A note on Sturm-type comparison theorems on a half-open interval

広島大・理 内藤 雄基 (Yūki Naito)

1. Introduction and statement of the results

In this note, we investigate comparison theorems of Sturm-type on a half-open interval $[a, \omega)$, $\omega \leq \infty$. We consider two differential equations

(1.1) \((p(t)x')' + q(t)x = 0, \quad a \leq t < \omega,\)

(1.2) \((P(t)y')' + Q(t)y = 0, \quad a \leq t < \omega,\)

where $p(t)$, $q(t)$, $P(t)$, and $Q(t)$ are continuous functions on $[a, \omega)$, and

\[
p(t) \geq P(t) > 0 \quad \text{and} \quad Q(t) \geq q(t) \quad \text{on} \quad [a, \omega).
\]

In this case, (1.2) is called a Sturm majorant for (1.1) on $[a, \omega)$ and (1.1) is called a Sturm minorant for (1.2).

Sturm's comparison theorem can be stated as follows: (See, e.g., [2, Chap.11, Theorem 3.1].)

**Theorem A.** Let $x(t) \not\equiv 0$ be a solution of (1.1) and let $x(t)$ has exactly $n \geq 1$ zeros $t = t_1 < t_2 < \cdots < t_n$ in $(a, b]$, $b < \omega$. Let $y(t)$ be a solution of (1.2). If either $x(a) = 0$ or $x(a) \neq 0$, $y(a) \neq 0$, and

\[
\frac{p(a)x'(a)}{x(a)} \geq \frac{P(a)y'(a)}{y(a)}.
\]

then $y(t)$ has one of the following properties:

(i) $y(t)$ has at least $n$ zeros in $(a, t_n)$;

(ii) $y(t)$ is a constant multiple of $x(t)$ on $[a, t_n]$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, t_n]$. 
Let \( x(t) > 0 \) in \( (t_n, \omega) \) in Theorem A. In this case, it seems interesting to ask the question whether a solution \( y(t) \) of (1.2) has at least one zero in \( (t_n, \omega) \) or not?

Assume that (1.1) is nonoscillatory at \( t = \omega \). It is well known [2, Chap.11, Theorem 6.4] that (1.1) has a principal solution \( x_0(t) \) which is essentially unique (up to a constant factor) such that

\[
\int_{\omega}^{\omega} \frac{ds}{p(s)[x_0(s)]^2} = \infty
\]

and for any solution \( x_1(t) \) linearly independent of \( x_0(t) \),

\[
\lim_{t \to \omega} \frac{x_0(t)}{x_1(t)} = 0.
\]

The solution \( x_1(t) \) is called a nonprincipal solution.

Our main results are the following.

**Theorem 1.** Assume that (1.1) is nonoscillatory at \( t = \omega \). Let \( x_0(t) \) be a principal solution of (1.1) satisfying \( x_0(t) > 0 \) in \( (a, \omega) \). Let \( y(t) \) be a solution of (1.2). If either \( x_0(a) = 0 \) or \( x_0(a) \neq 0, y(a) \neq 0, \) and

\[
\frac{p(a)x_0'(a)}{x_0(a)} \geq \frac{P(a)y'(a)}{y(a)}
\]

then \( y(t) \) has one of the following properties:

(i) \( y(t) \) has at least one zero in \( (a, \omega) \);

(ii) \( y(t) \) is a constant multiple of \( x_0(t) \) on \( [a, \omega) \) and \( p(t) \equiv P(t), q(t) \equiv Q(t) \) on \( [a, \omega) \).

**Theorem 2.** Assume that (1.1) is nonoscillatory at \( t = \omega \). Let \( x_0(t) \) be a principal solution of (1.1) and let \( z(t) \) has exactly \( n (\geq 1) \) zeros in \( (a, \omega) \). Let \( y(t) \) be a solution of (1.2). If either \( x_0(a) = 0 \) or \( x_0(a) \neq 0, y(a) \neq 0, \) and (1.3) holds, then \( y(t) \) has one of the following properties:

(i) \( y(t) \) has at least \( n + 1 \) zeros in \( (a, \omega) \);

(ii) \( y(t) \) is a constant multiple of \( x_0(t) \) on \( [a, \omega) \) and \( p(t) \equiv P(t), q(t) \equiv Q(t) \) on \( [a, \omega) \).

**Remark.** For other results concerning comparison theorems of Sturm-type on a half-open interval, we refer to [4] and [5].

When \( p(t) \equiv P(t) \) and \( q(t) \equiv Q(t) \) on \( [a, \omega) \), as a consequence of Theorems 1 and A, we have the following.

**Corollary 1.** Assume that (1.1) is nonoscillatory at \( t = \omega \). Let \( x_0(t) \) be a principal solution of (1.1) and let \( t_0 (\geq a) \) be the largest zero, i.e., \( x_0(t_0) = 0 \) and \( x_0(t) > 0 \) in \( (t_0, \omega) \). Then we have the following properties:
(i) every nonprincipal solution has exactly one zero in \((t_0, \omega)\);
(ii) every solution of (1.1) has exactly one zero on \([t_0, \omega)\).

Equation (1.1) is said to be disconjugate on an interval \(J\) if every solution of (1.1) has at most one zero on \(J\). (See [1] and [2].) By Corollary 1, we obtain a criterion for (1.1) to be disconjugate.

**Corollary 2.** Assume that (1.1) is nonoscillatory at \(t = \omega\). Let \(x_0(t)\) be a principal solution of (1.1) and let \(t_0 (\geq a)\) be the largest zero. Then (1.1) is disconjugate on \([t_1, \omega)\) if and only if \(t_0 \leq t_1\).

Finally, we give a comparison theorem for disconjugacy.

**Corollary 3.** Assume that (1.2) is nonoscillatory at \(t = \omega\). (Then (1.1) is nonoscillatory at \(t = \omega\).) Let \(x_0(t)\) and \(y_0(t)\) be principal solutions of (1.1) and (1.2), respectively. Let \(t_0\) and \(t_1\) \((t_0, t_1 \geq a)\) be the largest zeros of \(x_0(t)\) and \(y_0(t)\), respectively. Then, we have either (i) \(t_0 < t_1\) or (ii) \(t_0 = t_1\) and \(p(t) \equiv P(t), q(t) \equiv Q(t)\) on \([t_0, \omega)\). In particular, if (1.2) is disconjugate on an interval \(J\), then (1.1) is disconjugate on \(J\).

*Remark.* The comparison theorems for disconjugacy have been shown in [1] by different methods.

2. Proofs of Theorems

We prepare the following lemmas.

**Lemma 1.** Assume that \(q(t) \leq 0\) on \([a, \omega)\) in (1.1). Then (1.1) is nonoscillatory at \(t = \omega\) and a principal solution \(x_0(t)\) of (1.1) satisfies \(x_0(t) > 0\) and \(x_0'(t) \leq 0\) on \([a, \omega)\).

**Lemma 2.** Assume that (1.1) is nonoscillatory at \(t = \omega\). Let \(x_0(t)\) be a principal solution of (1.1) and let \(y(t)\) be a solution of (1.2) satisfying \(y(t) > 0\) on \([T, \omega), T \geq a\). Then \(x_0(t) > 0\) on \([T, \omega)\) and

\[
\frac{p(t)x_0'(t)}{x_0(t)} \leq \frac{P(t)y'(t)}{y(t)} \quad \text{on} \quad [T, \omega).
\]

Lemmas 1 and 2 are shown in [2, Chap.11, Corollary 6.4] and [2, Chap.11, Corollary 6.5], respectively. However, for the sake of the completeness, we give (slight simple) proofs of them.
Proof of Lemma 1. Let $x_i(t)$, $i = 1, 2$, be solutions of (1.1) determined by $x_i(a) = 1$ and $x_i'(a) = i$. It is easy to see that $(p(t)x_i(t))' \geq 0$ and $x_i(t) > 0$ on $[a, \omega)$, $i = 1, 2$. Since $x_1(t)$ and $x_2(t)$ are linearly independent, either $x_1(t)$ or $x_2(t)$ is a nonprincipal solution. Without loss of generality, we may assume that $x_1(t)$ is a nonprincipal solution. By [2, Chap. 11, Corollary 6.3],

$$x_0(t) = x_1(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2}, \quad a \leq t < \omega,$$

is well defined and a principal solution of (1.1). We see that $x_0(t) > 0$ on $[a, \omega)$. We obtain

$$x_0'(t) = x_1'(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2} - \frac{1}{p(t)x_1(t)}, \quad a \leq t < \omega.$$ 

Since $p(t)x_1'(t)$ is nondecreasing and $x_1(t)$ is positive,

$$p(t)x_0'(t) \leq \int_t^\omega \frac{x_1(s)}{[x_1(s)]^2} ds - \frac{1}{x_1(t)} = -\lim_{t \to \omega} \frac{1}{x_1(s)} \leq 0, \quad a \leq t < \omega.$$ 

Thus, we have $x_0'(t) \leq 0$ on $[a, \omega)$.

Proof of Lemma 2. Let

$$u(t) = \exp \left( \int_T^t \frac{P(s)y'(s)}{p(s)y(s)} ds \right), \quad T \leq t < \omega.$$ 

Then $u(t) > 0$ on $[T, \omega)$ and satisfies

$$\frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)} \quad \text{and} \quad (p(t)u')' + Q_0(t)u = 0 \quad \text{for} \ T \leq t < \omega,$$ 

where

$$Q_0(t) = Q(t) + \left( \frac{1}{P(t)} - \frac{1}{p(t)} \right) \left( \frac{P(t)y'(t)}{y(t)} \right)^2, \quad T \leq t < \omega.$$ 

Let $z(t) = x_0(t)/u(t)$ on $[T, \omega)$. Then $z(t)$ is a solution of

$$(p(t)[u(t)]^2z')' + [u(t)]^2 (q(t) - Q_0(t)) z = 0, \quad T \leq t < \omega.$$ 

Since $x_0(t)$ is a principal solution, we have

$$\int_T^\omega \frac{ds}{p(s)[x_0(s)]^2} = \int_T^\omega \frac{ds}{p(s)[u(s)]^2[z(s)]^2} = \infty.$$ 

Thus $z(t)$ is a principal solution of (2.2). We note that $Q_0(t) \geq Q(t) \geq q(t)$ on $[T, \omega)$. Then, by Lemma 1, we have $z(t) > 0$ and $z'(t) \leq 0$ on $[T, \omega)$, which implies $x_0(t) > 0$ on $[T, \omega)$. From the left side of (2.1) and

$$\frac{x'(t)}{x(t)} = \frac{u'(t)}{u(t)} + \frac{z'(t)}{z(t)}, \quad T \leq t < \omega,$$
we conclude that
\[ \frac{p(t)x'(t)}{x(t)} \leq \frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)}, \quad T \leq t < \omega. \]

\[ \square \]

Proof of Theorem 1. Assume that \( y(t) > 0 \) in \((a, \omega)\). By Picone's identity [3], we have
\[ (2.3) \quad \frac{d}{dt} \left[ \frac{x_0}{y} (px_0' - P_0 y') \right] = (Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2}. \]

We observe that if \( x_0(a) = 0 \) then
\[ \lim_{t \to a} \frac{x_0(t)}{y(t)} (p(t)x_0'(t)y(t) - P(t)x_0(t)y'(t)) = -P(a)x_0(a)y(a) \lim_{t \to a} \frac{x_0(t)}{y(t)} = 0, \]
and that if \( x_0(a) \neq 0 \), \( y(a) \neq 0 \), and (1.3) holds, then
\[ \lim_{t \to a} \frac{x_0(t)}{y(t)} (p(t)x_0'(t)y(t) - P(t)x_0(t)y'(t)) = x_0(a)^2 \left( \frac{p(a)x_0'(a)}{x_0(a)} - \frac{P(a)y'(a)}{y(a)} \right) \geq 0. \]

Therefore, integrating (2.3) over \([\tau, t]\) and letting \( \tau \to a \), it follows that
\[ [x_0(t)]^2 \left( \frac{p(t)x_0'(t)}{x_0(t)} - \frac{P(t)y'(t)}{y(t)} \right) \geq \int_\tau^t \left[ (Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2} \right] ds \]
for \( a < t < \omega \). From Lemma 2, we have
\[ \int_\tau^t \left[ (Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2} \right] ds \leq 0, \quad a < t < \omega, \]
which implies that \( q(t) \equiv Q(t) \), \( p(t) \equiv P(t) \), and \( x_0(t)y'(t) \equiv x_0'(t)y(t) \) on \([a, \omega)\). Hence, \( y(t) \) is a constant multiple of \( x_0(t) \) on \([a, \omega)\). This completes the proof of Theorem 1.

\[ \square \]

Proof of Theorem 2. Let \( t = t_1 < t_2 < \cdots < t_n \) be zeros of \( x_0(t) \) in \((a, \omega)\). We note that \( y(t) \) satisfies either (i) or (ii) in Theorem A on \([a, t_n]\).

By applying Theorem 1 on \([t_n, \omega]\), we have either \( y(t) \) has at least one zero in \((t_n, \omega)\) or \( y(t) \) is a multiple constant of \( x_0(t) \) on \([t_n, \omega)\) and \( p(t) \equiv P(t) \) and \( q(t) \equiv Q(t) \) on \([t_n, \omega)\). In the former case, \( y(t) \) has at least \( n + 1 \) zeros in \((a, \omega)\). In the latter case, since \( y(t_n) = 0 \), we have either \( y(t) \) has at least \( n + 1 \) zeros in \((a, \omega)\) or \( y(t) \) is a multiple constant of \( x_0(t) \) on \([a, \omega)\) and \( p(t) \equiv P(t) \) and \( q(t) \equiv Q(t) \) on \([a, \omega)\). This completes the proof of Theorem 2.

\[ \square \]
References


