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Kyoto University
Existence of Entire Solutions for Superlinear Elliptic Problems in $R^N$

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1. Introduction. In this talk, we are concerned with positive solutions of the following problem:

\[(P) \begin{cases} -\Delta u + u = g(x, u), & u > 0, \quad \text{in } R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}\]

where $f : R^N \to R$ and $g : \Omega \times R \to R$ is continuous with $g(x, 0) = 0$ for $x \in \Omega$. In the last decade, the existence and the properties of the solutions of problem $(P)$ has been studied by many authors. Recently, the existence of positive solutions of semilinear elliptic problem

\[(P_Q) \begin{cases} -\Delta u + u = Q(x) |u|^{p-1} u, & x \in R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}\]

has been studied by several authors, where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $Q(x)$ is positive bounded continuous function. If the function $Q(x)$ is a radial function, the existence of infinity many solutions of problem $(P_Q)$ can be shown by restricting our attention to the radial functions(cf. [1]). In case that $Q(x)$ is nonradial, we encounter a difficulty caused by lack of compact embedding of Sobolev type. In [6,7], P.L. Lions presented a method, called concentrate compactness method, which enable us to solve problems with lack of compactness, and established the following result: Assume that

\[\lim_{|x| \to \infty} Q(x) = \overline{Q}(> 0) \quad \text{and} \quad Q(x) \geq \overline{Q} \quad \text{on } R^N,\]
then problem \((P_Q)\) has a positive solution. This result is based on the observation that the ground state level \(c_Q\) of the functional

\[
I_Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(x)u^{p+1}dx
\]

is lower than the ground state level \(c_{\overline{Q}}\) of functional \(I_{\overline{Q}}\). We can apply the concentrate compactness method problem \((P)\) to the problem in case that \(g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies \(\lim_{|x| \rightarrow \infty} g(x, t) = t^p\) and the least critical level \(c_1\) of the functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \int_{\mathbb{R}^N} \int_0^{u(x)} g(x, t) dt dx,
\]

\(u \in H^1(\mathbb{R}^N)\), is lower than that of

\[
I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1} \int u^{p+1}dx.
\]

Under additional conditions on \(g\), the existence of positive solutions \((P)\) was established by Ding & Ni[4] and Stuart[10]. Recently, Cao[2] proved the existence of positive solution of \((P_Q)\) for the case that \(c_Q \leq c_{\overline{Q}}\) under the hypothesis that \(\lim_{||x||\rightarrow \infty} Q(x) = \overline{Q}\) and \(Q(x) \geq 2^{(1-p)/2} \overline{Q}\) on \(\mathbb{R}^N\). In case that \(c_Q = c_{\overline{Q}}\), we encounter a difficulty, because we can not apply the concentrate compactness method directly. On the other hand, in case that \(g\) is not given by the form \(Q(x)t^p\), we have to overcome another difficulty: that is, we can not use the Lagrange’s method of indeterminate coefficients. In the problem \((P_Q)\), we find a solution \(u\) of minimizing problem

\[
\inf\{I_Q(u) : u \in V_\lambda\},
\]

\(V_\lambda = \{u \in H^1(\mathbb{R}^N), u > 0, \int_{\mathbb{R}^N} Q(x)u^{p+1}dx = 1\}\)

Then \(cu\) is a solution of \((P_Q)\) for some \(c > 0\). The Lagrange’s method does not work if \(g\) is not the form \(Q(x)t^p\). Our approach enable us to treat the problem \((P)\) with \(g\) satisfying that \(g(0) = 0\) and \(g(t) \rightarrow t^p\) as \(t \rightarrow \infty\). We also consider the nonhomoginous case:

\[
(P_f) \quad \begin{cases} 
- \Delta u + u = |u|^{p-1} u + f, & x \in \mathbb{R}^N \\
 u \in H^1(\mathbb{R}^N), & N \geq 3
\end{cases}
\]
where \( p > 1 \) for \( N = 1 \) and \( 1 < p < (N + 2)/(N - 2) \) for \( N \geq 3 \).

The nonhomogeneous problem \((P_f)\) was studied by Zhu[12]. In [12], the existence of at least two solutions of \((P)\) was proved for nonnegative functions \( f \in L^2(R^N) \) with a small \( L^2\)–norm and a exponential decay

\[
f(x) \leq C \exp\{- (1 + \epsilon) |x|\}, \quad \text{for } x \in R^N.
\]

In the present paper, we consider multiple existence of solutions of \((P)\) for nonnegative functions \( f \in L^q(R^N) \), where \( q = (p + 1)/p \). Our result does not require that \( f \in L^\infty(R^N) \) or any condition for the decay of \( f \) at infinity.

In this talk, we show an approach for problems \((P)\) and \((P_f)\) based on arguments using singular homology theory. Throughout this paper, we denote by \( | \cdot |_q \) the norm of \( L^q(R^N) \). We impose the following conditions on the continuous mapping \( g : R^N \times R \to R \):

\begin{enumerate}[label=(g\arabic*)]
  \item There exists a positive number \( d < 1 \) such that
    \[ -dt + (1 - d)t^p \leq g(x, t) \leq dt + (1 + d)t^p \]
    for all \( (x, t) \in R^N \times [0, \infty) \);
  \item there exists a positive number \( C \) such that
    \[ |g_t(x, 0)| < 1 \quad \text{and} \quad 0 < t^2 g_{tt}(x, t) < C(1 + t^p) \]
    for all \( (x, t) \in R^N \times [0, \infty) \);
  \item \[
    \lim_{|x| \to \infty} g(x, t) = |t|^{p-1} t
  \]
    uniformly on bounded intervals in \([0, \infty)\),
\end{enumerate}

where \( 1 < p \) for \( N = 2 \) and \( 1 < p < (N + 2)/(N - 2) \) for \( N \geq 3 \), and \( g_t(\cdot, \cdot) \) stands for the derivative of \( g \) with respect to the second variable.

We can now state our main results.

**Theorem 1.** Suppose that \((g2)\) and \((g3)\) holds. Then there exists \( d_0 > 0 \) such that if \((g1)\) holds with \( d < d_0 \), then problem \((P)\) has a positive solution.

For problem \((P_f)\), we have

**Theorem 2.** There exists a positive number \( C \) such that for each \( f \in L^q(R^N) \), with \( f \geq 0 \) and \( |f|_q < C \), problem \((P_f)\) possesses at least two solutions.
2. Preliminaries. We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then $H$ is a Hilbert space with norm
\[
\|u\| = (\int_{R^N} (|\nabla u|^2 + |u|^2)dx)^{1/2}.
\]
The norm of the dual space $H^{-1}(R^N)$ of $H$ is also denoted by $\|\cdot\|$. $B_r$ stands for the open ball centered at 0 with radius $r$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $|\cdot|_r$. For simplicity, we write $|\cdot|_*$ instead of $|\cdot|_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by $C_p$ the minimal constant satisfying
\[
|u|_* \leq C_p \|u\| \quad \text{for} \quad u \in H. \tag{2.1}
\]
It is easy to check that critical points of $I$ are solutions of (P). It is also obvious that nonzero critical points of $I^\infty$ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional $F$ on $H$ and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put
\[
M = \{u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} ug(x, u)dx\},
\]
\[
M^\infty = \{u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} u^{p+1}dx\}.
\]
For the proof of the following two propositions are crucial:

Proposition 2.1. There exists positive number $d_0 < \tilde{d}_0$ and $\epsilon_0$ satisfying that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$,
\[
H_*(I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon}) = H_*(I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon})
\]
where $H_*(A, B)$ denotes the singular homology group for a pair $(A, B)$ of topological spaces(cf. Spanier[8]).

Proposition 2.2. For each positive number $\epsilon < \epsilon_0$,
\[
H_q(I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon}) = \begin{cases} 
2 & \text{if } q = 0, \\
0 & \text{if } q \neq 0.
\end{cases}
\]
Here we give a proof for Proposition 2.2.

We set
\[ T_{u_{\infty}}(M^{\infty}) = \{ \lim_{t \to 0} (c(t) - u_{\infty})/t : c \in C^{1}((-1, 1); M^{\infty}) \text{ with } c(0) = u_{\infty}, \} \]

\[ C = C_{-} \cup C_{+} = \{ -\tau_{x}u_{\infty} : x \in R^{N} \} \cup \{ \tau_{x}u_{\infty} : x \in R^{N} \} \]

and

\[ T_{u_{\infty}}(C) = \{ \lim_{t \to 0} (u_{\infty}(\cdot + tx) - u_{\infty}(\cdot))/t : x \in R^{N} \}. \]

It follows from the definition of \( f^{\infty} \) that the codimension of \( T_{u_{\infty}}(M^{\infty}) \) in \( H \) is one. It is also obvious that \( \dim T_{u_{\infty}}(C) = N \). We denote by \( \tilde{H} \) the subspace such that \( H = \tilde{H} \oplus T_{u_{\infty}}(C) \). For each \( r > 0 \), we set \( B_{r}^{0} = B_{r} \cap H \).

Here we consider the linearized equation

\[(L) \quad -\Delta u + u - h(x)u = \mu u, \quad u \in H, \mu \in R,\]

where \( h(x) = p \mid u_{\infty}(x) \mid^{p-1} \) for \( x \in R^{N} \). Since \( -\Delta \) is positive definite and \( h(x)I \) is compact, we find by Freidrich's theory that the negative spectrums of \( A = -\Delta - h(x)I \) are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of \( L = -\Delta + I - h(x)I \) is finite dimensional. Then there exists \( c_{0} > 0 \) and a decomposition \( H = H_{-} \oplus H_{0} \oplus H_{+} \) such that \( H_{0} = \ker(L) \) and \( L \) is positive(negative) definite on \( H_{+}(H_{-}) \) with

\[ \langle Lv, v \rangle \geq c_{0} \| v \|^{2} (\leq -c_{0} \| v \|^{2}) \quad \text{for } v \in H_{+}(H_{-}). \]

Since each \( u \in C \) is a solution of problem \((P_{\infty})\), we can see that \( T_{u_{\infty}}(C) \subset H_{0} \).

**Lemma 2.3.** \( \dim H_{-} = 1 \).

**Proof.** Since \( I^{\infty} \) attains its minimal on \( M^{\infty} \) at \( u_{\infty} \), we have that \( T_{u_{\infty}}(M^{\infty}) \subset H_{+} \oplus H_{0} \). Then since the codimension of \( M^{\infty} \) is one, we find that \( \dim H_{-} \leq 1 \). On the other hand, we have

\[ \langle Lu_{\infty}, u_{\infty} \rangle = \int_{R^{N}} (| \nabla u_{\infty} |^{2} + | u_{\infty} |^{2} - p | u_{\infty} |^{p+1})dx \]

\[ \leq \int_{R^{N}} (| \nabla u_{\infty} |^{2} + | u_{\infty} |^{2} - | u_{\infty} |^{p+1})dx = 0. \]
Then we have that \( \dim H_{-} \geq 1 \). This completes the proof.

In the following we denote by \( \varphi \) an element of \( H_{-} \) with \( \| \varphi \| = 1 \). Here we note that since \( h \in C^{\infty}(R^{N}) \), each solution \( u \) of (L) is in \( C^{1}(R^{N}) \). It then follows that if \( u \) has the form

\[
u(r, \theta) = \psi(r)\xi(\theta_1, \cdots, \theta_{n-1}), \quad \text{with} \ \xi \neq \text{const.,}
\]

in spherical coordinate, \( \psi \) satisfies that \( \psi(0) = 0 \).

We denote by \( H_{r} \) the set of all radial functions in \( H \) and by \( (L_{r}) \) the problem (L) restricted to \( H_{r} \). Then, in spherical coordinates, the problem \( (L_{r}) \) with \( \mu > 0 \) is reduced to

\[
\psi''(r) + \frac{n-1}{r} \psi'(r) + (h-1)\psi = -\mu \psi(r), \quad r > 0, \psi \in C_{r}, \tag{2.3}
\]

\[
\frac{d\psi(r)}{dr}(0) = 0, \tag{2.4}
\]

where \( C_{r} = \{ \psi \in C[0, \infty) : \lim_{r \to \infty} \psi(r) = 0 \} \).

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

\[
\psi''(r) + \frac{n-1}{r} \psi'(r) + \left((h-1) - \frac{\alpha_k}{r^2}\right)\psi(r) = -\mu \psi(r), \quad r > 0, \psi \in \mathcal{H}_{2R} \tag{2.6}
\]

where \( \alpha_k = k(k + n - 1), \ k = 1, 2, \cdots \). Note that \( \alpha_k \) are the eigenvalues of Laplacian \(-\Delta\) on \( S^{n-1} \), the unit sphere, and the dimension of the eigenspace \( S_k \) associate with \( \alpha_k \) is

\[
\rho_k = \binom{k + n - 2}{k} \frac{n + 2k - 2}{n + k - 2}.
\]

That is there exists smooth functions \( \{ \varphi_{k,i} : i = 1, \cdots, \rho_k \} \) defined on \( S^{n-1} \) such that \( S_k = \text{span}\{ \varphi_{k,1}, \cdots, \varphi_{k,\rho_k} \} \), and the functions \( u = \psi(r)\varphi_{k,i}(\theta) \) are the solutions of (L).

**Lemma 2.4.** \( \dim H_{0} \leq N + 1 \).

**Proof.** Since \( \dim H_{-} = 1 \) and \( u_{\infty} \in H_{r} \), we have by (2.2) that the problems (2.3), (2.4) has exactly one negative eigenvalue. We also note
that each nonpositive eigenvalue $\mu$ of problems (2.3), (2.4) is simple. Then the dimension of $H_{0,r} = H_0 \cap H_r$ is at most one.

We next consider nonradial cases. That is we will see that the eigenspace of the problem (2.5) with $\mu = 0$ is $N$-dimensional space. Recalling that $\nabla I(v) = 0$ on $C$, we can see that

$$-\Delta v + v - h(x)v = 0 \quad \text{for all } v \in T_{u_{\infty}}(C). \quad (2.7)$$

That is $T_{u_{\infty}}(C) \subset H_0$. Since $\dim T_{u_{\infty}}(C) = N$, we have that $\dim H_0 \geq N$. On the other hand, since $u_{\infty}$ satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + p |u_{\infty}|^{p-1} u(r) = 0, \quad (2.8)$$

we find that $v(r) = u'_{\infty}$ satisfies

$$v''(r) + \frac{n-1}{r}v'(r) + ((h(x) - 1) - \frac{\alpha_1}{r^2})v(r) = 0.$$

Then we find that the $N$-dimensional space $\bar{C} = \text{span}\{v(r)\varphi_{1,i} : i = 1, \cdots, n-1\}$ is a subspace of solution set of (L) with $\mu = 0$. We claim that there exists no nonradial solution of (L) with $\mu = 0$ which is not contained in $\bar{C}$. Suppose contrary, there exists a nonradial solution $z$ of (L) with $\mu = 0$ such that $z \perp \bar{C}$. Then there exists $\psi \in C_r$ such that

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{\alpha_k}{r^2})\psi(r) = 0$$

for some $k > 1$ and $z = \psi(r)\varphi_{k,i}$ are solutions of (L) with $\mu = 0$. The equality above can be rewritten as

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{(\alpha_k - \alpha_1)}{r^2})\psi(r) - \frac{\alpha_1}{r^2}\psi(r) = 0.$$

Then $u = \psi(r)\varphi_{1,1}$ is a solution of problem

$$-\Delta u + u - h(x)u = \frac{(\alpha_1 - \alpha_k)}{r^2} u.$$

It then follows that

$$< -\Delta u + u - h(x)u, u > > 0. \quad (2.9)$$
Since $u$ is orthogonal to $\varphi$, we obtain from (2.9) that $\dim H_\leq 2$. This is a contradiciton. Thus we obtain that $H_0 = T_{u_0}(C) \oplus H_{0,r}$ and then $\dim H_0 \leq N + 1$.

Here we recall that $H$ has a decomposition $H = \widetilde{H} \oplus T_{u_\infty}(C)$ and then $H = \tau_x \widetilde{H} \oplus \tau_x T_{u_\infty}(C)$ for each $x \in R^N$. Then since $C_\pm$ are smooth $N$–manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B_{r_0}^0) \cap \tau_y(u_\infty + B_{r_0}^0) = \phi$$

(2.10) for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty |_{u_\infty + \widetilde{H}}$ of $I^\infty$ on $u_\infty + \widetilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces $H_1, H_{2,1}, H_{2,2}$ of $\widetilde{H}$, a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B_{r_1}^0), R)$ and a homeomorphism $\psi : u_\infty + B_{r_1}^0 \to u_\infty + \widetilde{H}$ such that $\widetilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty |_{u_\infty + \widetilde{H}} (\psi(u)) = c - \|u_1\|^2 + \|u_{2,1}\|^2 + \beta(u_{2,2})$$

(2.11) for each $u \in u_\infty + B_{r_1}^0$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1$, $u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and $u_\infty$ is the minimal point of $I^\infty$ on $M$, we have by choosing $r_1$ sufficiently small that $\beta(t \varphi_2)$ is strictly increasing as $|t|$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\|\varphi_2\| = 1$.

Since $I^\infty$ is even, it is obvious that $I^\infty$ has the form (2.11) on $-(u_\infty + B_{r_1}^0)$. We also note that for each $x \in R^N$, (2.11) holds for each $u \in \tau_x(u_\infty + B_{r_0}^0)$ with $\psi$ replaced by $\tau_x \circ \psi$.

**Proof of Proposition 2.2.** By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I^\infty_{c+\epsilon}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_c, I^\infty_{c-\epsilon}), \text{ and}$$

$$H_q(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_{c-\epsilon}, I^\infty_{c-\epsilon}) \cong 0.$$ 

From the exactness of the singular homology groups ,

$$H_q(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_c \setminus C)$$

$$\to H_{q-1}(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \to \cdots$$
we find
\[ 0 \to H_q(I_c^\infty, I_c^\infty - \epsilon) \to H_q(I^\infty, I^\infty C) \to 0. \]
That is
\[ H_q(I_c^\infty, I_c^\infty - \epsilon) \cong H_q(I_c^\infty, I_c^\infty C). \]
Noting that \( \cup \{ \tau_x(\pm u_\infty + B_{r_1}^0) : x \in R^N \} \) are disjoint open neighborhoods of \( C_\pm \) respectively, and that \( I^\infty \) is invariant under the translations \( \tau_x \), we find from the excision property and (2.11) that
\[ H_*(I_c^\infty, I_c^\infty C) \cong H_*(I_c^\infty \cap (\bigcup_{i=\pm 1} \bigcup_{x} \tau_x(u_\infty + B_{r_1}^0) \setminus \{u_\infty\})) \]
\[ \oplus H_*(-u_\infty + B_{r_1}^1, (-u_\infty + B_{r_1}^1) \setminus \{u_\infty\}) \]
\[ \cong H_*(\{0, 1\}, \{0, 1\}) \oplus H_*(\{0, 1\}, \{0, 1\}). \]
This completes the proof.

3. Proof of Theorem 1. We next consider a triple \((U, K, \epsilon) \subset H \times H \times R^+\) satisfying the following conditions:

(1) \( U \cap (-U) = \emptyset; \)
(2) \( \{ \tau_xu_\infty : |x| \geq r \} \subset intK \) for some \( r > 0; \)
(3) \( cl(I_{c+\epsilon} \cap K) \subset int(I_{c+\epsilon} \cap U); \)
(4) \( H_{N-1}(I_{c+\epsilon} \cap U) = 1, \quad H_1(I_{c+\epsilon} \cap U) = 0; \)
(5) \( I_\epsilon \) is a strong deformation retract of \( I_{c+\epsilon} \setminus (K \cup (-K)); \)
(6) \( H_{0}(I_{c+\epsilon} \cap U) \setminus K) \geq 2 \) or \( H_0(I_{c+\epsilon} \cap U) \setminus K) \geq 2 \) holds.

**Proposition 3.1.** There exists a triple \((U, K, \epsilon) \subset H \times H \times R^+\) which satisfies (1) - (6).

We omit the proof of Proposition 3.1.
**Lemma 3.2.** Suppose that there exist a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying (1)-(6). Suppose in addition that $H_{N-1}(I_{c+\epsilon} \cap U) \backslash K) \geq 2$. Then $H_N(I_{c+\epsilon}, I_{\epsilon}) \geq 2$.

**Proof.** We put $\tilde{K} = K \cup (-K)$. Since $I_{\epsilon}$ is a strong deformation retract of $I_{c+\epsilon} \backslash \tilde{K}$, we find that

$$H_q(I_{c+\epsilon} \backslash \tilde{K}, I_{\epsilon}) \cong H_q(I_{\epsilon}, I_{\epsilon}) \cong 0.$$ 

Then we have from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon} \backslash \tilde{K}, I_{\epsilon})$ that

$$0 \rightarrow H_q(I_{c+\epsilon}, I_{\epsilon}) \rightarrow H_q(I_{c+\epsilon}, I_{c+\epsilon} \backslash \tilde{K}) \rightarrow 0.$$ 

That is

$$H_q(I_{c+\epsilon}, I_{\epsilon}) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon} \backslash \tilde{K}).$$ 

From (1), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon} \backslash \tilde{K}) \cong H_q(W, W \backslash \tilde{K}) \oplus H_q(-W, (-W) \backslash (-K))$$ 

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W \backslash K) \geq 2$, we have from (4) and the exactness of the sequence

$$\rightarrow H_q(W, W \backslash K) \rightarrow H_{q-1}(W \backslash K) \rightarrow H_{q-1}(W) \rightarrow H_{q-1}(W, W \backslash K) \rightarrow$$

with $q = N$ that $H_N(I_{c+\epsilon}, I_{\epsilon}) \cong H_N(W, W \backslash K) \oplus H_N(W, W \backslash K) \geq 2$.

**Lemma 3.3.** Suppose that $(U, K, \epsilon) \subset H \times H \times R^+$ satisfies (1) - (6). Suppose in addition that $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \backslash K) = 1$. Then $H_1(I_{c+\epsilon}, I_{\epsilon}) = 0$ or $H_0(I_{c+\epsilon}, I_{\epsilon}) = 2$ holds.

**Proof.** From the argument in the proof of Proposition 3.2, we have that $H_1(I_{c+\epsilon}, I_{\epsilon}) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \backslash K) \oplus H_0(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \backslash K)$. Then since $H_1(I_{c+\epsilon} \cap U) = 0$, and $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \backslash K) = 1$, the assertion follows from the exactness of the sequence (3.1) with $q = 1$.
We can now prove Theorem 1.

Proof of Theorem. Let \((U, K, \epsilon)\) be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that \(H_1(I_{c+\epsilon}, I_\epsilon) = 2\) and \(H_q(I_{c+\epsilon}, I_\epsilon) = 0\) for \(q \neq 1\). Now suppose that \((I_{c+\epsilon} \cap U) \setminus K\) is disconnected. Then since \(H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2\), we find by Lemma 3.2 that \(H_N(I_{c+\epsilon}, I_\epsilon) = 2\). This is a contradiction. On the other hand, if \(U \setminus K\) is connected, then \(H_0(U \setminus K) = 1\). Then by Lemma 3.3, we have \(H_1(I_{c+\epsilon}, I_\epsilon) = 0\) or \(H_0(I_{c+\epsilon}, I_\epsilon) = 2\). This is a contradiction. Thus we obtain that there exists a positive solution of (P).

REFERENCES
