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Kyoto University
On a Theoretically Conformable Duality for Semicontinuity of Set-Valued Mappings

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Abstract. The paper presents a study of semicontinuity of set-valued maps. The Berge definitions of upper and lower semicontinuities of set-valued maps are improved into weaker and slightly strong conditions, respectively, which have a theoretically conformable duality of semicontinuity of set-valued maps. The improved definitions of upper and lower semicontinuities are defined in terms of both neighborhoods and sequences in a metric space. As the result of this research, we understand the reason why certain conditions are needed to guarantee the equivalence between Berge's upper semicontinuity and Hogan's upper semicontinuity.

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Key words: Upper semicontinuity, lower semicontinuity, dual concepts, set-valued maps.

1. Introduction

A set-valued map (point-to-set map or multifunction) from a set $X$ into a set $Y$ is a map which associates a subset of $Y$ with each point of $X$. The notion of semicontinuity for a set-valued map is very important in the area of optimization as well as in other fields of applied mathematics. In particular, upper semicontinuity of a set-valued map is indispensable for fixed-point theorems and stability theory in mathematical programming.

The concepts of semicontinuous maps have been introduced in 1932 by G. Bouligand and K. Kuratowski; see [1] and the references therein. There are various different definitions of upper semicontinuity and lower semicontinuity so far, and they are classified into two categories; those defined in terms of neighborhoods and those defined in terms of sequences; see [1, 2, 3, 4] and [6, 7]. They are not always equivalent with each other without any conditions. Nevertheless, upper semicontinuity is considered to be the dual concept of lower semicontinuity in general. This is an incomplete point for theoretical duality, because Berge's upper...
semicontinuity and Hogan's upper semicontinuity are not coincident and Berge's lower semicontinuity and Hogan's lower semicontinuity are coincident in a metric space [3, 4] although those upper semicontinuities are considered to be the dual concepts of those lower semicontinuities, respectively. Hogan's upper semicontinuity and lower semicontinuity are called closed and open, and defined in terms of sequences in his paper [4], respectively. This paper points out the reason which causes the discrepancy between Berge's upper semicontinuity and Hogan's upper semicontinuity.

The purpose of this paper is to extend classical concepts of upper semicontinuity of set-valued maps and then to give a theoretical conformable duality between upper semicontinuity and lower semicontinuity. Also, this paper gives, in terms of sequences, equivalent definitions for the improved definitions of upper semicontinuity and lower semicontinuity.

The organization of this paper is given as follows. In Section 2, we points out the difference between Berge's upper semicontinuity and Hogan's upper semicontinuity. This difference is based on both the existence of pathological neighborhoods including an image set of a set-valued map and the possibility of an unbounded graph of a set-valued map. If a set-valued map $F$ is uniformly compact near $x$ and if $F$ is closed (Hogan's upper semicontinuous) at $x$, then $F$ is (Berge's) upper semicontinuous at $x$. Conversely, if $F(x)$ is a closed set and if $F$ is (Berge's) upper semicontinuous at $x$, then $F$ is closed (Hogan's upper semicontinuous) at $x$. Those two upper semicontinuities are extended into weaker notions, and then relationship among all of them is given. Moreover, we give an equivalent definition for the weakest one in the extended upper semicontinuities in terms of sequences. In Section 3, we observe improved notions of Berge's lower semicontinuity and Hogan's lower semicontinuity corresponding to extended upper semicontinuities, and then we give a theoretical conformable duality between upper semicontinuity and lower semicontinuity. These improved semicontinuities are defined in terms of both neighborhoods and sequences in a metric space.

2. Extensions of Upper Semicontinuity of Set-Valued Maps

Let $X$ and $Y$ be two topological spaces, and $F : X \rightarrow 2^Y$. The definitions of Berge's semicontinuities are meaningful in any topological spaces $X$ and $Y$, but Hogan's semicontinuities are meaningful in any spaces $X$ and $Y$ where the concept of convergence is defined in terms of nets or sequences. Since notions of semicontinuity having a theoretical conformable duality between upper semicontinuity and lower semicontinuity are defined in terms of open balls with radius $\varepsilon > 0$, we shall assume that $Y$ is a metric space in those definitions.

First, we begin with classical Berge's semicontinuities of set-valued maps.

**Definition 2.1. (Berge's u.s.c.)** A set-valued map $F : X \rightarrow 2^Y$ is said to be upper semicontinuous (u.s.c. for short) at $x_0$ if for any open set $U$ with $F(x_0) \subseteq U$, there exists a
neighborhood $V$ of $x_0$ such that $F(x) \subseteq U$ for all $x \in V$.

**Definition 2.2. (Berge's l.s.c.)** A set-valued map $F : X \rightarrow 2^Y$ is said to be lower semicontinuous (l.s.c. for short) at $x_0$ if for any open set $U$ with $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood $V$ of $x_0$ such that $F(x) \cap U \neq \emptyset$ for all $x \in V$.

The notions of upper semicontinuity and lower semicontinuity are distinct and not equivalent in general except on residual sets in a complete separable metric space $Y$ ([1, Th.1.4.13]). We provide the following example.

**Example 2.1.** Let $X = Y = R$ and $F_1, F_2$ set-valued maps from $R$ into $2^R$ defined by

$$F_1(x) := \begin{cases} 
  \{y \in R \mid 0 \leq y \leq 1\} & \text{for } x < 0; \\
  \{y \in R \mid 0 \leq y \leq x + 2\} & \text{for } x \geq 0,
\end{cases}$$

$$F_2(x) := \begin{cases} 
  \{y \in R \mid 0 \leq y < 1\} & \text{for } x \leq 0; \\
  \{y \in R \mid 0 \leq y < x + 2\} & \text{for } x > 0.
\end{cases}$$

One can verify that $F_1$ is u.s.c. at $x = 0$ but not l.s.c. at the point, and that $F_2$ is l.s.c. at $x = 0$ but not u.s.c. at the point.

In 1973, Hogan [4] gave an alternative to the Berge's semicontinuities of set-valued maps in the setting of open and closed maps. He also presented the relationship between Berge's semicontinuities and his ones; see [3, 4].

**Definition 2.3. (Hogan's u.s.c.)** Let $X$ and $Y$ be two metric spaces. A set-valued map $F : X \rightarrow 2^Y$ is said to be closed at $x_0$ if for any sequences $\{x_n\}$ with $x_n \rightarrow x_0$ and $\{y_n\}$ with $y_n \in F(x_n)$, $y_n \rightarrow y_0$ for some $y_0 \in Y$ implies that $y_0 \in F(x_0)$.

**Definition 2.4. (Hogan's l.s.c.)** Let $X$ and $Y$ be two metric spaces. A set-valued map $F : X \rightarrow 2^Y$ is said to be open at $x_0$ if for any sequence $\{x_n\}$ with $x_n \rightarrow x_0$ and $y_0 \in F(x_0)$, there exists a sequence $\{y_n\}$ such that $y_n \in F(x_n)$ and $y_n \rightarrow y_0$ (i.e., $d_Y(y_n, y_0) \rightarrow 0$).

The map $F_1$ in Example 2.1. is closed at $x = 0$ but not open at the point, and the map $F_2$ in Example 2.1. is open at $x = 0$ but not closed at the point. As known from this, the notions of closedness and openness have meanings similar to upper and lower semicontinuities, respectively. Actually, openness and lower semicontinuity are coincident, but closedness and upper semicontinuity are not equivalent; see [4, Th.1], [1, p.39], and the following example.

**Example 2.2.** Let $X = Y = R$ and $F_3, F_4$ set-valued maps from $R$ into $2^R$ defined by

$$F_3(x) := \{y \in R \mid 0 \leq y < 1\},$$

$$F_4(x) := \begin{cases} 
  \{y \in R \mid 0 \leq y \leq 1\} & \text{for } x \leq 0; \\
  \{y \in R \mid \frac{1}{2} \leq y \leq \frac{1}{2} + 1\} & \text{for } x > 0.
\end{cases}$$

One can verify that $F_3$ is u.s.c. at $x = 0$ but not closed at the point, and that $F_4$ is closed at $x = 0$ but not u.s.c. at the point.
We shall observe the reason why this discrepancy is caused although upper semicontinuity and closedness are considered to be the dual concepts of lower semicontinuity and openness, respectively. To this end, we extend the two notions into weaker ones, and then present the relationship among all of them.

**Definition 2.5. (w-u.s.c.)** A set-valued map $F: X \to 2^Y$ is said to be weakly upper semicontinuous (w-u.s.c. for short) at $x_0$ if for any open set $U$ with $\text{cl } F(x_0) \subset U$, there exists a neighborhood $V$ of $x_0$ such that $F(x) \subset U$ for all $x \in V$.

Of course, an upper semicontinuous map is also weakly upper semicontinuous. Conversely, if $F$ is weakly upper semicontinuous at $x_0$ and $F(x_0)$ is a closed set, then it is upper semicontinuous at the point. Weakly upper semicontinuity is a slight extension of upper semicontinuity, which takes in some pathological non-u.s.c. maps.

**Example 2.3.** Let $X = Y = R$ and $F_6$ a set-valued map from $R$ into $2^R$ defined by
\[
F_6(x) := \begin{cases} 
\{ y \in R \mid 0 \leq y < 1 \} & \text{for } x \leq 0; \\
\{ y \in R \mid 0 \leq y < x + 1 \} & \text{for } x > 0.
\end{cases}
\]

One can verify that $F_6$ is weakly u.s.c. at $x = 0$ but not u.s.c. at the point.

However, there is another example of a pathological set-valued map which is similar to an u.s.c. map in image values but not u.s.c.

**Example 2.4.** Let $X = R_+, Y = R^2$, and $F_6$ a set-valued map from $R_+$ into $2^{R^2}$ defined by
\[
F_6(x) := \left\{ (z_1, z_2) \in R^2 \mid z_2 \geq \frac{1}{x_1 + z_1}, z_1 \geq 0 \right\}.
\]

Consider an open set
\[
U := \left\{ (z_1, z_2) \in R^2 \mid z_2 > \frac{1}{2z_1}, z_1 \geq 0 \right\},
\]
which includes the set $\text{cl } F_6(0)$ but does not include any sets $F_6(x)$ for $x > 0$. This shows that $F_6$ is not weakly u.s.c. at $x = 0$ although it is similar to an u.s.c. map in image values.

To overcome this incompleteness, we introduce a more weaker notion of upper semicontinuity of set-valued maps. It is presented also in [1, p.39] when $F(x_0)$ is a compact set in $Y$.

**Definition 2.6. (equally w-u.s.c.)** Let $Y$ be a metric space. A set-valued map $F: X \to 2^Y$ is said to be equally weak upper semicontinuous (equally w-u.s.c. for short) at $x_0$ if for any $\varepsilon > 0$ there exists a neighborhood $V$ of $x_0$ such that $F(x) \subset B_Y (F(x_0), \varepsilon)$ for all $x \in V$, where $B_Y (F(x_0), \varepsilon) := \{ y \in Y \mid d_Y (y, F(x_0)) < \varepsilon \}$.

**Theorem 2.1.** Let $X$ and $Y$ be a topological space and a metric space, respectively. If a set-valued map $F$ from $X$ into $2^Y$ is w-u.s.c., then it is also equally w-u.s.c. Conversely, if $F$ is equally w-u.s.c. at $x_0$ and $\text{cl } F(x_0)$ is a compact set, then it is w-u.s.c. at the point.
Proof. The first part is obvious. We prove only the second part. Let $F$ be equally w-u.s.c. at $x_0$ and $\text{cl} \ F(x_0)$ a compact set, and let $U$ be an open set including $\text{cl} \ F(x_0)$. For each $y \in \text{cl} \ F(x_0)$, there is $\epsilon(y) > 0$ such that $B_Y(y, \epsilon(y)) \subset U$, and hence

$$\text{cl} \ F(x_0) \subset \bigcup_{y \in \text{cl} \ F(x_0)} B_Y(y, \epsilon(y)/2) \subset U.$$ 

Since $\text{cl} \ F(x_0)$ is compact, there exist $y_1, \ldots, y_m \in \text{cl} \ F(x_0)$ such that

$$\text{cl} \ F(x_0) \subset \bigcup_{i=1}^{m} B_Y(y_i, \epsilon(y_i)/2).$$ 

Let $\epsilon^* := \min_{i=1,\ldots,m} \epsilon(y_i) > 0$, then there is a neighborhood $V$ of $x_0$ such that $F(x) \subset B_Y(F(x_0), \epsilon^*/2)$ for all $x \in V$. Therefore, we have that $F(x) \subset U$ for all $x \in V$. In fact, let $z \in F(x)$, then it follows from $z \in B_Y(F(x_0), \epsilon^*/2)$ that there exists $z^* \in F(x_0)$ such that $d_Y(z, z^*) < \epsilon^*/2$. Since $z^* \in B_Y(y_i_0, \epsilon(y_i_0)/2)$ for some $i_0$, we have $d_Y(z, y_{i_0}) \leq d_Y(z, z^*) + d_Y(z^*, y_{i_0}) < \epsilon(y_{i_0})$, and hence $z \in B_Y(y_{i_0}, \epsilon(y_{i_0})) \subset U$. This completes the proof.

As known from this theorem, whenever $Y$ is a metric space and $F(x_0)$ is a compact set, three notions of u.s.c., w-u.s.c., and equally w-u.s.c. at $x_0$ are coincident with each other.

When $Y$ is a topological vector space or more generally a topological group, the notion of equally w-u.s.c. is coincident with the following one.

**Definition 2.7. (properly u.s.c.)** Let $Y$ be a topological vector space: A set-valued map $F : X \to 2^Y$ is said to be properly upper semicontinuous (p-u.s.c. for short) at $x_0$ if for any open neighborhood $G$ of the origin $\theta$, there exists a neighborhood $V$ of $x_0$ such that $F(x) \subset F(x_0) + G$ for all $x \in V$.

Next, we provide another definition of equally w-u.s.c. in terms of nets (sequences).

**Definition 2.8. (equally w-u.s.c.)** Let $Y$ be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally w-u.s.c. at $x_0$ if for any nets $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{y_\lambda\}$ with $y_\lambda \in F(x_\lambda)$, there exists a net (sequence) $\{z_\lambda\}$ such that $z_\lambda \in F(x_0)$ and $d_Y(z_\lambda, y_\lambda) \to 0$.

**Theorem 2.2.** Definitions 2.6. and 2.8. are coincident.

Proof. Assume that $F$ is equally w-u.s.c. at $x_0$ defined by Definition 2.6. Let $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{y_\lambda\}$ with $y_\lambda \in F(x_\lambda)$ be nets in $X$ and $Y$, respectively. By the assumption, for $\epsilon = 1/n$, $n = 1, 2, \ldots$, there exists a neighborhood $V_n$ of $x_0$ such that $F(x) \subset B_Y(F(x_0), 1/n)$ for all $x \in V_n$. Since $x_\lambda \to x_0$, for each $n = 1, 2, \ldots$ there exists $\lambda_n$ such that $x_\lambda \in V_n$ for all $\lambda \geq \lambda_n$, and hence $y_\lambda \in F(x_\lambda) \subset B_Y(F(x_0), 1/n)$ for all $\lambda \geq \lambda_n$. This means that $d_Y(y_\lambda, F(x_0)) < 1/n$ for all $\lambda \geq \lambda_n$. Therefore, we can take a net $\{z_\lambda\} \subset F(x_0)$ such that $d_Y(y_\lambda, z_\lambda) \to 0$. 


Conversely, assume that $F$ is equally w-u.s.c. at $x_0$ defined by Definition 2.8. Suppose to the contrary that there are $\varepsilon_0 > 0$ and a net $\{x_\lambda\}$ in $X$ such that $x_\lambda \to x_0$ and $F(x_\lambda) \not\subseteq B_Y(F(x_0), \varepsilon_0)$, which implies that there exists a net $\{y_\lambda\}$ such that $y_\lambda \in F(x_\lambda)$ and $d_Y(y_\lambda, F(x_0)) \geq \varepsilon_0$. By the assumption, we can take another net $\{z_\lambda\} \subset F(x_0)$ such that $d_Y(y_\lambda, z_\lambda) \to 0$, which is a contradiction to $d_Y(y_\lambda, F(x_0)) \geq \varepsilon_0$. This completes the proof.

Now, we turn to the discrepancy between upper semicontinuity and closedness of set-valued maps. In [4, Th.3] and [3], uniformly compactness near $x_0$ guarantees the coincidence between upper semicontinuity and closedness of the point as follows: if $F$ is uniformly compact near $x_0$, i.e., there is a neighborhood $V$ of $x_0$ such that the closure of the set $\cup_{x \in V} F(x)$ is compact, then $F$ is closed at $x_0$ if and only if $F(x_0)$ is compact and $F$ is u.s.c. at the point. To observe this, we extend Hogan’s closedness in Definition 2.3.

**Definition 2.9.** Let $X$ and $Y$ be two metric spaces. A set-valued map $F : X \to 2^Y$ is said to be weakly closed (w-closed for short) at $x_0$ if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $\{y_n\}$ with $y_n \in F(x_n)$, $y_n \to y_0$ for some $y_0 \in Y$ implies that $y_0 \in \text{cl} F(x_0)$.

**Remark 2.1.** This definition is a slight extension of closedness, and a closed set-valued map is also w-closed. Conversely, if $F$ is w-closed at $x_0$ and $F(x_0)$ is a closed set, then it is closed at the point. Moreover, we can verify, by using Definition 2.8. in terms of sequences, that any equally w-u.s.c. set-valued map is w-closed, and hence any u.s.c. map is also w-closed. If $F(x_0)$ is a closed set, then upper semicontinuity implies closedness.

Conversely, we can verify that closedness at $x_0$ implies upper semicontinuity at the point if the set-valued map is uniformly compact near $x_0$; see [4, Th.3]. Similarly, any w-closed map is also w-u.s.c. under the uniformly compactness, and therefore the three notions of weakly upper semicontinuity, equally weak upper semicontinuity, and weakly closedness at $x_0$ are equivalent with each other whenever the set-valued map is uniformly compact near the point.

**Remark 2.2.** We can verify that if a set-valued map $F$ is closed and w-u.s.c. at $x_0$, then $F$ is u.s.c. at the point. Actually, the graph of $F$ is closed by [4, Th.2], and hence $F$ is a closed-valued map, i.e., $F(x_0)$ is a closed set. Also, the maps $F_3$ in Example 2.2. and $F_5$ in Example 2.3. are w-closed and equally w-u.s.c. at $x = 0$ but not closed at the point. On the other hand, the map $F_4$ in Example 2.2. is closed at $x_0$ but not equally w-u.s.c. at the point, and the map $F_6$ in Example 2.4. is closed and equally w-u.s.c. but not w-u.s.c. at the point. Moreover, Example 2.6. shows the existence of a set-valued map which is equally w-u.s.c. but neither closed nor w-u.s.c.

Thus, the notion of weakly closedness of set-valued maps is a considerable large class of maps similar to upper semicontinuous maps.
Example 2.5. Let $X = Y = R$ and $F_7$ a set-valued map from $R$ into $2^R$ defined by

$$F_7(x) := \begin{cases} 
\{y \in R \mid 0 \leq y \leq 1\} & \text{for } x < 0; \\
\{y \in R \mid 0 \leq y < 1\} & \text{for } x = 0; \\
\{y \in R \mid \frac{1}{x} \leq y \leq \frac{1}{x} + 1\} & \text{for } x > 0.
\end{cases}$$

One can verify that $F_7$ is w-closed at $x = 0$ but neither closed nor equally w-u.s.c. at the point.

Example 2.6. Let $X = R_+, Y = R^2$, and $F_8$ a set-valued map from $R_+$ into $2^{R^2}$ defined by

$$F_8(x) := \left\{(z_1, z_2) \in R^2 \mid z_2 > \frac{1}{z_1 + x}, z_1 \geq 0\right\}.$$

One can verify that $F_8$ is equally w-u.s.c. at $x = 0$ but neither closed nor w-u.s.c. at the point.

We conclude this chapter with an illustration of the inclusion structure shown in Figure 1 where the number stands for that of each set-valued map in Examples 2.2. to 2.6..
3. Theoretically Duality between Upper Semicontinuity and Lower Semicontinuity

We shall consider the possibility of extension of Berge's lower semicontinuity and Hogan's lower semicontinuity corresponding to the extended upper semicontinuities in Section 2.

First, we begin with Berge's lower semicontinuity. Let $X$ and $Y$ be two topological spaces, and $F : X \to 2^Y$. Since conditions $F(x_0) \cap U \neq \emptyset$ and $\text{cl} F(x_0) \cap U \neq \emptyset$ are coincident for each open set $U \subset Y$, we can not extend lower semicontinuity into weaker notions in a similar way to that of Section 2. Then, we provide an improve notion of lower semicontinuity corresponding to equally weak upper semicontinuity in Definition 2.6.. The improved lower semicontinuity is precisely the dual concept of equally weak upper semicontinuity, and also stronger than lower semicontinuity.

Definition 3.1. (equally w-l.s.c.) Let $Y$ be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally weak lower semicontinuous (equally w-l.s.c. for short) at $x_0$ if for any $\varepsilon > 0$ there exists a neighborhood $V$ of $x_0$ such that $F(x_0) \subset B_Y (F(x), \varepsilon)$ for all $x \in V$.

Theorem 3.1. Let $X$ and $Y$ be a topological space and a metric space, respectively. If a set-valued map $F$ from $X$ into $2^Y$ is equally w-l.s.c., then it is also l.s.c. Conversely, if $F$ is l.s.c. at $x_0$ and $\text{cl} F(x_0)$ is a compact set, then it is equally w-l.s.c. at the point.

Proof. To prove the first part, let $U$ be an open set satisfying $F(x_0) \cap U \neq \emptyset$. Suppose to the contrary that there is a net $\{x_\lambda\}$ in $X$ such that $x_\lambda \to x_0$ and $F(x_\lambda) \cap U = \emptyset$. Hence there exist a vector $y_0 \in F(x_0)$ and a scalar $\varepsilon_0 > 0$ such that $B_Y (y_0, \varepsilon_0) \subset U$, and so $y_0 \not\in B_Y (F(x_\lambda), \varepsilon_0)$. By the assumption, there exists a neighborhood $V$ of $x_0$ such that $F(x_0) \subset B_Y (F(x), \varepsilon_0)$ for all $x \in V$. Since $x_\lambda \to x_0$, there exists $\lambda_0$ such that $x_\lambda \in V$ for all $\lambda \geq \lambda_0$, and hence $F(x_0) \subset B_Y (F(x_\lambda), \varepsilon_0)$ for all $\lambda \geq \lambda_0$, which is a contradiction to $y_0 \not\in B_Y (F(x_\lambda), \varepsilon_0)$.

Next, we prove the second part. Let $\varepsilon > 0$, and then

$$\text{cl} F(x_0) \subset \bigcup_{y \in \text{cl} F(x_0)} B_Y (y, \varepsilon/2).$$

Since $\text{cl} F(x_0)$ is a compact set, there exist $y_1, \ldots, y_m \in \text{cl} F(x_0)$ such that

$$\text{cl} F(x_0) \subset \bigcup_{i=1}^{m} B_Y (y_i, \varepsilon/2).$$

Let $U_i := B_Y (y_i, \varepsilon/2)$, then $F(x_0) \cap U_i \neq \emptyset$ for all $i = 1, \ldots, m$. By the assumption, there are neighborhoods $V_1, \ldots, V_m$ of $x_0$ such that $F(x) \cap U_i \neq \emptyset$ for all $x \in V_i$, $i = 1, \ldots, m$. Let $V := \bigcap_{i=1}^{m} V_i$, then we have that $F(x_0) \subset B_Y (F(x), \varepsilon)$ for all $x \in V$. In fact, let $z \in F(x_0)$, then it follows from $z \in \bigcup_{i=1}^{m} B_Y (y_i, \varepsilon/2)$ that there exists $i_0$ such that $z \in U_{i_0}$,
i.e., $d_Y(z, y_{i_0}) < \varepsilon/2$. By $x \in V$, we have $F(x) \cap U_{i_0} \neq \emptyset$, which implies that $d_Y(y_{i_0}, z^*) < \varepsilon/2$ for some $z^* \in F(x)$. Therefore, we have $d_Y(z, z^*) \leq d_Y(z, y_{i_0}) + d_Y(y_{i_0}, z^*) < \varepsilon$. This completes the proof.

We can provide another definition of equally w-l.s.c. in terms of nets (sequences), which is verified to be equivalent to Definition 3.1. in the same way as the proof of Theorem 2.2.

**Definition 3.2. (equally w-l.s.c.)** Let $Y$ be a metric space. A set-valued map $F : X \to 2^Y$ is said to be equally w-l.s.c. at $x_0$ if for any nets $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\{y_\lambda\}$ with $y_\lambda \in F(x_0)$, there exists a net (sequence) $\{y_{\lambda}\}$ such that $y_\lambda \in F(x_\lambda)$ and $d_Y(y_\lambda, y_{\lambda}) \to 0$.

This notion is precisely the dual concept of equally weak upper semicontinuity in terms of nets; see Definition 2.8. Moreover, we can verify easily that the notion is stronger than openness (Hogan's lower semicontinuity); an equally w-u.s.c. set-valued map is also open, and conversely if $F$ is open at $x_0$ and $\text{cl } F(x_0)$ is a compact set, then it is equally w-u.s.c. at the point. When $Y$ is a topological vector space or more generally a topological group, the notion of equally w-l.s.c. is coincident with the following one.

**Definition 3.3. (properly l.s.c.)** Let $Y$ be a topological vector space. A set-valued map $F : X \to 2^Y$ is said to be properly lower semicontinuous (p-l.s.c. for short) at $x_0$ if for any open neighborhood $G$ of the origin $\theta$, there exists a neighborhood $V$ of $x_0$ such that $F(x_0) \subset F(x) + G$ for all $x \in V$.

Finally, we obtain a theoretical conformable duality between upper semicontinuity and lower semicontinuity in terms of both neighborhoods and sequences in a metric space. Figure 2 illustrates the duality and the relationship among various semicontinuities of set-valued maps. From the figure we can observe that classical upper semicontinuity is precisely the dual concept of classical lower semicontinuity whenever the set-valued map is compact-valued in a metric space.

**References**


Figure 2: Theoretical Duality between Upper Semicontinuity and Lower Semicontinuity.

