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</thead>
<tbody>
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Second-order directional derivatives of sup-type functions

Sup-型関数の2次の方向微分について

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Abstract

In this paper, we deal with the following sup-type function:

\[ S(x) := \sup_{t \in T} G(x(t), t) \quad x \in X, \tag{1} \]

where \( T \) is a compact metric space, \( X \) is a subspace of the set of all \( n \)-dimensional vector-valued continuous functions \( C(T)^n \) equipped with the uniform norm. We denote by \( G_x \) and \( G_{xx} \) the gradient (row) vector and the Hesse matrix of \( f \) w.r.t. \( x \), respectively, and assume them to be continuous on \( R^n \times T \). This sup-type function is induced from a phase constraint

\[ G(x(t), t) \leq \forall t \in T \]

which appears in variational problems and optimal control problems [15].

On the other hand, another sup-type function has been deeply studied:

\[ S_0(x) := \sup_{t \in T} G(x(t), t) \quad x \in R^n, \tag{2} \]

Clarke[1], Correa and Seeger[2], Danskin [3], Dem’yanov and Malozemov[4] Demyanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9] [10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this sup-type function in Tchebycheff approximation. When \( T \) depends on \( x \), the minimization problem of \( S_0(x) \) becomes a parametric optimization problem. To tell the truth, \( S_0(x) \) is a special case of \( S(x) \). Indeed, if we take as \( X \) \( \{x \in C(T)^n \mid x(t) \equiv \text{constant vector} \in R^n \} \), then \( S(x) \) reduces to \( S_0(x) \). So \( S(x) \) inherits a lot of properties from \( S_0(x) \).
論文の概要

次の Sup-型関数の 1 次と 2 次の方向微分について考察する。

\[ S(x) := \sup_{t \in T} G(x(t), t) \quad x \in X, \quad (3) \]

ただし T はコンパクト距離空間, X は n 次元ベクトル値連続関数全体 C(T)^n の部分空間とする。

この Sup-型関数は変分問題や最適制御問題の相条件

\[ G(x(t), t) \leq \forall_{t \in T} \]

を考察するとき出会う。本論文では、この相条件から包絡線が生成されるかどうかを調べるために、sup-型関数 S(x) の 2 次の方向微分を表す公式を与える。

一方、従来よく研究されてきた Sup-型関数は次の関数である。

\[ X \quad (x) := \sup_{t \in T} G(x, t) \quad x \in R^n, \quad (4) \]

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 T が x に依存してよいとすれば、X(x) の最小化問題はパラメトリック最適化問題になる。S(x) が S0(x) と本質的に異なる点は、後者は x と t が独立に動けるのに対し、前者は x が t に依存することである。しかしながら、S0(x) は S(x) のスペシャルケースと見なすこともできる。つまり、X として n 次元ベクトル値定数関数全体 \( \{x(t) \equiv a \mid a \in R^n \} \) をとればよい。従って、S(x) は S0(x) の多くの性質を受け継ぐことになる。結論を先に述べると、相条件からも包絡線が生成される。

In the following, we denote by T(x) the set of all extreme points \( G(x(\cdot), \cdot) \), that is,

\[ T(x) := \{ t \in T ; G(x(t), t) = S(x) \}, \quad x \in C(T)^n. \]

**Theorem 1** The function S(x) is continuous.

**Theorem 2** The function S(x) is directionally differentiable in any direction \( y \in X \), and its directional derivative is given by

\[ S'(x; y) = \max \{ G_x(x(t), t)y(t); t \in T(x) \}. \quad (5) \]
Applying Theorem 2 to the sup-type function induced from the one-sided phase constraint:

\[ s(t) \leq x(t) \quad \forall t, \quad (6) \]

where \( s(t) \) is a given continuous function, we get the following result:

**Corollary 1** Let \( s \in C(T) \). Take \( G(x,t) := s(t) - x \) for any \( x \in \mathbb{R} \) and \( t \in T \). Then

\[ S'(x;y) = - \min_{t \in \tau(x)} y(t). \]

Taking constant functions as \( x(t) \) and \( y(t) \) in Theorem 2, we get Danskin’s formula.

**Corollary 2** (Danskin[3]) The function \( S_0(x) \) is directionally differentiable in any direction \( y \in \mathbb{R}^n \) and its directional derivative is given by

\[ S_0'(x;y) = \max \{ G_x(x,t)y; t \in T(x) \}. \quad (7) \]

Next, we consider a second-order directional derivative of \( S(x) \).

**Definition 1** The upper second-order directional derivative of \( S(x) \) at \( x \) in the direction \( y \) is defined by

\[ S''(x;y) = \limsup_{\varepsilon \to +0} \frac{S(x + \varepsilon y) - S(x) - \varepsilon S'(x;y)}{\varepsilon^2} \quad (8) \]

**Definition 2** ([9]) For any functions \( u, v \in C(T) \) satisfying

\[ \begin{align*}
  u(t) &\geq 0 \quad \forall t \in T, \\
  v(t) &\geq 0 \quad \text{if } u(t) = 0,
\end{align*} \quad (9) \]

we define a function \( E : T \to [-\infty, +\infty] \) by

\[ E(t) := \begin{cases}
  \sup \left\{ \limsup_{t_n} \frac{v(t_n)^2}{4u(t_n)} ; \{t_n\} \text{ satisfies (11)} \right\}, & \text{if } t \in T_0, \\
  0 & \text{if } u(t) = v(t) = 0 \text{ and } t \not\in T_0, \\
  -\infty & \text{otherwise},
\end{cases} \quad (10) \]

\[ T_0 := \left\{ t \in T; \exists t_n \to t \text{ s.t. } u(t_n) > 0, \frac{v(t_n)}{u(t_n)} \to +\infty \right\}. \quad (11) \]
\textsc{Theorem 3} \textit{Let }x\textit{ and }y\textit{ be arbitrary functions in }C(T)^n\textit{. Then it holds that}
\[ S''(x;y) = \max \left\{ \frac{y^T S_x(x(t),t)y(t)}{2} + E(t) \mid t \in T(x;y) \right\}, \tag{12} \]
where \( T(x;y) := \{ t \in T(x) \mid S'(x;y) = G_{x}(x(t), t)y(t) \} \) and \( E(t) \) is defined via Definition 2 by taking
\[ u(t) = S(x) - G(x(t),t), \quad v(t) = S'(x;y) - G_{x}(x(t), t)y(t). \tag{13} \]

Taking constant functions as \( x(t) \) and \( y(t) \) in Theorem 3, we get the following formula due to [9].

\textsc{Corollary 3} \textit{Let }x\textit{ and }y\textit{ be arbitrary points in }R^n\textit{. Then it holds that}
\[ S''(x;y) = \max \left\{ \frac{y^T G_{xx}(x,t)y}{2} + E(t) \mid t \in T(x;y) \right\}, \tag{14} \]
where \( E(t) \) is defined via Definition 2 by taking
\[ u(t) = S(x) - G(x, t), \quad v(t) = S'(x;y) - G_{x}(x(t), t)y. \tag{15} \]

We proved in [9] and [10] that an envelope is formed from \( G(x,t) \) when \( E(t) > 0 \) at some point \( t \). Similarly, an envelope is formed from \( G(x(t),t) \) when \( E(t) > 0 \) at some \( t \).

\textbf{Example} We can find an envelope even in the simplest one-sided phase constraint:
\[ x(t) \geq 0 \quad \forall t, \]
that is, \( G(x,t) = -x \). Let \( x(t) := t^2, \; T := [-1,1] \) and \( y(t) := -2t \). Then
\[ \phi(\varepsilon) := S(x + \varepsilon y) \]
\[ = \max_{|t| \leq 1} \{-x(t) - \varepsilon y(t)\} \]
\[ = \max_{|t| \leq 1} \{2t\varepsilon - t^2\} \]
\[ = \begin{cases} \varepsilon^2 & |\varepsilon| \leq 1 \\ |2\varepsilon| - 1 & |\varepsilon| \geq 1 \end{cases} \]
For each \( t \in [-1,1] \), the function \( 2t\varepsilon - t^2 \) is affine w.r.t. \( \varepsilon \). However, these affine functions form the envelope \( \phi(\varepsilon) = \varepsilon^2 \) near \( \varepsilon = 0 \).
It is clear from the definition of the upper second-order directional derivative that
\[
\overline{S}''(x; y) = \limsup_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0) - \epsilon \phi'(0)}{\epsilon^2} = 1.
\]
On the other hand, the functions \(u(t)\) and \(v(t)\) defined by (13) become
\[
u(t) = S'(x; y) - G_x(x(t), t)y(t) = -y(0) - (-y(t)) = -2t,
\]
respectively. Hence
\[
E(t) = \begin{cases} 
1, & t = 0, \\
-\infty, & t \neq 0.
\end{cases}
\]
Since \(G(x, t)\) is affine w.r.t. \(x\), its second partial derivative vanishes. So the right hand side of (12) equals 1.

參考文献


