Second-order directional derivatives of sup-type functions

Sup-型関数の2次方向微分について

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Abstract

In this paper, we deal with the following sup-type function:

\[ S(x) := \sup_{t \in T} G(x(t), t) \quad x \in X, \]  

where \( T \) is a compact metric space, \( X \) is a subspace of the set of all \( n \)-dimensional vector-valued continuous functions \( C(T)^n \) equipped with the uniform norm. We denote by \( G_x \) and \( G_{xx} \) the gradient (row) vector and the Hesse matrix of \( f \) w.r.t. \( x \), respectively, and assume them to be continuous on \( R^n \times T \). This sup-type function is induced from a phase constraint

\[ G(x(t), t) \leq \forall t \in T \]

which appears in variational problems and optimal control problems [15].

On the other hand, another sup-type function has been deeply studied:

\[ S_0(x) := \sup_{t \in T} G(x, t) \quad x \in R^n, \]  

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Demyanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9][10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this sup-type function in Tchebycheff approximation. When \( T \) depends on \( x \), the minimization problem of \( S_0(x) \) becomes a parametric optimization problem. To tell the truth, \( S_0(x) \) is a special case of \( S(x) \). Indeed, if we take as \( X \) \{ \( x \in C(T)^n \mid x(t) \equiv \) constant vector \( \in R^n \} \), then \( S(x) \) reduces to \( S_0(x) \). So \( S(x) \) inherits a lot of properties from \( S_0(x) \).
論文の概要

次の Sup-型関数の1次と2次の方程式微分について考察する。

\[ S(x) := \sup_{t \in \tau} G(x(t), t) \quad x \in X, \]

ただし \( T \) はコンパクト距離空間, \( X \) は \( n \) 次ベクトル値連続関数全体 \( C(T)^n \) の部分空間とする。

この Sup-型関数は変分問題や最適制御問題の相対条件

\[ G(x(t), t) \leq \forall_{t \in T} \]

を考察するとき出会う。本論文では、この相対条件から包絡線が生成されるかどうかを調べるために、sup-型関数 \( S(x) \) の2次の方程式微分を表す公式を与える。

一方、従来よく研究されてきた Sup-型関数は次の関数である。

\[ S_0(x) := \sup_{t \in T} G(x(t), t) \quad x \in R^n, \]

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 \( T \) が \( x \) に依存してよいとすれば、\( S_0(x) \) の最小化問題はパラメトリック最適化問題になる。\( S(x) \) が \( S_0(x) \) と本質的に異なる点は、後者は \( x \) と \( t \) が独立に動けるのに対し、前者は \( x \) が \( t \) に依存することである。しかしながら、\( S_0(x) \) は \( S(x) \) のスペシャルケースと見なすこともできる。つまり、\( X \) として \( n \) 次ベクトル値定数関数全体 \( \{x(t) \equiv a \mid a \in R^n\} \) をとればよい。従って、\( S(x) \) は \( S_0(x) \) の多くの性質を受け継ぐことになる。結論を先に述べると、相対条件からも包絡線が生成される。

In the following, we denote by \( T(x) \) the set of all extreme points \( G(x(\cdot), \cdot) \), that is,

\[ T(x) := \{t \in T \mid G(x(t), t) = S(x)\}, \quad x \in C(T)^n. \]

**Theorem 1** The function \( S(x) \) is continuous.

**Theorem 2** The function \( S(x) \) is directionally differentiable in any direction \( y \in X \), and its directional derivative is given by

\[ S'(x; y) = \max\{G_x(x(t), t)y(t) ; t \in T(x)\}. \]
Applying Theorem 2 to the sup-type function induced from the one-sided phase constraint:

\[ s(t) \leq x(t) \quad \forall t, \quad (6) \]

where \( s(t) \) is a given continuous function, we get the following result:

**Corollary 1** Let \( s \in C(T) \). Take \( G(x, t) := s(t) - x \) for any \( x \in \mathbb{R} \) and \( t \in T \). Then

\[ S'(x; y) = - \min_{t \in \tau(x)} y(t). \]

Taking constant functions as \( x(t) \) and \( y(t) \) in Theorem 2, we get Danskin's formula.

**Corollary 2** (Danskin[3]) The function \( S_0(x) \) is directionally differentiable in any direction \( y \in \mathbb{R}^n \) and its directional derivative is given by

\[ S_0'(x; y) = \max \{ G_x(x, t)y; t \in T(x) \}. \quad (7) \]

Next, we consider a second-order directional derivative of \( S(x) \).

**Definition 1** The upper second-order directional derivative of \( S(x) \) at \( x \) in the direction \( y \) is defined by

\[ S''(x; y) = \limsup_{\varepsilon \to 0^+} \frac{S(x + \varepsilon y) - S(x) - \varepsilon S'(x; y)}{\varepsilon^2}. \quad (8) \]

**Definition 2** ([9]) For any functions \( u, v \in C(T) \) satisfying

\[ \begin{cases} u(t) \geq 0 & \forall t \in T, \\ v(t) \geq 0 & \text{if } u(t) = 0, \end{cases} \quad (9) \]

we define a function \( E : T \to [-\infty, +\infty] \) by

\[ E(t) := \begin{cases} \sup \left\{ \limsup_\frac{v(t_n)^2}{2u(t_n)} ; \{t_n\} \text{ satisfies } (11) \right\}, & \text{if } t \in T_0, \\ 0 & \text{if } u(t) = v(t) = 0 \text{ and } t \not\in T_0, \\ -\infty & \text{otherwise}, \end{cases} \quad (10) \]

\[ T_0 := \left\{ t \in T ; \exists t_n \to t \text{ s.t. } u(t_n) > 0, \frac{v(t_n)}{u(t_n)} \to +\infty \right\}. \quad (11) \]
Theorem 3 Let \( x \) and \( y \) be arbitrary functions in \( C(T)^n \). Then it holds that
\[
\overline{S}''(x;y) = \max \left\{ \frac{y(t)^T G_{xx}(x(t), t)y(t)}{2} + E(t) \ ; \ t \in T(x;y) \right\},
\]
(12)
where \( T(x;y) := \{ t \in T(x) \ ; \ S'(x;y) = G_x(x(t), t)y(t) \} \) and \( E(t) \) is defined via Definition 2 by taking
\[
u(t) = S(x) - G(x(t), t), \quad v(t) = S'(x;y) - G_x(x(t), t)y(t).
\]
(13)

Taking constant functions as \( x(t) \) and \( y(t) \) in Theorem 3, we get the following formula due to [9].

Corollary 3 Let \( x \) and \( y \) be arbitrary points in \( \mathbb{R}^n \). Then it holds that
\[
\overline{S}''(x;y) = \max \left\{ \frac{y^T G_{xx}(x, t)y}{2} + E(t) \ ; \ t \in \tau(x;y) \right\},
\]
(14)
where \( E(t) \) is defined via Definition 2 by taking
\[
u(t) = S(x) - G(x, t), \quad v(t) = S'(x;y) - G_x(x, t)y.
\]
(15)

We proved in [9] and [10] that an envelope is formed from \( G(x, t) \) when \( E(t) > 0 \) at some point \( t \). Similarly, an envelope is formed from \( G(x(t), t) \) when \( E(t) > 0 \) at some \( t \).

Example We can find an envelope even in the simplest one-sided phase constraint:
\[
x(t) \geq 0 \ \forall t,
\]
that is, \( G(x, t) = -x \). Let \( x(t) := t^2, \ T := [-1, 1] \) and \( y(t) := -2t \). Then
\[
\phi(\epsilon) := S(x + \epsilon y)
\]
\[
= \max_{|t| \leq 1} \{-x(t) - \epsilon y(t)\}
\]
\[
= \max_{|t| \leq 1} \{2t\epsilon - t^2\}
\]
\[
= \begin{cases} 
\epsilon^2 & |\epsilon| \leq 1 \\
|2\epsilon| - 1 & |\epsilon| > 1
\end{cases}
\]
For each \( t \in [-1, 1] \), the function \( 2t\epsilon - t^2 \) is affine w.r.t. \( \epsilon \). However, these affine functions form the envelope \( \phi(\epsilon) = \epsilon^2 \) near \( \epsilon = 0 \).
It is clear from the definition of the upper second-order directional derivative that
\[
\bar{S}''(x;y) = \limsup_{\varepsilon \to 0} \frac{\phi(\varepsilon) - \phi(0) - \varepsilon \phi'(0)}{\varepsilon^2} = 1.
\]

On the other hand, the functions \( u(t) \) and \( v(t) \) defined by (13) become
\[
u(t) = S'(x;y) - G_x(x(t), t)y(t) = -y(0) - (-y(t)) = -2t,
\]
respectively. Hence
\[
E(t) = \begin{cases} 
  1, & t = 0, \\
  -\infty, & t \neq 0.
\end{cases}
\]

Since \( G(x,t) \) is affine w.r.t. \( x \), its second partial derivative vanishes. So the right hand side of (12) equals 1.

\[\text{參考文献}\]


