7. Bimodular Type Simple K3 Singularities

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7.1 Introduction

Let $f_1, f_2, \ldots, f_r$ be holomorphic functions defined in an open set $U$ of the complex space $\mathbb{C}^n$. Let $X$ be the analytic set $f_1^{-1}(0) \cap \ldots \cap f_r^{-1}(0)$. Let $x \in X$, and let $g_1, g_2, \ldots, g_s$ be a system of generators of ideal $I(X)_{x_0}$ of the generators of the holomorphic functions which vanish identically on a neighborhood of $x_0$ in $X$. $x_0$ is called a simple point of $X$ if the matrix $(\partial g_i / \partial x_j)$ attains its maximal rank. Otherwise, $x_0$ is called a singular point(singularity) of $X$. For $r = 1$, $x_0$ is called a Hypersurface singularity of $X$. Let $V$ be an analytic set in $\mathbb{C}^n$. A singular point $x_0$ of $V$ is said to be isolated if, for some open neighborhood $W$ of $x_0$ in $\mathbb{C}^n$, $W \cap V - \{x_0\}$ is a smooth submanifold of $W - \{x_0\}$.

Example
For a holomorphic function $f(z_0, \ldots, z_n)$ defined in a neighborhood $U$ of the origin in $\mathbb{C}^{n+1}$, let $X = \{(z_0, \ldots, z_n) \in U | f(z_0, \ldots, z_n) = 0\}$. Then if

$$\{x = (0, \ldots, 0)\} = \{\frac{\partial f}{\partial z_0} = \ldots = \frac{\partial f}{\partial z_n} = 0\} \cap X,$$

$X$ has an isolated singularity at $x$. 
Let \((X, x)\) be a germ of normal isolated singularity of dimension \(n\). Suppose that \(X\) is a Stein space. Let \(\pi : (M, E) \rightarrow (X, x)\) be a resolution of singularity. Then for \(1 \leq i \leq n - 1\), 
\(dim(R^i\pi, \partial_M)_X\) is finite. \(R^i\pi, \partial_M\) has support on \(x\). They are independent of the resolution.
In fact
\[
dim(R^i\pi, \partial_M)_X = \dim H^{i+1}_X(X, \partial_M) \quad (1 \leq i \leq n - 2)
\]
and
\[
dim(R^{n-1}\pi, \partial_M)_X = \frac{\dim \Gamma(X - \{x\}, \partial K)}{L^2(X - \{x\})}.
\]
where \(L^2(X - \{x\})\) is the subspace of \(\Gamma(X - \{x\}, \partial K)\) consisting of \(n\)-form on \(X - \{x\}\) which are square integrable near \(x\).
We denote them by
\[
h^i(X, x) := \dim(R^i\pi, \partial_M)_X \quad (1 \leq i \leq n - 2)
\]
and
\[
P_g(X, x) := \dim(R^n\pi, \partial_M)_X.
\]
The invariant \(P_g(X, x)\) is called the geometric genus of \((X, x)\).

In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. What are natural generalizations in three-dimensional case of those singularities. They are purely elliptic singularities. We define the purely elliptic singularities.

Definition ([2])
For each positive integer \(m\), the \(m\)-genus of a normal isolated singularity \((X, x)\) in an \(n\)-dimensional analytic space is defined to be
\[
\delta_m(X, x) = \frac{\dim \Gamma(X - \{x\}, \partial (mK))}{L^{2/m}(X - \{x\})},
\]
where \(K\) is the canonical line bundle on \(X - \{x\}\), and \(L^{2/m}(X - \{x\})\) is the set of all holomorphic \(m\)-ple \(n\)-forms on \(X - \{x\}\) which are \(L^{2/m}\)-integrable at \(x\).

The \(m\)-genus \(\delta_m\) is finite and does not depend on the choice of a Stein neighborhood \(X\).
A singularity $(X, x)$ is said to be purely elliptic if $\delta_m(X, x) = 1$ for every positive integer $m$.

When $X$ is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities. In higher dimensions, however, purely elliptic singularities are not always quasi-Gorenstein.

Theorem (4)
Let $(X, x)$ be a quasi-Gorenstein normal isolated singularity of dimension 3, then

$$2\{P_g(X, x) - \frac{-K_{MC_2}(M)}{24}\} = \dim \mathcal{H}^1(M, \mathcal{O}).$$

Consequently quasi-Gorenstein purely elliptic singularities of dimension 3 are classified into 6 classes.

1. $h^1(X, x) = 2p$, $(0,0)$-type.
   
   $p = 1 \rightarrow$ Hilbert modular cusp singularities.
   
   $p > 1 \rightarrow$ Tsuchihashi cusp singularities (1).
2. $h^1(X, x) = 2$, $(0,1)$-type.
3. $h^1(X, x) = 2$, $(0,2)$-type.
4. $h^1(X, x) = 0$, $(0,0)$-type.
5. $h^1(X, x) = 0$, $(0,1)$-type.
6. $h^1(X, x) = 0$, $(0,2)$-type.

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of $(0,1)$-type and of $(0,0)$-type, respectively.

Definition ([2])
A three-dimensional singularity $(X, x)$ is a simple $K3$ singularity if the following two equivalent (Watanabe-Ishii[5]) conditions are satisfied:

1. $(X, x)$ is a Gorenstein purely elliptic singularity of $(0,2)$-type.
2. The exceptional divisor $D$ is a normal $K3$ surface for any $Q$-factorial terminal modification $\delta : (Y, D) \rightarrow (X, x)$.

The notion of a simple $K3$ singularity is defined as a three-dimensional isolated Gorenstein purely elliptic singularity of $(0,2)$-type.
Example
Let \( f(x, y, z, w) \) be a quasi-homogeneous polynomial of type \((p, q, r, s : h)\) with \( p + q + r + s = h \), and suppose \( f(x, y, z, w) = 0 \) defines an isolated singularity at the origin in \( C^4 \). Then the origin is a simple K3 singularity.

Next we consider the case where \((X, x)\) is a hypersurface singularity defined by a nondegenerate polynomial

\[
f = \sum a_v x^v \in C[x_0, x_1, \ldots, x_n],
\]

and \( x = 0 \in C^{n+1} \). Recall that the Newton boundary \( \Gamma(f) \) of \( f \) is the union of the compact faces of \( \Gamma_+(f) \), where \( \Gamma_+(f) \) is the convex hull of \( \bigcup_{a_n \neq 0} (n + R_0^{n+1}) \) in \( R^{n+1} \). For any face \( \Delta \) of \( \Gamma_+(f) \), set \( f_\Delta := \sum_{n \in \Delta} a_v x^v \). We say \( f \) to be nondegenerate, if

\[
\frac{\partial f_\Delta}{\partial x_0} = \frac{\partial f_\Delta}{\partial x_1} = \ldots = \frac{\partial f_\Delta}{\partial x_n} = 0
\]

has no solution in \((C^*)^{n+1}\) for any face \( \Delta \).

When \( f \) is nondegenerate, the condition for \((X, x)\) to be a purely elliptic singularity is given as follows:

Theorem ([3])
Let \( f \) be a nondegenerate polynomial and suppose \( X = f = 0 \) has an isolated singularity at \( x = 0 \in C^{n+1} \).
(1) \((X, x)\) is purely elliptic if and only if \((1, 1, \ldots, 1) \in \Gamma(f)\).
(2) Let \( n = 3 \) and let \( \Delta_0 \) be the face of \( \Gamma(f) \) containing \((1, 1, 1, 1)\) in the relative interior of \( \Delta_0 \). Then \((X, x)\) is a simple K3 singularity if and only if \( \dim_R \Delta_0 = 3 \).

Thus if \( f \) is nondegenerate and defines a simple K3 singularity, then \( f_{\Delta_0} \) is a quasi-homogeneous polynomial of a uniquely determined weight \( a \) called the weight of \( f \).

Yonemura([6]) classified nondegenerate hypersurface simple K3 singularities into ninety five classes in terms of the weight of \( f \).
7.2 Parameters in a defining equation

Yonemura calculate the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained examples such that the polynomial $f$ is quasi-homogeneous and that $\{f = 0\} \subset C^4$ has a simple $K3$ singularity at the origin. The minimum number of parameters in the polynomial is less than or equal to 19 and is associated with the moduli of the $K3$ surface with singularities. The need for a unique form may be questioned. However, defining equations were not unique. So, in this section, we try to impose a condition to construct a unique form for quasi-homogeneous polynomials and decide conditions of their parameters.

We can take the following form for a weighted quasi-homogeneous polynomial $f$ in $C^{n+1}$ with the coordinate $[x_0, x_1, x_2, \ldots, x_n]$:

$$f = f_0 + f_1 + f_2 + \ldots + f_m$$

where $f_i(0 \leq i \leq m)$ is a homogeneous polynomial of degree $i$ in $C^{n+1}$. And let $W = (w_0, w_1, \ldots, w_n)$ be the weight. Then we can take the following form for the homogeneous polynomial of each degree $i$:

$$\sum_{k_0 + k_1 + \ldots + k_n = i} a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n} \quad (k_i \in \mathbb{N}_0, 0 \leq i \leq n).$$

Let $\ll$ be the lexical linear ordering of the terms of the homogeneous polynomials for $0 \leq i \leq m$ in turn from the minimal term to the maximal term given below:

Definition

Let $K = (k_0, k_1, \ldots, k_n)$ ($k_i \in \mathbb{N}_0, 0 \leq i \leq n$) and let $a_K X^K$ denote the term

$$a_K X^K = a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}.$$

Then $a_K X^K \ll b_L X^L$ if there exists an integer $s(0 \leq s \leq n)$ such that $k_i = l_i$ for $m = 0, 1, \ldots, s - 1$ and $k_s < l_s$.

Example

$$x_0^3 \ll x_0^2 x_1 \ll x_0^4 \ll x_1^4 \ll x_0^5$$

Hereafter, for the sake of simplicity, we shall sometimes omit the coefficients in indicating terms.
We will consider the following procedure by using this ordering.

**Step 1**
We try a term $X^{K_i}$ to eliminate by a suitable analytic transformation with respect to $X$. We find a condition of the coefficient of term $X^{K_i}$ where we can make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. We classify the following two cases by the above condition.

Case 1: We can make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll x^{K_i}$. We classify the following two cases by the above condition.

Case 1: We can make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll X^{K_i}$.

Case 2: Otherwise for case 1.

For the condition of case 1, we make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. For the condition of case 2, we don’t use the analytic transformation and go to next step.

**Step 2**
We make the coefficient of the term which determine the weight $(w_0, w_1, \ldots, w_{n-1})$ for the quasi-homogeneous polynomial equal to 1 by the magnification of the coordinate.

Let $W_4$ be the set of defining equations which has a nondegenerate hypersurface simple $K3$ singularity at the origin and let $\#m(f)$ be the minimum number of parameters of defining equation for any $f \in W_4$. Then for $\#m(f) = i(1 \leq i \leq 3)$, there exists 3 types, 8 types, 7 types, respectively. In general, the relation of the parameters is a simultaneous equation of them.

### 7.3 Relation of parameters

For $\#m(f) = 2$, we consider the relation of parameters in a defining equation of nondegenerate hypersurface simple $K3$ singularity which is constructed by the procedure in section 2. Then we obtain the following results:

**Result**

\[
\begin{align*}
f_{84} : & x_0^3 + \lambda x_0 x_1 x_2 x_3 + x_0 x_2^3 + x_1^2 x_2 + x_1 x_3 + \mu x_2^2 x_3^2 = 0 \quad ((\lambda^2 + 27\mu)\mu - \lambda^2)^2 \neq (2(9\lambda\mu - 8))^2, \\
f_{88} : & x_0^2 x_1 + x_1^2 x_3 + x_0 x_3 + x_2^3 + \lambda x_1 x_2^2 x_3 + \mu x_2 x_3^2 = 0 \\
& 16\mu(125\lambda^2 + 8\lambda 4\mu + 225\lambda\mu^2 + 4\lambda^3\mu^2 + 108\mu^4) \neq 3125 - 64\lambda^5.
\end{align*}
\]

The number $n$ of $f_n$ denotes the number of the defining equation in the classification by Yonemura.
We show a file (program) of Mathematica for the above computation. The file name is the same name as the $f_n$ in the above result.

File: f84

File:

f84 = x0^3 + p x0 x1 x2 x3 + x0 x2 + x3 + x0 x2^2 + x1 x3^4 + q x2^2 + x1 x3^2 + x1^3;
dx0 = D[f84, x0]; dx1 = D[f84, x1]; dx2 = D[f84, x2]; dx3 = D[f84, x3];
e0 = dx0; e1 = Expand[(9 dx0 x0 - 8 dx1 x1 - 3 dx2 x2 + 2 dx3 x3)/27];
e2 = Expand[(-8 dx1 x1 + 6 dx2 x2 + 2 dx3 x3)/(-18 x2)];
e3 = Expand[(dx1 x1 - 3 dx2 x2 + 2 dx3 x3)/9];
f0 = Expand[e0 x2^2]; f1 = Expand[e1 x2^2]; f2 = Expand[e2 x2^2]; f3 = Expand[e3 x2^2];
g0 = 3 a - 2 + p b d; g1 = a - 3 c - b - 3; g2 = b - 3 - q c - 2 d - 3 - a - 2; g3 = b d - 4 - a c;
a = (b d - 4)/c; j0 = y^3 + 3 x^2 z + p x y^2; j1 = y - 2 z^2; j2 = x^3 - x y z - q y^2 z;
y = z; s0 = Expand[j0/(z - 3)]; s1 = Expand[j2/(z - 3)];
u1 = 3 t^2 p + t; v2 = - 3 - t - q;
y = - z; 
t0 = Expand[j0/(z - 3)]; t2 = Expand[j2/(z - 3)];
v1 = 3 t^2 p + t; t = t^3 - t - q;
Timing[Reduce[{u1, u2} == {0, 0}, t]]

Timing[Reduce[{v1, v2} == {0, 0}, t]]

File: f86

f86 = x2^2 x1 + x1^3 x3 + x0 x3^4 + x2^5 + p x1 x2^2 + x3^2 + x2 x3^5 + q x2 x3^4 + x2^5 + p x1 x2^2 + x3^2 + x2 x3^5;
dx0 = D[f86, x0]; dx1 = D[f86, x1]; dx2 = D[f86, x2]; dx3 = D[f86, x3];
g0 = dx0; g1 = dx1; g2 = Expand[(dx3 x3 - 5 dx2 x2 + 8 dx1 x1 - 4 dx0 x0)/25]; g3 = dx3;
x0 = (- (x3^4)/(x1^3)); x3 = (x2^5)/(x1^3);
h1 = Expand[(g1 4 x1^26)/(x2^5)]; h2 = Expand[g3 x1^22];
u1 = 12 a^5 + 4 p a^4 b + b^5; u2 = a^5 + 2 p a^4 b + 5 q a^2 b^3 - 2 b^5;
v1 = 12 t^5 + 4 p t^4 + 1; v2 = v t^5 + 2 p t^4 + 5 q t^2 - 2;
Timing[Reduce[{v1, v2} == {0, 0}, t]]

Timing[Reduce[{v1, v2} == {0, 0}, t]]

File: 1

File 1: f84

Timing[Reduce[{u1, u2} == {0, 0}, t]]

Timing[Reduce[{v1, v2} == {0, 0}, t]]

Timing[Reduce[{u1, u2} == {0, 0}, t]]

2.966666666667152*Second,

2000*p^2*q + 128*p^4*q^2 + 3600*p*q^3 + 64*p^3*q^4 +
1728*q^5 == 3125 - 64*p^5 &&
t == (-369140625*p - 585000*p^6 - 2048*p^11 +

Timing[Reduce[{u1, u2} == {0, 0}, t]]
参考文献