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Existence of Entire Solutions for Superlinear Elliptic Problems in $R^N$

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1. **Introduction**. In this talk, we are concerned with positive solutions of the following problem:

$$(P) \quad \begin{cases} - \Delta u + u = g(x, u), & u > 0, \quad \text{in } R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

where $f : R^N \to R$ and $g : \Omega \times R \to R$ is continuous with $g(x, 0) = 0$ for $x \in \Omega$. In the last decade, the existence and the properties of the solutions of problem $(P)$ has been studied by many authors. Recently, the existence of positive solutions of semilinear elliptic problem

$$(P_Q) \quad \begin{cases} - \Delta u + u = Q(x) |u|^{p-1} u, & x \in R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

has been studied by several authors, where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $Q(x)$ is positive bounded continuous function. If the function $Q(x)$ is a radial function, the existence of infinity many solutions of problem $(P_Q)$ can be shown by restricting our attention to the radial functions(cf. [1]). In case that $Q(x)$ is nonradial, we encounter a difficulty caused by lack of compact embedding of Sobolev type. In [6,7], P.L. Lions presented a method, called concentrate compactness method, which enable us to solve problems with lack of compactness, and established the following result: Assume that

$$\lim_{|x| \to \infty} Q(x) = \overline{Q}(>0) \quad \text{and} \quad Q(x) \geq \overline{Q} \quad \text{on } R^N,$$
then problem \((P_Q)\) has a positive solution. This result is based on the observation that the ground state level \(c_Q\) of the functional

\[
I_Q(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1} \int_{R^N} Q(x)u^{p+1}dx
\]

is lower than the ground state level \(c_{\overline{Q}}\) of functional \(I_{\overline{Q}}\). We can apply the concentrate compactness method problem \((P)\) to the problem in case that \(g : R^N \times R \to R\) satisfies \(\lim_{|x| \to \infty} g(x, t) = t^p\) and the least critical level \(c_1\) of the functional

\[
I(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2)dx - \int_{R^N} \int_0^{u(x)} g(x, t)dt\,dx,
\]

\(u \in H^1(R^N),\) is lower than that of

\[
I^\infty(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1} \int u^{p+1}dx.
\]

Under additional conditions on \(g\), the existence of positive solutions \((P)\) was established by Ding & Ni[4] and Stuart[10]. Recently, Cao[2] proved the existence of positive solution of \((P_Q)\) for the case that \(c_Q \leq c_{\overline{Q}}\) under the hypothesis that \(\lim_{|x| \to \infty} Q(x) = \overline{Q}\) and \(Q(x) \geq 2^{(1-p)/2}\overline{Q}\) on \(R^N\). In case that \(c_Q = c_{\overline{Q}}\), we encounter a difficulty, because we can not apply the concentrate compactness method directly. On the other hand, in case that \(g\) is not given by the form \(Q(x)t^p\), we have to overcome another difficulty: that is, we can not use the Lagrange's method of indeterminate coefficients. In the problem \((P_Q)\), we find a solution \(u\) of minimizing problem

\[
\inf\{I_Q(u) : u \in V_\lambda\},
\]

\[
V_\lambda = \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x)u^{p+1}dx = 1\}
\]

Then \(cu\) is a solution of \((P_Q)\) for some \(c > 0\). The Lagrange's method does not work if \(g\) is not the form \(Q(x)t^p\). Our approach enable us to treat the problem \((P)\) with \(g\) satisfying that \(g(0) = 0\) and \(g(t) \to t^p\) as \(t \to \infty\). We also consider the nonhomginous case:

\[
(P_f) \begin{cases}
- \Delta u + u = |u|^{p-1} u + f, & x \in R^N \\
u \in H^1(R^N), & N \geq 3
\end{cases}
\]
where \( p > 1 \) for \( N = 1 \) and \( 1 < p < (N + 2)/(N - 2) \) for \( N \geq 3 \).

The nonhomogeneous problem \((P_f)\) was studied by Zhu[12]. In [12], the existence of at least two solutions of \((P)\) was proved for nonnegative functions \( f \in L^2(R^N) \) with a small \( L^2 \)-norm and a exponential decay

\[
f(x) \leq C \exp\{- (1 + \epsilon) |x|\}, \quad \text{for } x \in R^N.
\]

In the present paper, we consider multiple existence of solutions of \((P)\) for nonnegative functions \( f \in L^q(R^N) \), where \( q = (p+1)/p \). Our result does not require that \( f \in L^\infty(R^N) \) or any condition for the decay of \( f \) at infinity.

In this talk, we show an approach for problems \((P)\) and \((P_f)\) based on arguments using singular homology theory. Throughout this paper, we denote by \(| \cdot |_q\) the norm of \( L^q(R^N) \). We impose the following conditions on the continuous mapping \( g : R^N \times R \rightarrow R \):

(g1) There exists a positive number \( d < 1 \) such that
\[
-dt + (1 - d)t^p \leq g(x, t) \leq dt + (1 + d)t^p
\]
for all \((x, t) \in R^N \times [0, \infty)\);

(g2) there exists a positive number \( C \) such that
\[
|g_t(x, 0)| < 1 \quad \text{and} \quad 0 < t^2 g_{tt}(x, t) < C(1 + t^p)
\]
for all \((x, t) \in R^N \times [0, \infty)\);

(g3) \[
\lim_{|x| \rightarrow \infty} g(x, t) = |t|^{p-1} t
\]
uniformly on bounded intervals in \([0, \infty)\),

where \( 1 < p \) for \( N = 2 \) and \( 1 < p < (N + 2)/(N - 2) \) for \( N \geq 3 \), and \( g_t(\cdot, \cdot) \) stands for the derivative of \( g \) with respect to the second variable.

We can now state our main results.

**Theorem 1.** Suppose that (g2) and (g3) holds. Then there exists \( d_0 > 0 \) such that if (g1) holds with \( d < d_0 \), then problem \((P)\) has a positive solution.

For problem \((P_f)\), we have

**Theorem 2.** There exists a positive number \( C \) such that for each \( f \in L^q(R^N) \), with \( f \geq 0 \) and \( |f|_q < C \), problem \((P_f)\) possesses at least two solutions.
2. Preliminaries. We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then $H$ is a Hilbert space with norm

$$
\| u \| = \left( \int_{R^N} (|\nabla u|^2 + |u|^2)dx \right)^{1/2}.
$$

The norm of the dual space $H^{-1}(R^N)$ of $H$ is also denoted by $\| \cdot \|$. $B_r$ stands for the open ball centered at 0 with radius $r$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $| \cdot |_r$. For simplicity, we write $| \cdot |_*$ instead of $| \cdot |_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by $C_p$ the minimal constant satisfying

$$
| u |_* \leq C_p \| u \| \quad \text{for} \quad u \in H. \tag{2.1}
$$

It is easy to check that critical points of $I$ are solutions of (P). It is also obvious that nonzero critical points of $I^\infty$ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional $F$ on $H$ and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put

$$
\begin{align*}
M &= \{u \in H \setminus \{0\} : \| u \|^2 = \int_{R^N} ug(x, u)dx \}, \\
M^\infty &= \{u \in H \setminus \{0\} : \| u \|^2 = \int_{R^N} u^{p+1}dx \}.
\end{align*}
$$

For the proof of the following two propositions are crucial:

**Proposition 2.1.** There exists positive number $d_0 < \tilde{d}_0$ and $\epsilon_0$ satisfying that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$,

$$
H_* (I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon}) = H_* (I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon})
$$

where $H_* (A, B)$ denotes the singular homology group for a pair $(A, B)$ of topological spaces(cf. Spanier[8]).

**Proposition 2.2.** For each positive number $\epsilon < \epsilon_0$,

$$
H_q (I^\infty_{c+\epsilon}, I^\epsilon_{c+\epsilon}) = \begin{cases} 
2 & \text{if} \ q = 0, \\
0 & \text{if} \ q \neq 0.
\end{cases}
$$
Here we give a proof for Proposition 2.2.

We set
\[ T_{u_{\infty}}(M^\infty) = \{ \lim_{t \to 0} (C(t) - u_{\infty})/t : c \in C^1((-1,1); M^\infty) \text{ with } c(0) = u_{\infty} \}, \]
\[ C = C_- \cup C_+ = \{-\tau_x u_{\infty} : x \in \mathbb{R}^N\} \cup \{\tau_x u_{\infty} : x \in \mathbb{R}^N\} \]
and
\[ T_{u_{\propto}}(C) = \{ \lim_{t \to 0} (u_{\infty}(\cdot + tx) - u_{\infty}(\cdot))/t : x \in \mathbb{R}^N \}. \]

It follows from the definition of \( f_{VI}^\infty \) that the codimension of \( T_{u_{\infty}}(M^\infty) \) in \( H \) is one. It is also obvious that \( \dim T_{u_{\infty}}(C) = N \). We denote by \( \tilde{H} \) the subspace such that \( H = \tilde{H} \oplus T_{u_{\infty}}(C) \). For each \( r > 0 \), we set \( B_r^0 = B_r \cap H \).

Here we consider the linealized equation
\[ (L) \quad -\Delta u + u - h(x)u = \mu u, \quad u \in H, \mu \in \mathbb{R}, \]
where \( h(x) = p \mid u_{\infty}(x) \mid^{p-1} \) for \( x \in \mathbb{R}^N \). Since \( -\Delta \) is positive definite and \( h(x)I \) is compact, we find by Freidrich’s theory that the negative spectrums of \( A = -\Delta - h(x)I \) are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of \( L = -\Delta + I - h(x)I \) is finite dimensional. Then there exists \( c_0 > 0 \) and a decomposition \( H = H_- \oplus H_0 \oplus H_+ \) such that \( H_0 = \ker(L) \) and \( L \) is positive(negative) definite on \( H_+ (H_-) \) with
\[ \langle Lu, v \rangle \geq c_0 \|v\|^2 (\leq -c_0 \|v\|^2) \quad \text{ for } v \in H_+(H_-). \]

Since each \( u \in C \) is a solution of problem \((P_\infty)\), we can see that \( T_{u_{\infty}}(C) \subset H_0 \).

**Lemma 2.3.** \( \dim H_- = 1 \).

**Proof.** Since \( I^\infty \) attains its minimal on \( M^\infty \) at \( u_{\infty} \), we have that \( T_{u_{\infty}}(M^\infty) \subset H_+ \oplus H_0 \). Then since the codimension of \( M^\infty \) is one, we find that \( \dim H_- \leq 1 \). On the other hand, we have
\[ \langle Lu_{\infty}, u_{\infty} \rangle = \int_{\mathbb{R}^N} (\|\nabla u_{\infty}\|^2 + \|u_{\infty}\|^2 - p \mid u_{\infty} \mid^{p+1})dx. \]
\[ < \int_{\mathbb{R}^N} (\|\nabla u_{\infty}\|^2 + \|u_{\infty}\|^2 - \mid u_{\infty} \mid^{p+1})dx = 0. \]
Then we have that $\dim H_- \geq 1$. This completes the proof.

In the following we denote by $\varphi$ an element of $H_-$ with $\| \varphi \| = 1$. Here we note that since $h \in C^\infty(R^N)$, each solution $u$ of (L) is in $C^1(R^N)$. It then follows that if $u$ has the form

$$u(r, \theta) = \psi(r)\xi(\theta_1, \cdots, \theta_{n-1}), \quad \text{with} \ \xi \neq \text{const.},$$

in spherical coordinate, $\psi$ satisfies that $\psi(0) = 0$.

We denote by $H_r$ the set of all radial functions in $H$ and by $(L_r)$ the problem (L) restricted to $H_r$. Then, in spherical coordinates, the problem $(L_r)$ with $\mu > 0$ is reduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + (h-1)\psi = -\mu\psi(r), \quad r > 0, \psi \in C_r, \quad (2.3)$$

$$\frac{d\psi(r)}{dr}(0) = 0, \quad (2.4)$$

where $C_r = \{ \psi \in C[0, \infty) : \lim_{r \to \infty} \psi(r) = 0 \}$.

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h-1) - \frac{\alpha_k}{r^2})\psi(r) = -\mu\psi(r), \quad r > 0, \psi \notin 2R$$

$$\psi(0) = 0 (2.6)$$

where $\alpha_k = k(k+n-1), \ k = 1, 2, \cdots$. Note that $\alpha_k$ are the eigenvalues of Laplacian $-\Delta$ on $S^{n-1}$, the unit sphere, and the dimension of the eigenspace $S_k$ associate with $\alpha_k$ is

$$\rho_k = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$ 

That is there exists smooth functions $\{ \varphi_{k,i} : i = 1, \cdots, \rho_k \}$ defined on $S^{n-1}$ such that $S_k = \text{span}\{ \varphi_{k,1}, \cdots, \varphi_{k,\rho_k} \}$, and the functions $u = \psi(r)\varphi_{k,i}(\theta)$ are the solutions of (L).

**Lemma 2.4.** $\dim H_0 \leq N + 1$.

**Proof.** Since $\dim H_- = 1$ and $u_\infty \in H_r$, we have by (2.2) that the problems (2.3), (2.4) has exactly one negative eigenvalue. We also note
that each nonpositive eigenvalue \( \mu \) of problems (2.3), (2.4) is simple. Then the dimension of \( H_{0,r} = H_0 \cap H_r \) is at most one.

We next consider nonradial cases. That is we will see that the eigenspace of the problem (2.5) with \( \mu = 0 \) is \( N \)-dimensional space. Recalling that \( \nabla I(v) = 0 \) on \( C \), we can see that

\[
-\Delta v + v - h(x)v = 0 \quad \text{for all } v \in T_{u_\infty}(C). \tag{2.7}
\]

That is \( T_{u_\infty}(C) \subset H_0 \). Since \( \dim T_{u_\infty}(C) = N \), we have that \( \dim H_0 \geq N \). On the other hand, since \( u_\infty \) satisfies

\[
u''(r) + \frac{n-1}{r}\nu'(r) + p |u_\infty|^{p-1} u(r) = 0, \tag{2.8}\]

we find that \( v(r) = u'_\infty \) satisfies

\[
u''(r) + \frac{n-1}{r}\nu'(r) + ((h(x) - 1) - \frac{\alpha_1}{r^2})\nu(r) = 0.
\]

Then we find that the \( N \)-dimensional space \( \bar{C} = \text{span}\{v(r)\varphi_{1,i} : i = 1, \cdots, n-1\} \) is a subspace of solution set of (L) with \( \mu = 0 \). We claim that there exists no nonradial solution of (L) with \( \mu = 0 \) which is not contained in \( \bar{C} \). Suppose contrary, there exists a nonradial solution \( z \) of (L) with \( \mu = 0 \) such that \( z \perp \bar{C} \). Then there exists \( \psi \in C_r \) such that

\[
\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{\alpha_k}{r^2})\psi(r) = 0
\]

for some \( k > 1 \) and \( z = \psi(r)\varphi_{k,i} \) are solutions of (L) with \( \mu = 0 \). The equality above can be rewritten as

\[
\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{(\alpha_k - \alpha_1)}{r^2})\psi(r) - \frac{\alpha_1}{r^2}\psi(r) = 0.
\]

Then \( u = \psi(r)\varphi_{1,1} \) is a soluiton of problem

\[
-\Delta u + u - h(x)u = \frac{(\alpha_1 - \alpha_k)}{r^2}u.
\]

It then follows that

\[
< -\Delta u + u - h(x)u, u > < 0. \tag{2.9}
\]
Since $u$ is orthogonal to $\varphi$, we obtain from (2.9) that $\dim H_- \geq 2$. This is a contradiction. Thus we obtain that $H_0 = T_{u_0}(C) \oplus H_{0,r}$ and then $\dim H_0 \leq N + 1$.

Here we recall that $H$ has a decomposition $H = \tilde{H} \oplus T_{u_\infty}(C)$ and then $H = \tau_x \tilde{H} \oplus \tau_x T_{u_\infty}(C)$ for each $x \in R^N$. Then since $C_\pm$ are smooth $N$–manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B_{r_0}^0) \cap \tau_y(u_\infty + B_{r_0}^0) = \varphi$$ (2.10)

for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty |_{u_\infty + \tilde{H}}$ of $I^\infty$ on $u_\infty + \tilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces $H_1, H_{2,1}, H_{2,2}$ of $\tilde{H}$, a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B_{r_1}^0), R)$ and a homeomorphism $\psi : u_\infty + B_{r_1}^0 \to u_\infty + \tilde{H}$ such that $\tilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty |_{u_\infty + \tilde{H}} (\psi(u)) = c - \| u_1 \|^2 + \| u_{2,1} \|^2 + \beta(u_{2,2})$$ (2.11)

for each $u \in u_\infty + B_{r_1}^0$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1, u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and $u_\infty$ is the minimal point of $I^\infty$ on $M$, we have by choosing $r_1$ sufficiently small that $\beta(t \varphi_2)$ is strictly increasing as $|t|$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\| \varphi_2 \| = 1$.

Since $I^\infty$ is even, it is obvious that $I^\infty$ has the form (2.11) on $-(u_\infty + B_{r_1}^0)$. We also note that for each $x \in R^N$, (2.11) holds for each $u \in \tau_x(u_\infty + B_{r_0}^0)$ with $\psi$ replaced by $\tau_x \circ \psi$.

**Proof of Proposition 2.2.** By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I^\infty_{c+\epsilon}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_c, I^\infty_{c-\epsilon}),$$

and

$$H_q(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_{c-\epsilon}, I^\infty_{c-\epsilon}) \cong 0.$$  

From the exactness of the singular homology groups ,

$$H_q(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_c \setminus C) \to H_{q-1}(I^\infty_c \setminus C, I^\infty_{c-\epsilon}) \to \cdots$$
we find
\[ 0 \rightarrow H_q(I_{c}^{\infty}, I_{c}^{\infty}-\epsilon) \rightarrow H_q(I_{c}^{\infty}, I_{c}^{\infty}\backslash C) \rightarrow 0. \]
That is
\[ H_q(I_{c}^{\infty}, I_{c}^{\infty}-\epsilon) \cong H_q(I_{c}^{\infty}, I_{c}^{\infty}\backslash C). \]
Noting that \( \cup\{\tau_x(\pm u_{\infty} + B_{r_1}^0) : x \in R^N\} \) are disjoint open neighborhoods of \( C_{\pm} \) respectively, and that \( I^{\infty} \) is invariant under the translations \( \tau_x \), we find from the excision property and (2.11) that
\[
H_*(I_{c}^{\infty}+\epsilon, I_{c}^{\infty}) \\
\cong H_*(I_{c}^{\infty}, I_{c}^{\infty}\backslash C) \\
\cong H_*(I_{c}^{\infty} \cap (\bigcup_{i=1}^{\pm 1} \cup_{x} \tau_x(iu_{\infty} + B_{r_1}^0)), I_{c}^{\infty} \cap (\bigcup_{i=1}^{\pm 1} \cup_{x} \tau_x(iu_{\infty} + B_{r_1}^0)\backslash C)) \\
\cong H_*(u_{\infty} + B_{r_1}^1, (u_{\infty} + B_{r_1}^1)\backslash \{u_{\infty}\}) \\
\oplus H_*(-u_{\infty} + B_{r_1}^1, (-u_{\infty} + B_{r_1}^1)\backslash \{u_{\infty}\}) \\
\cong H_*([0,1], \{0,1\}) \oplus H_*([0,1], \{0,1\}).
\]
This completes the proof.

3. Proof of Theorem 1. We next consider a triple \((U, K, \epsilon) \subset H \times H \times R^+\) satisfying the following conditions:

(1) \( U \cap (-U) = \emptyset \);
(2) \( \{\tau_xu_{\infty} : x \geq r\} \subset intK \) for some \( r > 0 \);
(3) \( cl(I_{c+\epsilon} \cap K) \subset int(I_{c+\epsilon} \cap U) \);
(4) \( H_{N-1}(I_{c+\epsilon} \cap U) = 1, \quad H_1(I_{c+\epsilon} \cap U) = 0 \);
(5) \( I_{\epsilon} \) is a strong deformation retract of \( I_{c+\epsilon}\backslash (K \cup (-K)) \);
(6) \( H_{N-1}((I_{c+\epsilon} \cap U)\backslash K) = 2 \) or \( H_0((I_{c+\epsilon} \cap U)\backslash K) \geq 2 \) holds.

Proposition 3.1. There exists a triple \((U, K, \epsilon) \subset H \times H \times R^+\) which satisfies (1) - (6).

We omit the proof of Proposition 3.1.
Lemma 3.2. Suppose that there exist a triple \((U, K, \epsilon) \subset H \times H \times R^+\) satisfying (1)-(6). Suppose in addition that \(H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) \geq 2\). Then \(H_N(I_{c+\epsilon}, I_\epsilon) \geq 2\).

Proof. We put \(\tilde{K} = K \cup (-K)\). Since \(I_\epsilon\) is a strong deformation retract of \(I_{c+\epsilon} \setminus \tilde{K}\), we find that

\[
H_q(I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.
\]

Then we have from the exactness of the singular homology groups of the triple \((I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon)\) that

\[
0 \rightarrow H_q(I_{c+\epsilon}, I_\epsilon) \rightarrow H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \rightarrow 0.
\]

That is

\[
H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}).
\]

From (1), we find

\[
H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \cong H_q(W, W \setminus K) \oplus H_q(-W, (-W \setminus (-K))
\]

where \(W = I_{c+\epsilon} \cap U\). Then since \(H_{N-1}(W \setminus K) \geq 2\), we have from (4) and the exactness of the sequence

\[
\rightarrow H_q(W, W \setminus K) \rightarrow H_{q-1}(W \setminus K) \rightarrow H_{q-1}(W) \rightarrow H_{q-1}(W, W \setminus K) \rightarrow
\]

with \(q = N\) that \(H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W \setminus K) \oplus H_N(W, W \setminus K) \geq 2\).

Lemma 3.3. Suppose that \((U, K, \epsilon) \subset H \times H \times R^+\) satisfies (1) - (6). Suppose in addition that \(H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1\). Then \(H_1(I_{c+\epsilon}, I_\epsilon) = 0\) or \(H_0(I_{c+\epsilon}, I_\epsilon) = 2\) holds.

Proof. From the argument in the proof of Proposition 3.2, we have that \(H_1(I_{c+\epsilon}, I_\epsilon) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K) \oplus H_N(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K)\). Then since \(H_1(I_{c+\epsilon} \cap U) = 0\), and \(H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1\), the assertion follows from the exactness of the sequence (3.1) with \(q = 1\).
We can now prove Theorem 1.

**Proof of Theorem.** Let \((U, K, \epsilon)\) be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that \(H_1(I_{c+\epsilon}, I_\epsilon) = 2\) and \(H_q(I_{c+\epsilon}, I_\epsilon) = 0\) for \(q \neq 1\). Now suppose that \((I_{c+\epsilon} \cap U) \setminus K\) is disconnected. Then since \(H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2\), we find by Lemma 3.2 that \(H_N(I_{c+\epsilon}, I_\epsilon) = 2\). This is a contradiction. On the other hand, if \(U \setminus K\) is connected, then \(H_0(U \setminus K) = 1\). Then by Lemma 3.3, we have \(H_1(I_{c+\epsilon}, I_\epsilon) = 0\) or \(H_0(I_{c+\epsilon}, I_\epsilon) = 2\). This is a contradiction. Thus we obtain that there exists a positive solution of (P).

**REFERENCES**


