7. Bimodular Type Simple K3 Singularities

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7.1 Introduction

Let $f_1, f_2, \ldots, f_r$ be holomorphic functions defined in an open set $U$ of the complex space $C^n$. Let $X$ be the analytic set $f_1^{-1}(0) \cap \ldots \cap f_r^{-1}(0)$. Let $x \in X$, and let $g_1, g_2, \ldots, g_s$ be a system of generators of ideal $I(X)_{x_0}$ of the generators of the holomorphic functions which vanish identically on a neighborhood of $x_0$ in $X$. $x_0$ is called a simple point of $X$ if the matrix $(\partial g_i/\partial x_j)$ attains its maximal rank. Otherwise, $x_0$ is called a singular point(singularity) of $X$. For $r = 1$, $x_0$ is called a Hypersurface singularity of $X$. Let $V$ be an analytic set in $C^n$. A singular point $x_0$ of $V$ is said to be isolated if, for some open neighborhood $W$ of $x_0$ in $C^n$, $W \cap V - \{x_0\}$ is a smooth submanifold of $W - \{x_0\}$.

Example
For a holomorphic function $f(z_0, \ldots, z_n)$ defined in a neighborhood $U$ of the origin in $C^{n+1}$, let $X = \{(z_0, \ldots, z_n) \in U | f(z_0, \ldots, z_n) = 0\}$.

Then if

$$\{x = (0, \ldots, 0)\} = \left\{ \frac{\partial f}{\partial z_0} = \ldots = \frac{\partial f}{\partial z_n} = 0 \right\} \cap X,$$

$X$ has an isolated singularity at $x$. 
Let \((X, x)\) be a germ of normal isolated singularity of dimension \(n\). Suppose that \(X\) is a Stein space. Let \(\pi : (M, E) \to (X, x)\) be a resolution of singularity. Then for \(1 \leq i \leq n - 1\), \(\dim(R^i\pi, \partial_M)_X\) is finite. \(R^i\pi, \partial_M\) has support on \(x\). They are independent of the resolution. In fact

\[
\dim(R^i\pi, \partial_M)_X = \dim H^{i+1}_X(X, \partial_M) \quad (1 \leq i \leq n-2)
\]

and

\[
\dim(R^{n-1}\pi, \partial_M)_X = \frac{\dim \Gamma(X - \{x\}, \partial K)}{L^2(X - \{x\})}.
\]

where \(L^2(X - \{x\})\) is the subspace of \(\Gamma(X - \{x\}, \partial K)\) consisting of \(n\)-form on \(X - \{x\}\) which are square integrable near \(x\).

We denote them by

\[
h^i(X, x) := \dim(R^i\pi, \partial_M)_X \quad (1 \leq i \leq n - 2)
\]

and

\[
P_g(X, x) := \dim(R^i\pi, \partial_M)_X.
\]

The invariant \(P_g(X, x)\) is called the geometric genus of \((X, x)\).

In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. What are natural generalizations in three-dimensional case of those singularities. They are purely elliptic singularities. We define the purely elliptic singularities.

Definition ([2])

For each positive integer \(m\), the \(m\)-genus of a normal isolated singularity \((X, x)\) in an \(n\)-dimensional analytic space is defined to be

\[
\delta_m(X, x) = \frac{\dim \Gamma(X - \{x\}, \partial(mK))}{L^{2/m}(X - \{x\})},
\]

where \(K\) is the canonical line bundle on \(X - \{x\}\), and \(L^{2/m}(X - \{x\})\) is the set of all holomorphic \(m\)-ple \(n\)-forms on \(X - \{x\}\) which are \(L^{2/m}\)-integrable at \(x\).

The \(m\)-genus \(\delta_m\) is finite and does not depend on the choice of a Stein neighborhood \(X\).
Definition ([3])

A singularities \((X, x)\) is said to be purely elliptic if \(\delta_m(X, x) = 1\) for every positive integer \(m\).

When \(X\) is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities. In higher dimensions, however, purely elliptic singularities are not always quasi-Gorenstein.

Theorem ([4])

Let \((X, x)\) be a quasi-Gorenstein normal isolated singularity of dimension 3, then

\[
2 \left\{ P_g(X, x) - \frac{-K_{MC_2}(M)}{24} \right\} = \dim_c H^1(NI, \theta).
\]

Consequently quasi-Gorenstein purely elliptic singularities of dimension 3 are classified into 6 classes.

1. \(h^1(X, x) = 2p\), (0,0)-type.
   - \(p = 1\) → Hilbert modular cusp singularities.
   - \(p > 1\) → Tsuchihashi cusp singularities ([1]).
2. \(h^1(X, x) = 2\), (0,1)-type.
3. \(h^1(X, x) = 2\), (0,2)-type.
4. \(h^1(X, x) = 0\), (0,0)-type.
5. \(h^1(X, x) = 0\), (0,1)-type.
6. \(h^1(X, x) = 0\), (0,2)-type.

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of (0,1)-type and of (0,0)-type, respectively.

Definition ([2])

A three-dimensional singularity \((X, x)\) is a simple \(K3\) singularity if the following two equivalent (Watanabe-Ishii[5]) conditions are satisfied:

1. \((X, x)\) is a Gorenstein purely elliptic singularity of (0,2)-type.
2. The exceptional divisor \(D\) is a normal \(K3\) surface for any \(Q\)-factorial terminal modification \(\delta : (Y, D) \to (X, x)\).

The notion of a simple \(K3\) singularity is defined as a three-dimensional isolated Gorenstein purely elliptic singularity of (0,2)-type.
Example
Let $f(x,y,z,w)$ be a quasi-homogeneous polynomial of type $(p,q,r,s:h)$ with $p+q+r+s=h$, and suppose $f(x,y,z,w)=0$ defines an isolated singularity at the origin in $C^4$. Then the origin is a simple $K3$ singularity.

Next we consider the case where $(X,x)$ is a hypersurface singularity defined by a nondegenerate polynomial

$$f = \sum a_v x^v \in C[x_0,x_1,\ldots,x_n],$$

and $x=0 \in C^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of $f$ is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_n \neq 0} (n+R_0^{n+1})$ in $R^{n+1}$. For any face $\Delta$ of $\Gamma_+(f)$, set $f_\Delta := \sum_{n\in\Delta} a_v x^v$. We say $f$ to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial x_0} = \frac{\partial f_\Delta}{\partial x_1} = \ldots = \frac{\partial f_\Delta}{\partial x_n} = 0$$

has no solution in $(C^*)^{n+1}$ for any face $\Delta$.

When $f$ is nondegenerate, the condition for $(X,x)$ to be a purely elliptic singularity is given as follows:

Theorem ([3])
Let $f$ be a nondegenerate polynomial and suppose $X=f=0$ has an isolated singularity at $x=0 \in C^{n+1}$.
(1) $(X,x)$ is purely elliptic if and only if $(1,1,\ldots,1) \in \Gamma(f)$.
(2) Let $n=3$ and let $\Delta_0$ be the face of $\Gamma(f)$ containing $(1,1,1,1)$ in the relative interior of $\Delta_0$. Then $(X,x)$ is a simple $K3$ singularity if and only if $\dim_R \Delta_0 = 3$.

Thus if $f$ is nondegenerate and defines a simple $K3$ singularity, then $f_{\Delta_0}$ is a quasi-homogeneous polynomial of a uniquely determined weight $\alpha$ called the weight of $f$.

Yonemura([6]) classified nondegenerate hypersurface simple $K3$ singularities into ninety five classes in terms of the weight of $f$. 
7.2 Parameters in a defining equation

Yonemura calculate the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained examples such that the polynomial $f$ is quasi-homogeneous and that \( \{ f = 0 \} \subset C^4 \) has a simple $K3$ singularity at the origin. The minimum number of parameters in the polynomial is less than or equal to 19 and is associated with the moduli of the $K3$ surface with singularities. The need for a unique form may be questioned. However, defining equations were not unique. So, in this section, we try to impose a condition to construct a unique form for quasi-homogeneous polynomials and decide conditions of their parameters.

We can take the following form for a weighted quasi-homogeneous polynomial $f$ in $C^{n+1}$ with the coordinate $[x_0, x_1, x_2, \ldots, x_n]$:

$$f = f_0 + f_1 + f_2 + \ldots + f_m$$

where $f_i(0 \leq i \leq m)$ is a homogeneous polynomial of degree $i$ in $C^{n+1}$. And let $W = (w_0, w_1, \ldots, w_n)$ be the weight. Then we can take the following form for the homogeneous polynomial of each degree $i$:

$$\sum_{k_0+k_1+\ldots+k_n=1} a_{k_0k_1\ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n} \quad (k_i \in \mathbb{N}_0, 0 \leq i \leq n).$$

Let \( \ll \) be the lexical linear ordering of the terms of the homogeneous polynomials for $0 \leq i \leq m$ in turn from the minimal term to the maximal term given below:

Definition

Let $K = (k_0, k_1, \ldots, k_n)$ ($k_i \in \mathbb{N}_0, 0 \leq i \leq n$) and let $a_K X^K$ denote the term

$$a_K X^K = a_{k_0k_1\ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}.$$

Then $a_K X^K \ll b_L X^L$ if there exists an integer $s(0 \leq s \leq n)$ such that $k_i = l_i$ for $m = 0, 1, \ldots, s - 1$ and $k_s < l_s$.

Example

$$x_0^3 \ll x_0^2 x_1 \ll x_0^4 \ll x_1^4 \ll x_0^5$$

Hereafter, for the sake of simplicity, we shall sometimes omit the coefficients in indicating terms.
We will consider the following procedure by using this ordering.

Step 1
We try a term $X^{K_i}$ to eliminate by a suitable analytic transformation with respect to $X$. We find a condition of the coefficient of term $X^{K_i}$ where we can make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. We classify the following two cases by the above condition.

Case 1: We can make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll x^{K_i}$.
Case 2: Otherwise for case 1.

For the condition of case 1, we make the term $X^{K_i}$ to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. For the condition of case 2, we don’t use the analytic transformation and go to next step.

Step 2
We make the coefficient of the term which determine the weight $(w_0, w_1, \ldots, w_{n-1})$ for the quasi-homogeneous polynomial equal to 1 by the magnification of the coordinate.

Let $W_4$ be the set of defining equations which has a nondegenerate hypersurface simple $K3$ singularity at the origin and let $\# m(f)$ be the minimum number of parameters of defining equation for any $f \in W_4$. Then for $\# m(f) = i (1 \leq i \leq 3)$, there exists 3 types, 8 types, 7 types, respectively. In general, the relation of the parameters is a simultaneous equation of them.

7.3 Relation of parameters

For $\# m(f) = 2$, we consider the relation of parameters in a defining equation of nondegenerate hypersurface simple $K3$ singularity which is constructed by the procedure in section 2. Then we obtain the following results:

Result
$f_{84} : x_0^3 + \lambda x_0 x_1 x_2 + x_0 x_2^3 + x_1^2 x_2 + x_1 x_4 + \mu x_2 x_3 = 0 \quad (\lambda^2 + 27 \mu) \mu - \lambda^2)^2 \neq (2(9 \lambda \mu - 8))^2$,
$f_{86} : x_0^3 x_1 + x_1^2 x_3 + x_0 x_3^3 + x_2^3 + \lambda x_1 x_2^2 x_3 + \mu x_2 x_3^3 = 0$

$16 \mu (125 \lambda^2 + 8 \lambda 4 \mu + 225 \lambda \mu^2 + 4 \lambda^3 \mu^3 + 108 \mu^4) \neq 3125 - 64 \lambda^5$.

The number $n$ of $f_n$ denotes the number of the defining equation in the classification by Yonemura.
We show a file (program) of Mathematica for the above computation. The file name is the same name as the \( f_n \) in the above result.

**File: f84**

\[
f84=x0^3+p \cdot x0 \cdot x1 \cdot x2 \cdot x3+q \cdot x0^2 \cdot x2 \cdot x3^3; \\
dx0 = D[f84,x0]; dx1 = D[f84,x1]; dx2 = D[f84,x2]; dx3 = D[f84,x3]; \\
e0=dx0; e1=Expand[(9dx0 x0-8dx1 x1-3dx2 x2+2dx3 x3)/27]; \\
e2=Expand[(-8dx1 x1+6dx2 x2+2dx3 x3)/(-18x2)]; \\
e3=Expand[(dx1 x1-3dx2 x2+2dx3 x3)/9]; \\
f0=Expand[e0 x2^2]; f1=Expand[e1 x2^8]; f2=Expand[e2 x2^9]; f3=Expand[e3 x2^3]; \\
g0=3a^2+c\cdot p \cdot b \cdot d; g1=a^3-c*b^3; g2=b^3-q \cdot c^2 \cdot d^3-a^2; g3=b^4-a \cdot c; \\
a=(b \cdot d^4)/c; j0=y^3+3x^2 \cdot z^2 \cdot p \cdot x \cdot y^2; j1=y^2-z^2; j2=x^3+x \cdot y \cdot z^2 \cdot z; \\
y^2; s0=Expand[j0/(x^3)]; s1=Expand[j2/(z^3)]; \\
u1=3t^2+p \cdot t+1; u2=t^3-t-q; \\
y=-z; t0=Expand[j0/(z^3)]; t2=Expand[j2/(z^3)]; \\
v1=3t^2+p \cdot t-1; v2=t^3+t^2; \\
Timing[Reduce[{u1, u2}=={0, 0}, t]] \\
\{0.13333333333332121 \ast Second, p*(-18 + p^2) \cdot q + 27 \cdot q^2 == -16 + p^2 \&\& \\
t == (-p + 9 \cdot q)/(12 + p^2)\} \\
Timing[Reduce[{v1, v2}=={0, 0}, t]] \\
\{0.13333333333332121 \ast Second, p*(18 + p^2) \cdot q + 27 \cdot q^2 == 16 + p^2 \&\& \\
t == (p + 9 \cdot q)/(12 + p^2)\} \\

**File 1: f86**

\[
f86=x0^2 \cdot x1+2 \cdot x1+3 \cdot x3+q \cdot x0^3+q \cdot x0^2 \cdot x2+5 \cdot p \cdot x1 \cdot x2 \cdot x3+q \cdot x2 \cdot x3^5; \\
dx0 = D[f86,x0]; dx1 = D[f86,x1]; dx2 = D[f86,x2]; dx3 = D[f86,x3]; \\
g0=dx0; g1=dx1; g2=Expand[(dx3 x3-5dx2 x2+8dx1 x1-4dx0 x0)/25]; g3=dx3; \\
x0=(x3^3)/(2x1); x3=(x2^5)/(x1^3); \\
h1=Expand[(g1 4x1^26)/(x^25)]; h2=Expand[g3 x1^22]; \\
u1=12a^5+4p \cdot a^4 \cdot b \cdot b^5; u2=a^5+2p \cdot a^4 \cdot b \cdot 5q \cdot a^2 \cdot b^3-2b^5; \\
v1=12t^5+4p \cdot t^4+1; v2=t^5+2p \cdot t^4+5q \cdot t^2-2; \\
Timing[Reduce[{v1, v2}=={0, 0}, t]] \\
\{2.9666666667152 \ast Second, \\
2000*p^2*q + 128*p^4*q^2 + 3600*p*q^3 + 64*p^3*q^4 + \\
1728*q^5 == 3125 - 64*p^5 \&\& \\
t == (-369140625*p - 585000*p^6 - 2048*p^11 +
7312500*p^{-3}*q + 960*p^{-8}*q - 379687500*q^2 +
924000*p^{-5}*q^2 - 2048*p^{-10}*q^2 + 4050000*p^{-2}*q^{-3} +
704*p^{-7}*q^3 + 1512000*p^{-4}*q^{-4})/
(79101625 + 4212500*p^{-5} + 4096*p^{-10})}

参考文献


