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The affine algebraic curve of Newton maps in Przytycki's cubic rational maps

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4.1 Introduction

Przytycki studies a family of cubic rational maps:

$$\mathcal{PC}(a, b, c) = \left\{ f(z) = z^2 + c + \frac{b}{z-a} : a, b, c \in \mathbf{C} \right\},$$

and defines the exotic map to be a map of $\mathcal{PC}(a, b, c)$ having two super-attracting fixed points and a critical point of period 2. An example of exotic map is given in Przytycki [Prz94], obtained by computer experiment.

Example: $f(z) = z^2 + c + b/(z-a) : c = -3.121092, a = 1.719727, b = 0.3142117$.

He considers 1-parameter families joining exotic examples with Newton maps for degree 3 polynomials. The subfamily with a super-attracting fixed point except of ∞ can be parametrized by two parameters (k, w) as follows:

let w be super-attracting fixed point, and $a = kw$. Then

$$a = kw, \quad b = 2w^3(1-k)^2, \quad c = w^2(2k-3) + w,$$

and

$$\mathcal{PC}(w, k) = \left\{ f(z) = z^2 + w^2(2k - 3) + w + \frac{2w^3(1 - k)^2}{z - kw} \right\}.$$

Four critical points are ∞ , w and

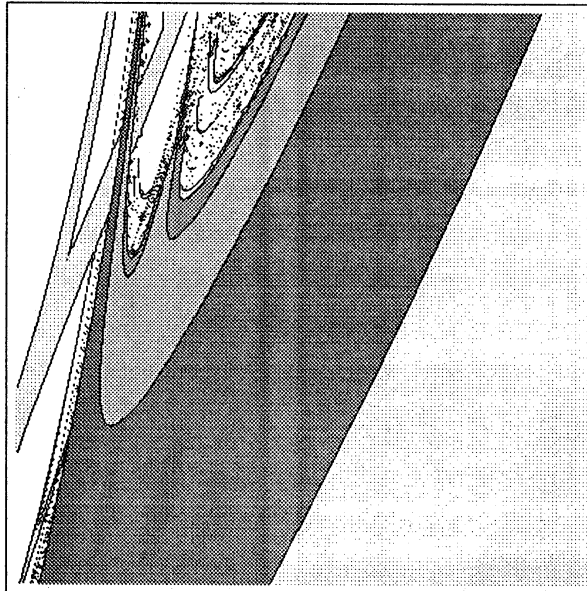
$$u = \frac{w(-1 + 2k - \sqrt{4k - 3})}{2}, \quad v = \frac{w(-1 + 2k + \sqrt{4k - 3})}{2}$$

Przytycki gives two questions in [Prz94]:

Question 1 In the set of Newton maps for the polynomials $P_\lambda = z^3 + (\lambda - 1)z - \lambda$, there exist Mandelbrot-like sets where the free critical point converges to a periodic attracting orbit. For a critical point $v = g(k, w)$, what happens to $\mathcal{M}(v)$ -sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones? : namely, do these sets move to $\mathcal{M}(v)$ (or $\mathcal{M}(u)$)-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones?

Question 2 Describe precisely how does the dynamics bifurcate for real parameters k, w .

Concerning about these questions, we obtain following results: to the first question, there is an affine algebraic curve consisting of Newton maps for degree 3 polynomials, and to the second one, we partly give an answer. Namely, for any fixed parameter k , we consider the bifurcation of this family as the parameter w varies monotonely. We observe complex bifurcations for $3/4 < k < 1$.



⊠ 1 Bifurcations in real (k, w) -plane

4.2 Cubic Newton Curve

We show that there is an affine algebraic curve consisted by all Newton maps in $\mathcal{PC}(a, b, c)$. We call hereafter this curve **cubic Newton curve**, denoted by $\mathcal{N}(w, k)$.

Proposition 1 *The defining equation of the cubic Newton curve is the following:*

$$\mathcal{N}(w, k) : k^2 w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1 = 0.$$

The cubic Newton curve is irreducible of genus one with two singular points

$$(0, 0, 1), \quad (0, 1, 0)$$

on $L_\infty : Z = 0$ of $\mathbb{P}^2\mathbb{C} : (Z, k, w)$.

Outline of proof:

$$f(k, w, x) = x^2 + w^2(2k - 3) + w + \frac{2w^3(1 - k)^2}{(x - kw)},$$

$$f'(k, w, x) = 2x - \frac{2(1 - k)^2 w^3}{(x - kw)^2}.$$

The critical points are ∞ , w , and

$$u = \frac{-w\sqrt{4k-3} - (1-2k)w}{2}, \quad v = \frac{w\sqrt{4k-3} - (1-2k)w}{2}.$$

For f to be a Newton map, we claim $f(k, w, u) = u$, or $f(k, w, v) = v$.

Therefore the equation of the Newton curve is

$$(k^2 w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1) = 0.$$

Let

$$F(Z, k, w) = Z^4 - 3wZ^3 - 2z^2kw + 9Z^2w^2 - 6Zkw^2 + k^2w^2 = 0,$$

and $P_k : (0, 0, 1)$, $P_w : (0, 1, 0)$. The singular points are P_k , P_w . The principal part of F is $(3Z - k)^2$. Therefore by Plücker's formula, we can calculate the genus one.

We define dynamical curves in the parameter space:

Definition Let $Per_p(\mu)$ consist of all parameter pairs (k, w) for which the associated cubic rational map $f(k, w, x)$ has a periodic orbit of period p with multiplier $(f^p)'$ equal to μ .

In particular, $Per_1(0)$ consists of all parameter pairs with a super-attracting fixed point. The real part of the curve $Per_1(1)$ consists of all parameter pairs for which the graph of f is tangent to the diagonal. Such points of tangency are called **saddle nodes** of period 1. On $Per_1(-1)$ attracting period one orbits bifurcate into attracting period two orbits.

It is clear that (k, w) of $Per_1(-1)$ belongs to $Per_2(1)$, but it is not known whether inverse inclusion holds. In the quadratic rational maps, we have $Per_2(1) = Per_1(-1)$.

Proposition 2 We obtain defining equations of two dynamical curves:

$$Per_1(1) : k^2 w^2 - 6k w^2 + 9w^2 - 2k w - 2w + 1 = 0,$$

$$Per_1(-1) : 3k^2 w^2 - 18k w^2 + 27w^2 - 6k w - 14w + 3 = 0.$$

See Figures for plots of these curves in the real case.

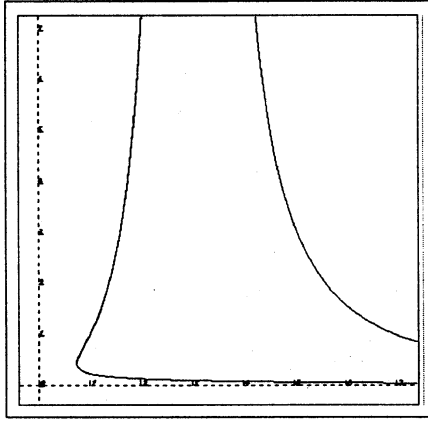


Figure 2 Newton

$$\text{curve : } (k^2 - 6k + 9)w^2 - (2k + 3)w + 1 = 0.$$

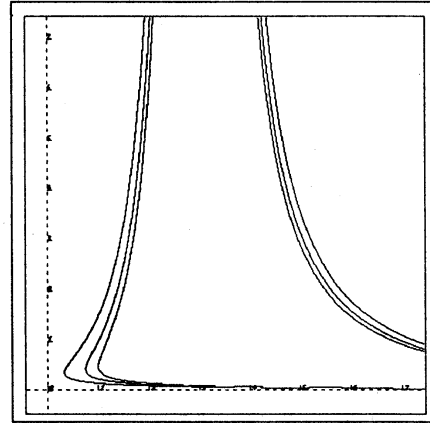


Figure 3 Dynamical curves : $Per_1(1)$, $Per_1(-1)$.

Outline of proof:

The fixed points of $f(k, w, x)$ are ∞ , w and

$$x_1 = \frac{(1-k)w - 1 - \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2},$$

$$x_2 = \frac{-(1-k)w + 1 + \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2}.$$

From $f'(k, w, x_i) = 1$, (resp. $= -1$) for $i = 1$, or 2 , we obtain the equation of $Per_1(1)$ (resp. $Per_1(-1)$).

4.3 bifurcations

Proposition 3 For the real parameter k , we can roughly divide (k, w) -plane into four distinct classes as follows: (1) $k < 3/4$, (2) $3/4 \leq k < 1$, (3) $1 < k < 3/2$, (4) $3/2 \leq k$.

Outline of Proof:

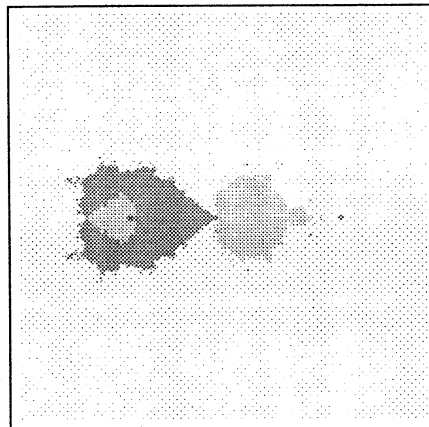
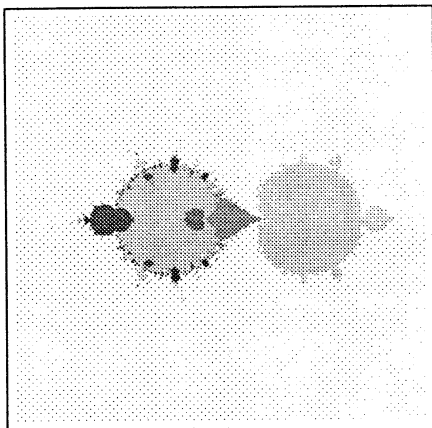


图 4 Bifurcation: $k = 1.15$, $w : (-1.5, -2) - (2.5, 2)$ 图 5 Bifurcation: $k = 1.5$, $w : (-2, -4) - (6, 4)$

- For $k : 3/4 \leq k \leq 1$, the critical points u, v, w except ∞ are ordered as $u < v < a < w$.
- For $k : 1 \leq k < 3/2$, $u < w < a < v$.
- For $k : 3/2 \leq k$, $w < u < a < v$. in case (3) (resp.(4)), the dynamics of u (resp. v) is analyzed as the dynamics of a quadratic map by Douady-Hubbard-Sullivan's theory([D-H85]). In case (2), the dynamical behavior is very complex. In case (1), two critical points u, v are complex numbers, mutually complex conjugate. The dynamical behavior is very complex: namely we can observe "swallow" and "tri-corn" configurations ([Mil90]).

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