Second-order directional derivatives of sup-type functions

Sup-型関数の2次の方向微分について

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Abstract

In this paper, we deal with the following sup-type function:

\[ S(x) := \sup_{t \in T} G(x(t), t) \quad x \in X, \]  \hspace{1cm} (1)

where \( T \) is a compact metric space, \( X \) is a subspace of the set of all \( n \)-dimensional vector-valued continuous functions \( C(T)^n \) equipped with the uniform norm. We denote by \( G_x \) and \( G_{xx} \) the gradient (row) vector and the Hesse matrix of \( f \) w.r.t. \( x \), respectively, and assume them to be continuous on \( R^n \times T \). This sup-type function is induced from a phase constraint

\[ G(x(t), t) \leq \forall t \in T \]

which appears in variational problems and optimal control problems [15].

On the other hand, another sup-type function has been deeply studied:

\[ S_0(x) := \sup_{t \in T} G(x, t) \quad x \in R^n, \]  \hspace{1cm} (2)

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Dem'yanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9][10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this sup-type function in Tchebycheff approximation. When \( T \) depends on \( x \), the minimization problem of \( S_0(x) \) becomes a parametric optimization problem. To tell the truth, \( S_0(x) \) is a special case of \( S(x) \). Indeed, if we take as \( X \) \{ \( x \in C(T)^n \mid x(t) \equiv \) constant vector \( \in R^n \} \), then \( S(x) \) reduces to \( S_0(x) \). So \( S(x) \) inherits a lot of properties from \( S_0(x) \).
論文の概要

次の Sup-型関数の1次と2次の方程式について考察する。

\[ S(x) := \sup_{t \in \tau} G(x(t), t) \quad x \in X, \]  
\[ (3) \]

ただし \( T \) はコンパクト距離空間, \( X \) は \( n \)次元ベクトル値連続関数全体 \( C(T)^n \) の部分空間とする。
この Sup-型関数は変分問題や最適制御問題の相関である。

\[ G(x(t), t) \leq \forall t \in T \]

考察するとき出会う。本論文では、この相関条件から包絡線が生成されるかどうかを調べるために、sup-型関数 \( S(x) \) の2次の方程式を表す公式を与える。

一方、従来よく研究されてきた Sup-型関数は次の関数である。

\[ S_0(x) := \sup_{t \in \tau} G(x, t) \quad x \in R^n, \]
\[ (4) \]

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 \( T \) が \( x \) に依存してよいと思うれば、\( S_0(x) \) の最小化問題はパラメトリック最適化問題になる。\( S(x) \) が \( S_0(x) \) と本質的に異なる点は、後者は \( x \) と \( t \) が独立に動けるのに対し、前者は \( x \) が \( t \) に依存することである。しかしながら、\( S_0(x) \) は \( S(x) \) のスペシャルケースと見なすこともできる。つまり、\( X \) として \( n \) 次元ベクトル値定数関数全体 \( \{x(t) \equiv a | a \in R^n\} \) をとればよい。従って、\( S(x) \) は \( S_0(x) \) の多くの性質を受け継ぐことになる。結論を先に述べると、相関条件からも包絡線が生成される。

In the following, we denote by \( T(x) \) the set of all extreme points \( G(x(\cdot), \cdot) \), that is,

\[ T(x) := \{t \in T ; G(x(t), t) = S(x)\}, \quad x \in C(T)^n. \]

**Theorem 1** The function \( S(x) \) is continuous.

**Theorem 2** The function \( S(x) \) is directionally differentiable in any direction \( y \in X \), and its directional derivative is given by

\[ S'(x; y) = \max\{G_x(x(t), t)y(t); t \in T(x)\}. \]
\[ (5) \]
Applying Theorem 2 to the sup-type function induced from the one-sided phase constraint:

\[ s(t) \leq x(t) \quad \forall t, \]

(6)

where \( s(t) \) is a given continuous function, we get the following result:

**COROLLARY 1** Let \( s \in C(T) \). Take \( G(x, t) := s(t) - x \) for any \( x \in \mathbb{R} \) and \( t \in T \). Then

\[ S'(x; y) = - \min_{t \in \mathcal{T}(x)} y(t). \]

Taking constant functions as \( x(t) \) and \( y(t) \) in Theorem 2, we get Danskin's formula.

**COROLLARY 2** (Danskin[3]) The function \( S_0(x) \) is directionally differentiable in any direction \( y \in \mathbb{R}^n \) and its directional derivative is given by

\[ S_0'(x; y) = \max \{ G_x(x, t) y ; t \in T(x) \}. \]

(7)

Next, we consider a second-order directional derivative of \( S(x) \).

**DEFINITION 1** The upper second-order directional derivative of \( S(x) \) at \( x \) in the direction \( y \) is defined by

\[ \overline{s}''(x; y) = \lim_{\epsilon \to +0} \sup_{0} \frac{S(x + \epsilon y) - S(x) - \epsilon S'(x; y)}{\epsilon^2} \]

(8)

**DEFINITION 2** ([9]) For any functions \( u, v \in C(T) \) satisfying

\[ \{ \begin{array}{l}
 u(t) \geq 0 \quad \forall t \in T, \\
 v(t) \geq 0 \quad \text{if} \quad u(t) = 0,
\end{array} \]

(9)

we define a function \( E : T \to [-\infty, +\infty] \) by

\[ E(t) := \begin{cases} 
 \sup \left\{ \limsup_{n \to +\infty} \frac{v(t_n)^2}{u(t_n)} ; \{ t_n \} \text{ satisfies (11)} \right\}, & \text{if } t \in T_0, \\
 0 & \text{if } u(t) = v(t) = 0 \text{ and } t \not\in T_0, \\
 -\infty & \text{otherwise},
\end{cases} \]

(10)

\[ T_0 := \left\{ t \in T ; \exists t_n \to t \text{ s.t. } u(t_n) > 0, \frac{v(t_n)}{u(t_n)} \to +\infty \right\}. \]

(11)
**Theorem 3** Let $x$ and $y$ be arbitrary functions in $C(T)^n$. Then it holds that

$$S''(x; y) = \max \left\{ \frac{y(t)^T G_{xx}(x(t), t)y(t)}{2} + E(t) ; t \in T(x; y) \right\}, \quad (12)$$

where $T(x; y) := \{ t \in T(x) ; S'(x; y) = G_x(x(t), t)y(t) \}$ and $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x(t), t), \quad v(t) = S'(x; y) - G_x(x(t), t)y(t). \quad (13)$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 3, we get the following formula due to [9].

**Corollary 3** Let $x$ and $y$ be arbitrary points in $R^n$. Then it holds that

$$S''(x; y) = \max \left\{ \frac{y^T G_{xx}(x, t)y}{2} + E(t) ; t \in \tau(x; y) \right\}, \quad (14)$$

where $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x, t), \quad v(t) = S'(x; y) - G_x(x, t)y. \quad (15)$$

We proved in [9] and [10] that an envelope is formed from $G(x, t)$ when $E(t) > 0$ at some point $t$. Similarly, an envelope is formed from $G(x(t), t)$ when $E(t) > 0$ at some $t$.

**Example** We can find an envelope even in the simplest one-sided phase constraint:

$$x(t) \geq 0 \quad \forall t,$$

that is, $G(x, t) = -x$. Let $x(t) := t^2$, $T := [-1, 1]$ and $y(t) := -2t$. Then

$$\phi(\epsilon) := S(x + \epsilon y)$$

$$= \max_{|t| \leq 1} \{-x(t) - \epsilon y(t)\}$$

$$= \max_{|t| \leq 1} \{2t\epsilon - t^2\}$$

$$= \begin{cases} 
\epsilon^2 & |\epsilon| \leq 1 \\
|2\epsilon| - 1 & |\epsilon| \geq 1
\end{cases}$$

For each $t \in [-1, 1]$, the function $2t\epsilon - t^2$ is affine w.r.t. $\epsilon$. However, these affine functions form the envelope $\phi(\epsilon) = \epsilon^2$ near $\epsilon = 0$. 
It is clear from the definition of the upper second-order directional derivative that
\[
\overline{S}''(x; y) = \limsup_{\varepsilon \to +0} \frac{\phi(\varepsilon) - \phi(0) - \varepsilon \phi'(0)}{\varepsilon^2} = 1.
\]
On the other hand, the functions \(u(t)\) and \(v(t)\) defined by (13) become
\[
u(t) = S'(x; y) - G_x(x(t), t)y(t) = y(0) - (y(t)) = -2t,
\]
respectively. Hence
\[
E(t) = \begin{cases} 
1, & t = 0, \\
-\infty, & t \neq 0.
\end{cases}
\]
Since \(G(x, t)\) is affine w.r.t. \(x\), its second partial derivative vanishes. So the right hand side of (12) equals 1.

参考文献


