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An algorithm of computing b-functions

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8.1 Introduction

Let $f(x) \in K[x] = K[x_1, \ldots, x_n]$ be a polynomial with coefficients in a field K of characteristic zero. Let us denote by

$$\hat{\mathcal{D}}_n := K[[x_1, \ldots, x_n]]\langle \partial_1, \ldots, \partial_n \rangle$$

the ring of differential operators with formal power series coefficients with $\partial_i = \partial/\partial x_i$ and $\partial = (\partial_1, \ldots, \partial_n)$. (If K is a subfield of the field $\mathbf{C}$ of complex numbers, then we can use the ring $\mathcal{D}_n$ of differential operators with convergent power series coefficiets instead of $\hat{\mathcal{D}}_n$. This makes no difference in the definition below.) Let s be a parameter. Then the (local) b-function (or the Bernstein-Sato polynomial) $b_f(s)$ associated with $f(x)$ is the monic polynomial of the least degree $b(s) \in K[s]$ satisfying

$$P(s, x, \partial) f(x)^{s+1} = b(s) f(x)^s$$

with some $P(s, x, \partial) \in \hat{\mathcal{D}}_n[s]$.

We present an algorithm of computing the b-function $b_f(s)$ for an arbitrary $f(x) \in K[x]$. A system *Kan* of N. Takayama [T2] is available for actual execution of our algorithm.

An algorithm of computing $b_f(s)$ was first given by M. Sato et al. [SKKO] when $f(x) \in \mathbf{C}[x]$ is a relative invariant of a prehomogeneous vector space. J. Briançon et al. [BGMM], [M] have given an algorithm of computing $b_f(s)$ for $f(x) \in \mathbf{C}\{x\}$ with isolated singularity. Also note that T. Yano [Y] worked out many interesting examples of b-functions systematically.

8.2 Algorithm

Notation

$\bullet K$: a field of characteristic 0;

$\bullet A_{n+1} := K[t, x_1, \ldots, x_n]\langle \partial_t, \partial_1, \ldots, \partial_n \rangle \qquad (\partial_t := \partial/\partial t, \quad \partial_i = \partial/\partial x_i)$

$\bullet \prec_F$: a (total) order on $\mathbf{N}^{2n+2}$ with $\mathbf{N} := \{0, 1, 2, \ldots\}$ that satisfies the following conditions:

(A-1) $\alpha \succ_F \beta \Longrightarrow \alpha + \gamma \succ_F \beta + \gamma \ (\forall \alpha, \beta, \gamma \in \mathbf{N}^{2n+2})$;

(A-3) $\nu - \mu > \nu' - \mu' \quad \Longrightarrow \quad (\mu, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta') \quad (\forall \mu, \nu, \mu', \nu' \in \mathbf{N}, \ \forall \alpha, \beta, \alpha', \beta' \in \mathbf{N}^n)$;

(A-4) $(\mu, \mu, \alpha, \beta) \succeq_F (0, 0, 0, 0) \quad (\forall \mu \in \mathbf{N}, \ \forall \alpha, \beta \in \mathbf{N}^n)$,

where $(\mu, \nu, \alpha, \beta)$ corresponds to the 'monomial' $t^\mu x^\alpha \partial_t^\nu \partial^\beta$. Note that $\succ_F$ does not satisfy

(A-2) $\alpha \succeq_F 0 \quad (\forall \alpha \in \mathbf{N}^{2n+2})$.

For each integer m, define a K-subspace of A_{n+1} by

$$F_m(A_{n+1}) := \{P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in A_{n+1} \mid a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m\}.$$

If $P \neq 0$, its *F-order* $\mathrm{ord}_F(P)$ is defined as the minimum integer m such that $P \in F_m(A_{n+1})$. Then

$$\hat{\sigma}(P) = \hat{\sigma}_m(P) := \sum_{\nu - \mu = m} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta$$

is called the *formal symbol* of P. We define $\psi(P)(s) \in A_n[s]$ by

$$\hat{\sigma}_0(t^m P) = \psi(P)(t\partial_t) \quad \text{if} \quad m \geq 0,$$

$$\hat{\sigma}_0(\partial_t^{-m} P) = \psi(P)(t\partial_t) \quad \text{if} \quad m < 0.$$

Definition 1 For $i, j, \mu, \nu, \mu', \nu' \in \mathbf{N}$, $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, an order $\prec_H$ on $\mathbf{N}^{2n+3}$ is defined by

$$\begin{aligned} (i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta') \quad &\Longleftrightarrow \quad (i > j) \\ \text{or} \quad &(i = j \text{ and } (\mu + \ell, \nu, \alpha, \beta) \succ_F (\mu' + \ell', \nu', \alpha', \beta')) \\ \text{or} \quad &(i = j, \ (\nu, \alpha, \beta) = (\nu', \alpha', \beta'), \ \mu > \mu') \end{aligned}$$

with $\ell, \ell' \in \mathbf{N}$ s.t. $\nu - \mu - \ell = \nu' - \mu' - \ell'$, where $(i, \mu, \nu, \alpha, \beta)$ corresponds to $t^\mu x_0^i x^\alpha \partial^\beta$. This definition is independent of the choice of ℓ, ℓ', and $\succ_H$ satisfies (A-1) and (A-2).

In the following algorithm, we also use an order $\prec$ on $\mathbf{N}^{2n+1}$ satisfying (A-1), (A-2) (with $2n+2$ replaced by $2n+1$) and

(A-5) if $|\beta| > |\beta'|$, then $(\mu, \alpha, \beta) \succ (\mu', \alpha', \beta')$ for any $\mu, \mu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, where (μ, α, β) corresponds to $s^\mu x^\alpha \partial^\beta$.

Algorithm 2

Input: $f(x) \in K[x]$;

1. Let $\mathbf{G}$ be a Gröbner basis of the left ideal of $A_{n+1}[x_0]$ generated by $t - x_0 f(x)$ and $\partial_i + x_0(\partial f/\partial x_i)\partial_t$ $(i = 1, \ldots, n)$ with respect to $\prec_H$;

2. Compute a Gröbner basis $\mathbf{H}$ of the left ideal of $A_n[s]$ generated by $\psi(\mathbf{G}) := \{\psi(P(1)) \mid P(x_0) \in \mathbf{G}\}$ w.r.t. an order satisfying (A-1), (A-2), (A-5);

3. Let J be the ideal of $K[x, s]$ generated by $\mathbf{H} \cap K[x, s] = \{f_1(x, s), \ldots, f_k(x, s)\}$;

4. Compute the monic generator $f_0(s)$ of the ideal of $K[s]$ generated by $f_1(0, s), \ldots, f_k(0, s)$ by Gröbner basis or GCD computation; if $f_0(s) = 1$, then put $b(s) := 1$ and quit;

5. Compute the factorization $f_0(s) = (s - s_1)^{\mu_1} \ldots (s - s_m)^{\mu_m}$ in $\overline{K}[s]$ ($\overline{K}$: the algebraic closure of K);

6. Put $\overline{J} := \overline{K}[x, s]J$.

   For $i := 1$ to m do

   By computing the ideal quotient $\overline{J} : (s - s_i)^\ell$ for $\ell = \mu_i, \mu_i + 1, \ldots$ repeatedly, determine the least $\ell \geq \mu_i$ such that $\overline{J} : (s - s_i)^\ell$ contains an element which does not vanish at $(x, s) = (0, s_i)$. Denote this ℓ by ℓ_i;

7. Put $b(s) := (s - s_1)^{\ell_1} \ldots (s - s_m)^{\ell_m}$;

Output: $b_f(s) := b(-s - 1) \in K[s]$;

Remark 3 A theorem of Kashiwara [K] states that the roots of $b_f(s)$ are negative rational numbers. Hence in steps 5 and 6, there is no need of field extension.

We have implemented the steps 1 and 2 of the above algorithm in Kan/sm1 [T2], and the steps 3–7 in Risa/Asir [NS]. In the following table, the timing data refer to the computation time of steps 1 and 2, which are naturally the most expensive part of our algorithm.

$f(x)$	$b_f(s)$	timing data by Kan on S-4/20
$x^3 - y^2$	$(s+1)(s+\frac{5}{6})(s+\frac{7}{6})$	0.2s
$(x^3 - y^2)^2$	$(s+1)(s+\frac{1}{12})(s+\frac{5}{12})(s+\frac{1}{2})(s+\frac{7}{12})(s+\frac{11}{12})$	0.7s
$x^5 - y^2$	$(s+1)(s+\frac{7}{10})(s+\frac{9}{10})(s+\frac{11}{10})(s+\frac{13}{10})$	0.2s
$x^5 + y^5$	$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})(s+\frac{8}{5})$	0.8s
$x^5 + y^5 + x^3y^3$	$(s+1)^2(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})$	180s
$x^3y + y^3 + z^2$	$(s+1)(s+\frac{35}{18})(s+\frac{31}{18})(s+\frac{29}{18})$ $\times(s+\frac{3}{2})(s+\frac{25}{18})(s+\frac{23}{18})(s+\frac{19}{18})$	5s
$x^5 + y^3 + z^2$	$(s+1)(s+\frac{59}{30})(s+\frac{53}{30})(s+\frac{49}{30})(s+\frac{47}{30})$ $\times(s+\frac{43}{30})(s+\frac{41}{30})(s+\frac{37}{30})(s+\frac{31}{30})$	7s
$x^3 + y^2z^2$	$(s+1)(s+\frac{5}{6})^2(s+\frac{7}{6})^2(s+\frac{4}{3})(s+\frac{5}{3})$	0.5s
$x^3 + y^3 - 3xyz$	$(s+1)^3(s+\frac{4}{3})(s+\frac{5}{3})$	2.5s
$x^3 + xyz$	$(s+1)^3(s+\frac{4}{3})(s+\frac{5}{3})$	0.5s
$x^4 + y^2z^2 + x^3y^3$	$(s+1)^3(s+\frac{3}{4})^2(s+\frac{5}{6})^2(s+\frac{7}{6})^2(s+\frac{5}{4})^2$ $\times(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{4}{3})(s+\frac{17}{12})(s+\frac{3}{2})$ $\times(s+\frac{19}{12})(s+\frac{5}{3})(s+\frac{7}{4})$	180s

In the above table, the last four examples have non-isolated singularities. Hence, as far as the author knows, no algorithm has been known for computing b-functions for these polynomials. See [Y, pp. 198–200] for estimates of the b-functions of $x^3 + y^2z^2$, $x^3 + y^3 - 3xyz$, $x^3 + xyz$.

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