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<tr>
<td>Author(s)</td>
<td>Oaku, Toshinori</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 941: 52-56</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60136">http://hdl.handle.net/2433/60136</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
8.

An algorithm of computing \( b \)-functions

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8.1 Introduction

Let \( f(x) \in K[x] = K[x_1, \ldots, x_n] \) be a polynomial with coefficients in a field \( K \) of characteristic zero. Let us denote by

\[
\hat{D}_n := K[[x_1, \ldots, x_n]][\partial_1, \ldots, \partial_n]
\]

the ring of differential operators with formal power series coefficients with \( \partial_i = \partial/\partial x_i \) and \( \partial = (\partial_1, \ldots, \partial_n) \). (If \( K \) is a subfield of the field \( \mathbb{C} \) of complex numbers, then we can use the ring \( \mathcal{D}_n \) of differential operators with convergent power series coefficients instead of \( \hat{D}_n \). This makes no difference in the definition below.) Let \( s \) be a parameter. Then the (local) \( b \)-function (or the Bernstein-Sato polynomial) \( b_f(s) \) associated with \( f(x) \) is the monic polynomial of the least degree \( b(s) \in K[s] \) satisfying

\[
P(s, x, \partial)f(x)^{s+1} = b(s)f(x)^s
\]

with some \( P(s, x, \partial) \in \hat{D}_n[s] \).

We present an algorithm of computing the \( b \)-function \( b_f(s) \) for an arbitrary \( f(x) \in K[x] \). A system \( \text{Kan} \) of N. Takayama [T2] is available for actual execution of our algorithm.

An algorithm of computing \( b_f(s) \) was first given by M. Sato et al. [SKKO] when \( f(x) \in \mathbb{C}[x] \) is a relative invariant of a prehomogeneous vector space. J. Briançon et al. [BGMM], [M] have given an algorithm of computing \( b_f(s) \) for \( f(x) \in \mathbb{C}[x] \) with isolated singularity. Also note that T. Yano [Y] worked out many interesting examples of \( b \)-functions systematically.
8.2 Algorithm

Notation

- $K$: a field of characteristic 0;
- $A_{n+1} := K[t, x_1, \ldots, x_n][\partial_t, \partial_1, \ldots, \partial_n]$ (where $\partial_t := \partial/\partial t$, $\partial_\ast := \partial/\partial x_i$);
- $\prec_F$: a (total) order on $\mathbb{N}^{2n+2}$ with $\mathbb{N} := \{0, 1, 2, \ldots\}$ that satisfies the following conditions:
  (A-1) $\alpha \succ_F \beta \implies \alpha + \gamma \succ_F \beta + \gamma$ ($\forall \alpha, \beta, \gamma \in \mathbb{N}^{2n+2}$);
  (A-3) $\nu - \mu \succ \nu' - \mu'$ ($\forall \mu, \nu, \nu', \mu' \in \mathbb{N}$, $\forall \alpha, \beta, \alpha', \beta' \in \mathbb{N}^n$);
  (A-4) $(\mu, \mu, \alpha, \beta) \succeq_F (0, 0, 0, 0)$ ($\forall \mu \in \mathbb{N}$, $\forall \alpha, \beta \in \mathbb{N}^n$),

where $(\mu, \nu, \alpha, \beta)$ corresponds to the 'monomial' $t^\mu x^\alpha \partial_\nu \partial^\beta$.

Note that $\succ_F$ does not satisfy (A-2).

For each integer $m$, define a $K$-subspace of $A_{n+1}$ by

$F_m(A_{n+1}) := \{ P = \sum_{\mu, \nu, \alpha, \beta} a_{\mu, \nu, \alpha, \beta} t^\mu x^\alpha \partial_\nu \partial^\beta \in A_{n+1} \mid a_{\mu, \nu, \alpha, \beta} = 0 \text{ if } \nu - \mu > m \}.$

If $P \neq 0$, its $F$-order $\text{ord}_F(P)$ is defined as the minimum integer $m$ such that $P \in F_m(A_{n+1})$.

Then

$\hat{\sigma}(P) = \hat{\sigma}_m(P) := \sum_{\nu - \mu = m} a_{\mu, \nu, \alpha, \beta} t^\mu x^\alpha \partial_\nu \partial^\beta$

is called the formal symbol of $P$. We define $\psi(P)(s) \in A_n[s]$ by

$\hat{\sigma}_0(t^m P) = \psi(P)(t\partial_t)$ if $m \geq 0$,

$\hat{\sigma}_0(t^{-m} P) = \psi(P)(t\partial_t)$ if $m < 0$.

Definition 1 For $i, j, \mu, \nu, \nu', \nu'' \in \mathbb{N}$, $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^n$, an order $\prec_H$ on $\mathbb{N}^{2n+3}$ is defined by

$(i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta') \iff (i > j)$

or $(i = j$ and $(\mu + \ell, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta')$)

or $(i = j, (\nu, \alpha, \beta) = (\nu', \alpha', \beta'), \mu > \mu')$

with $\ell, \ell' \in \mathbb{N}$ s.t. $\nu - \mu - \ell = \nu' - \mu' - \ell'$, where $(i, \mu, \nu, \alpha, \beta)$ corresponds to $t^\mu x^\nu \partial^\beta$. This definition is independent of the choice of $\ell, \ell'$, and $\succ_F$ satisfies (A-1) and (A-2).

In the following algorithm, we also use an order $\prec$ on $\mathbb{N}^{2n+1}$ satisfying (A-1), (A-2) (with $2n+2$ replaced by $2n+1$) and

(A-5) if $|\beta| > |\beta'|$, then $(\mu, \alpha, \beta) \succ (\mu', \alpha', \beta')$ for any $\mu, \mu' \in \mathbb{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^n$. 

where \((\mu, \alpha, \beta)\) corresponds to \(s^{\mu}x^{\alpha}\partial^{\beta}\).

**Algorithm 2**

**Input:** \(f(x) \in K[x]\);

1. Let \(G\) be a Gröbner basis of the left ideal of \(A_{n+1}[x_0]\) generated by \(t - x_0f(x)\) and \(\partial_i + x_0(\partial f/\partial x_i)\partial_i\) \((i = 1, \ldots, n)\) with respect to \(\prec_H\);
2. Compute a Gröbner basis \(H\) of the left ideal of \(A_n[s]\) generated by \(\psi(G) := \{\psi(P(1)) | P(x_0) \in G\}\) w.r.t. an order satisfying (A-1), (A-2), (A-5);
3. Let \(J\) be the ideal of \(K[x, s]\) generated by \(H \cap K[x, s] = \{f_1(x, s), \ldots, f_k(x, s)\}\);
4. Compute the monic generator \(f_0(s)\) of the ideal of \(K[s]\) generated by \(f_1(0, s), \ldots, f_k(0, s)\) by Gröbner basis or GCD computation; if \(f_0(s) = 1\), then put \(b(s) := 1\) and quit;
5. Compute the factorization \(f_0(s) = (s - s_1)^{\mu_1}\ldots(s - s_m)^{\mu_m}\) in \(\overline{K}[s]\) (\(\overline{K}\) : the algebraic closure of \(K\));
6. Put \(\overline{J} := \overline{K}[x, s]J\).

For \(i := 1\) to \(m\) do

By computing the ideal quotient \(\overline{J} : (s - s_i)^{\ell}\) for \(\ell = \mu_1, \mu_1 + 1, \ldots\) repeatedly, determine the least \(\ell \geq \mu_i\) such that \(\overline{J} : (s - s_i)^{\ell}\) contains an element which does not vanish at \((x, s) = (0, s_i)\). Denote this \(\ell\) by \(\ell_i\);
7. Put \(b(s) := (s - s_1)^{\ell_1}\ldots(s - s_m)^{\ell_m}\);

**Output:** \(b_f(s) := b(-s - 1) \in K[s]\);

**Remark 3** A theorem of Kashiwara [K] states that the roots of \(b_f(s)\) are negative rational numbers. Hence in steps 5 and 6, there is no need of field extension.

We have implemented the steps 1 and 2 of the above algorithm in Kan/sml [T2], and the steps 3–7 in Risa/Asir [NS]. In the following table, the timing data refer to the computation time of steps 1 and 2, which are naturally the most expensive part of our algorithm.
In the above table, the last four examples have non-isolated singularities. Hence, as far as the author knows, no algorithm has been known for computing $b$-functions for these polynomials. See [Y, pp. 198–200] for estimates of the $b$-functions of $x^3 + y^2 z^2$, $x^3 + y^2 - 3xyz$, $x^3 + xyz$.

Acknowledgement: The author would like to express his gratitude to Professor N. Takayama of Kobe University for kind assistance in using Kan, without which implementation of our algorithm...

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$b_f(s)$</th>
<th>timing data by Kan on S-4/20</th>
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<tbody>
<tr>
<td>$x^3 - y^2$</td>
<td>$(s + 1)(s + \frac{5}{6})(s + \frac{7}{6})$</td>
<td>0.2s</td>
</tr>
<tr>
<td>$(x^3 - y^2)^2$</td>
<td>$(s + 1)(s + \frac{1}{12})(s + \frac{5}{12})(s + \frac{1}{2})(s + \frac{7}{12})(s + \frac{11}{12})$</td>
<td>0.7s</td>
</tr>
<tr>
<td>$x^5 - y^2$</td>
<td>$(s + 1)(s + \frac{7}{10})(s + \frac{9}{10})(s + \frac{11}{10})(s + \frac{13}{10})$</td>
<td>0.2s</td>
</tr>
<tr>
<td>$x^5 + y^5$</td>
<td>$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})(s+\frac{8}{5})$</td>
<td>0.8s</td>
</tr>
<tr>
<td>$x^5 + y^5 + x^2 y^3$</td>
<td>$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})$</td>
<td>180s</td>
</tr>
<tr>
<td>$x^3 y + y^3 + z^2$</td>
<td>$(s + 1)(s + \frac{35}{18})(s + \frac{31}{18})(s + \frac{29}{18}) \times (s + \frac{3}{2})(s + \frac{25}{18})(s + \frac{23}{18})(s + \frac{19}{18})$</td>
<td>5s</td>
</tr>
<tr>
<td>$x^5 + y^3 + z^2$</td>
<td>$(s + 1)(s + \frac{59}{30})(s + \frac{53}{30})(s + \frac{49}{30})(s + \frac{47}{30}) \times (s + \frac{43}{30})(s + \frac{41}{30})(s + \frac{37}{30})(s + \frac{31}{30})$</td>
<td>7s</td>
</tr>
<tr>
<td>$x^3 + y^2 z^2$</td>
<td>$(s + 1)(s + \frac{5}{6})^2(s + \frac{7}{6})^2(s + \frac{4}{3})(s + \frac{5}{3})$</td>
<td>0.5s</td>
</tr>
<tr>
<td>$x^3 + y^3 - 3xyz$</td>
<td>$(s + 1)^3(s + \frac{4}{3})(s + \frac{5}{3})$</td>
<td>2.5s</td>
</tr>
<tr>
<td>$x^3 + xyz$</td>
<td>$(s + 1)^3(s + \frac{4}{3})(s + \frac{5}{3})$</td>
<td>0.5s</td>
</tr>
<tr>
<td>$x^4 + y^2 z^2 + x^3 y^3$</td>
<td>$(s + 1)^3(s + \frac{3}{4})^2(s + \frac{5}{6})^2(s + \frac{7}{6})^2(s + \frac{5}{4})^2 \times (s + \frac{11}{12})(s + \frac{13}{12})(s + \frac{4}{3})(s + \frac{17}{12})(s + \frac{3}{2}) \times (s + \frac{19}{12})(s + \frac{5}{3})(s + \frac{7}{4})$</td>
<td>180s</td>
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would have been much more difficult.

参考文献


