}\cdot i_k} \quad ((i_1, i_2, \cdots, i_k) \in J_k).$$

Since, for $\underline{i} = (i_1, \cdots, i_k)$, $\underline{i}' = (i'_1, \cdots, i'_k) \in J_k$

$$\begin{aligned}&D_{\underline{i},\ i_{k+1}} \subset D_{\underline{i}} \\ &D_{\underline{i}} \cap D_{\underline{i}'} = \varnothing \text{ if } \underline{i} \neq \underline{i}' \\ &\cup_j D_j^k = \cup_{J_k} D_{\underline{i}}\end{aligned}$$

(the notation $\underline{i}, i_{k+1}$ means that $\underline{i}, i_{k+1} = (i_1, \cdots, i_k, i_{k+1})$), we have that

$$\{D_i^k | 1 \le i \le \ell^k\} = \{D_{i_1,\cdots,i_k} | (i_1, \cdots, i_k) \in J_k\}.$$

For $(i_1, i_2, \cdots) \in I$ there exists a unique point $x_{i_1,i_2,\cdots} \in K_1$ such that $x_{i_1,i_2,\cdots} = \cap_{k=1}^{\infty} D_{i_1,\cdots,i_k}$. Thus, by $h((i_1, i_2, \cdots)) = x_{i_1,i_2,\cdots}$ a continuous bijection $h : I \to K_1$ is defined, and thus for any ball B

$$\begin{aligned} \mu(h^{-1}B) &= \mu(h^{-1}(B \cap K_1)) \\ &= \mu(\{(i_1, i_2, \cdots) | x_{i_1,i_2,\cdots} \in K_1 \cap B\}) \end{aligned} \qquad (\dagger\dagger)$$

Let B be an ball of radius $u < 1$ and let $k_0 = \min\{k \ge 0 | r^k \le u\}$. Then we have

$$r \cdot u \le \text{radius of } D_{i_1,\cdots,i_{k_0}} \le u,$$

and the cardinarity of

$$Q_1 = \{(i_1, \cdots, i_{k_0}) \in J_{k_0} | D_{i_1,\cdots,i_{k_0}} \cap B \ne \varnothing\}$$

is equal to 9/4 (see Lemma 9.2 of [F]). Thus,

$$\begin{aligned} \mu(h^{-1}B) &\le \mu(\bigcup_{Q_1} I_{i_1,\cdots,i_{k_0}}) \quad (\text{from } (\dagger\dagger)) \\ &\le \sum_{Q_1} r^{k_0 s} \le \sum_{Q_1} u^s \le (9/4)u^s, \end{aligned}$$

from which we have $\mathcal{H}^s(K_1) > 0$ (see Mass distribution principle [F]), and therefore $\dim_H(K_1) \ge s$. □

Remark 7. The C^2 diffeomorphism $f : D \to D$ constructed in [F-Y] belongs to the class of infinitely renormalizable diffeomorphisms defined by using isotopys. Thus the example of Franks-Young satisfies Theorem B.

Proof of Theorem B. Proof of (1) is very similar to that of (2). Thus we give the proof of (2) and will be omit it of (1).

By Remark 6 we know that $\dim_H(K_1) = -\log \ell / \log r$, and so write

$$s = -\log \ell / \log r.$$

We claim that C^∞-diffeomorphisms $g(i, \ell^n)$ have the property that for $n \ge 1$ and $1 \le i \le \ell^n$ there is $\tilde{C} > 0$ satisfying

$$\begin{aligned} ||D_x^2 g(i, \ell^n)|| &\le \tilde{C}/\ell^n \\ ||D_x^3 g(i, \ell^n)|| &\le \tilde{C}/\ell^n \end{aligned} \qquad (\ddagger)$$

This is obtained by applying Lemma 3 of [F-Y]. From the construction of f it follows that for $x \in D_i^n \setminus (\bigcup_{j=1}^{\ell^{n+1}} D_j^{n+1})$ and $n \ge 1$, $1 \le i \le \ell^n$

$$f(x) = f_{n+1}(x) = r^n \cdot g(i, \ell^n)\left((x - p_i^n)/r^n\right) + p_{(i+1)\text{mod}\ell^n}^n$$

and thus

$$D_x^2 f = D_x^2 f_{n+1} = r^{-n} \cdot D^2_{\{(x-p_i^n)/r^n\}} g(i, \ell^n)$$
$$D_x^3 f = D_x^3 f_{n+1} = r^{-2n} \cdot D^3_{\{(x-p_i^n)/r^n\}} g(i, \ell^n).$$

We use (‡) and have then

$$||D_x^2 f|| \leq \tilde{C}/(r\ell)^n, \quad ||D_x^3 f|| \leq \tilde{C}/(r^2\ell)^n. \qquad (‡‡)$$

Therefore we can conclude that f is of C^2, since $r\ell > 1$.

Next we prove that if $\varepsilon + 1 \leq \dim_H(K_1)$ then f is of $C^{2+\varepsilon}$. To do so we divid into three parts the proof. Pick up points x, y from D.

Part (a): If $x, y \in K_1$, then we have $D_x^2 f = D_y^2 f = 0$. Obviously

$$||D_x^2 f - D_y^2 f|| = 0 \leq |x-y|^{s-1}.$$

Part (b): If there exist $n \geq 1$ and $1 \leq i \leq \ell^n$ such that $x, y \in D_i^n \setminus (\bigcup_{j=1}^{\ell^{n+2}} D_j^{n+2})$, by the mean value theorem

$$||D_x^2 f - D_y^2 f|| = ||D_x^2 f_{n+2} - D_y^2 f_{n+2}|| \leq \sup_{z \in D_i^n} ||D_z^3 f_{n+2}|| \cdot |x-y|.$$

Since $\dim_H(K_1) \leq 2$, we have $\ell r^2 \leq 1$ and so by (‡‡)

$$\sup_{z \in D_i^n} ||D_z^3 f_{n+2}|| \leq \tilde{C}/(r^2\ell)^{n+1},$$

from which

$$\begin{aligned} ||D_x^2 f - D_y^2 f|| &\leq \left(\tilde{C}/(r^2\ell)^{n+1}\right) |x-y|^{2-s} \cdot |x-y|^{s-1} \\ &\leq \left(\tilde{C}/r^2\ell\right) \frac{1}{(r^s\ell)^n} \frac{|x-y|^{2-s}}{(r^n)^{2-s}} |x-y|^{s-1}. \end{aligned}$$

Since $r^s\ell = 1$ and $|x-y| \leq r^n$, we have $\frac{1}{(r^s\ell)^n} \frac{|x-y|^{2-s}}{(r^n)^{2-s}} \leq 1$ and therefore

$$||D_x^2 f - D_y^2 f|| \leq \left(\tilde{C}/r^2\ell\right) |x-y|^{s-1}. \qquad (‡‡‡)$$

Part (c):When the points x, y do not satisfy (a) and (b), we have $x \notin K_1$ or $x \notin K_1$. Define the integer

$$n_0 = \min\left\{ n > 0 \,\middle|\, x \in \bigcup_{i=1}^{\ell^n} D_i^n \setminus (\bigcup_{j=1}^{\ell^{n+1}} D_j^{n+1}), \text{ or } y \in \bigcup_{i=1}^{\ell^n} D_i^n \setminus (\bigcup_{j=1}^{\ell^{n+1}} D_j^{n+1}) \right\},$$

and for the case when $x \in \bigcup_{i=1}^{\ell^{n_0}} D_i^{n_0} \setminus (\bigcup_{j=1}^{\ell^{n_0+1}} D_j^{n_0+1})$ it suffices to show (‡‡‡). In fact it follows that $x \in D_{i_0}^{n_0}$ for some $1 \leq i_0 \leq \ell^{n_0}$. The point y satisfies one of the two cases

(C-1) $y \in D_{i_1}^{n_0}$ for some $1 \leq i_1 \leq \ell^{n_0}$ with $i_1 \neq i_0$,

(C-2) $y \in D_{i_0}^{n_0} \cap (\bigcup_{j=1}^{\ell^{n_0+2}} D_j^{n_0+2})$.

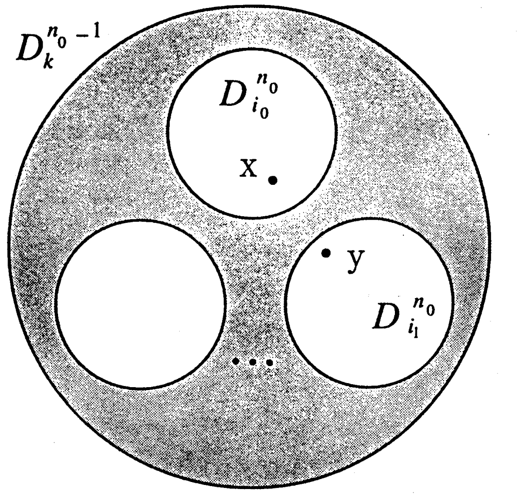

Figure 5 (Case (C-1)).

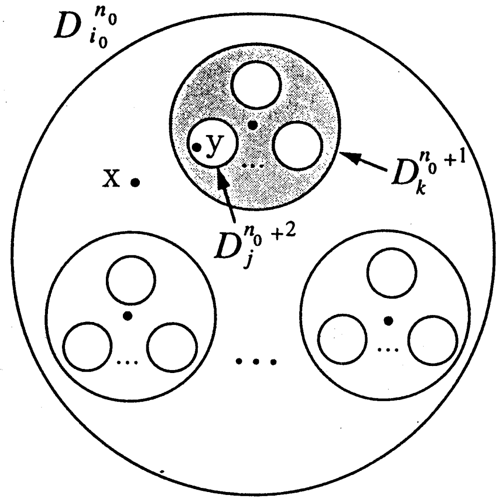

Figure 6 (Case (C-2)).

Let $h_t : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the C^∞-isotopy and $\{D_i \mid 1 \le i \le \ell\}$ be the subdisks appeared in the construction of f. We define

$$\delta = \min_{t \in [0,1]} \{\min \{d(h_t(D_i), h_t(D_j)),\ d(D^c, h_t(D_i)) \mid 1 \le i \ne j \le \ell\}\}.$$

Obviously, $\delta > 0$. The shadow parts of Figures 5 and 6 is a copy shrinking the shadow part of the following Figure.

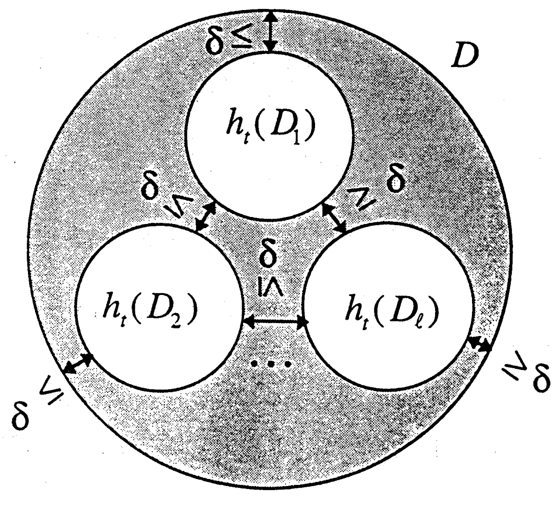

Figure 7.

The Figures 5 and 6 made r^{n_0-1} and r^{n_0+1} times as large respectively. Thus we have

$$\delta r^{n_0+1} \le |x-y|.$$

In any case of (C-1) and (C-2) we have by using (‡‡) that

$$\begin{aligned}||D_x^2 f - D_y^2 f|| &\le ||D_x^2 f|| + ||D_y^2 f|| \\ &\le 2\tilde{C}/(r\ell)^{n_0} \le \frac{2}{r^{s-1}} \frac{\tilde{C}}{(r^s \ell)^{n_0}} (r^{n_0+1})^{s-1} \\ &\le \frac{2\tilde{C}}{(r\delta)^{s-1}} |x-y|^{s-1}.\end{aligned}$$

Therefore we conclude that f is of $C^{2+\varepsilon}$ when $\varepsilon \le s-1 = \dim_H(K_1) - 1$. The part of "only if" was proved. The proof of "if part" is clear by Theorem A. □

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