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Modular varieties associated to quaternion unitary groups of degree 2

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We treat quaternion unitary groups of degree 2, which were studied by Arakawa in [1]. The purpose of this note is to report that modular varieties associated to those unitary groups with fully large levels are of general type.

1 Modular varieties

Let \( B \) be an indefinite division quaternion algebra over the rational number field \( \mathbb{Q} \), and \( \overline{\cdot} : B \to B \quad (a \mapsto \bar{a}) \) the canonical involution of \( B \). Since \( B_{\infty} = B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_{2}(\mathbb{R}) \), we identify \( B_{\infty} \) and \( M_{2}(\mathbb{R}) \) by fixing an isomorphism. Let \( G \) be the \( B \)-unitary group of degree 2. We put

\[
G_{\mathbb{Q}} := \{ g \in M_{2}(B) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{t} \bar{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \},
\]

where \( ^{t} \bar{g} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \) for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( G_{\mathbb{Q}} \) is \( \mathbb{Q} \)-rational points of \( G \). Let \( N \) be a natural number, and \( \mathfrak{O} \) a maximal order of \( B \). Set

\[
\Gamma(N) := \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}} \mid a - 1, b, c, d - 1 \in N\mathfrak{O} \}.
\]

Let

\[
\mathfrak{H}_{2} := \{ Z \in M_{2}(\mathbb{C}) \mid ^{t}Z = Z, \quad \text{Im}(Z) > 0 \}
\]

be the Siegel upper half plane of degree 2, and set

\[
\mathfrak{H} := \{ Z \in M_{2}(\mathbb{C}) \mid ZJ^{-1} \in \mathfrak{H}_{2} \}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

For the group \( G_{\mathbb{R}} \) of \( \mathbb{R} \)-rational points of \( G \), we have

\[
qG_{\mathbb{R}}q^{-1} = Sp_{2}(\mathbb{R}) := \{ g \in M_{4}(\mathbb{R}) \mid g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{t}g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \},
\]
where $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $q := \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$. The group $G_{\mathbb{R}}$ acts on $\mathfrak{s}$ by $g(Z) = (aZ + b)(cZ + d)^{-1}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}, Z \in \mathfrak{s}$. Though pairs $(G_{\mathbb{R}}, \mathfrak{H})$ and $(Sp_{2}(\mathbb{R}), \mathfrak{s}_{2})$ are the same essentially, we here consider the pair $(G_{\mathbb{R}}, \mathfrak{s})$.

Since the $Q$-rank of $G_{\mathbb{Q}}$ is 1, $\Gamma(N)$ has only point cusps. Let $Y(N)$ be a toroidal compactification of $\Gamma(N) \setminus \mathfrak{s}$.

2 Modular forms

In this section, we remember modular forms with respect to $\Gamma(N)$. See Arakawa [1] and Hashimoto [2] for details. For any positive integer $k$, let $M_{k}(\Gamma(N))$ be the $\mathbb{C}$-vector space of modular forms of weight $k$ with respect to $\Gamma(N)$. Namely, $M_{k}(\Gamma(N))$ is the space of holomorphic functions $f(Z)$ on $\mathfrak{s}$ satisfying $f(g(Z)) = \det(cZ + d)^{k}f(Z)$ for all $g \in \Gamma(N)$.

An element $f(Z)$ in $M_{k}(\Gamma(N))$ is called a cusp form if $|f(Z)\det({\rm Im}(Z))^{k/2}|$ is bounded on $\mathfrak{s}$. We denote by $S_{k}(\Gamma(N))$ the $\mathbb{C}$-vector space of cusp forms of weight $k$ with respect to $\Gamma(N)$.

Let $\mathbb{B}^{-}$ be the set of pure quaternions in $\mathbb{B}$. We put $L := \mathfrak{D} \cap \mathbb{B}^{-}$ and $L^{*} := \{ y \in \mathbb{B}^{-} | \text{tr}(xy) \in \mathbb{Z} \text{ for all } x \in L \}$.

Then Arakawa showed the following Proposition and Theorem.

**Proposition (Arakawa).** Each modular form $f(Z) \in M_{k}(\Gamma(N))$ has the following Fourier expansion

$$f(Z) = a(0) + \sum_{\substack{t \in L^{*} \\
|t| \geq 1}} a(t)e\left(\frac{1}{N}\text{tr}(tZ)\right),$$

where $e[ \cdot ] = \exp(2\pi i \cdot )$. In particular, $f(Z) \in S_{k}(\Gamma(N))$ is equivalent to $a(0) = 0$.

Let $\mathfrak{D}^{\times}$ be the group of units in $\mathfrak{D}$. For any element $\varepsilon \in \mathfrak{D}^{\times}$ and $x \in L$, we have $\varepsilon x \varepsilon^{-1} \in L$. The lattice $L^{*}$ also has this property. The Fourier coefficients $a(t)$ in Proposition satisfy $a(\varepsilon t \varepsilon^{-1}) = (Ne)^{k}a(t)$ for $\varepsilon \in \mathfrak{D}^{\times}$.

**Theorem (Arakawa).** Assume $k \geq 5, N \geq 3$. Then we have

$$\text{dim}_{\mathbb{C}}S_{k}(\Gamma(N)) = 2^{-7}3^{-3}5^{-1}[\Gamma : \Gamma(N)](k - 1)(k - \frac{3}{2})(k - 2) \prod_{p | d(\mathbb{B})} (p - 1)(p^{2} + 1)$$

$$+ 2^{-4}3^{-1}[\Gamma : \Gamma(N)]N^{-3} \prod_{p | d(\mathbb{B})} (p - 1),$$

where $d(\mathbb{B})$ is the discriminant of $\mathbb{B}$. 


3 The result

Let \( \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix} \) be the coordinates of \( S_3 \). Set \( \omega := dz_1 \wedge dz_2 \wedge dz_3 \). Arakawa showed that \( \Gamma(N) \) is torsion-free if \( N \geq 3 \). We here consider the case \( N \geq 3 \). For any cusp form \( f \in S_{3k}(\Gamma(N)) \), we would like to know the extendability of a \( \Gamma(N) \)-invariant form \( f\omega^{\otimes k} \) over the resolution of a point cusp.

We set
\[
L^*_+ := \{ y \in L^* \mid yJ > 0 \}, \quad L_+ := \{ x \in L \mid J^{-1} > 0 \}.
\]

Put
\[
\Lambda_m(\infty) := \{ y \in L^*_+ \mid \text{tr}(yx) \leq m \} \quad \text{for some } x \in L_+ \}, \quad d_m(\infty) := \Lambda_m(\infty)/\sim,
\]
where we write \( y_1 \sim y_2 \) when \( y_1 = \epsilon y_2 \overline{\epsilon} \) holds for some norm 1 unit \( \epsilon \) in \( \mathcal{O}^\times \). This number \( d_m(\infty) \) shows us the extendability of \( f\omega^{\otimes m} \).

Put \( N(L^*_+) := \min\{ N(x) \mid x \in L_+ \} \). The following is the main result:

**Theorem.** Assume \( N \geq 3 \). If
\[
3\sqrt{2}N^3[\mathcal{O}^\times : (1 + N\mathcal{D})^\times]N(L^*_+)3/2d(B) \prod_{p|d(B)}(p^2 + 1) > 2^75\pi,
\]
then \( Y(N) \) is a modular variety of general type.

**Sketch of proof:** The number of cusps for \( \Gamma(N) \) is \([\Gamma(1) : \Gamma(N)]/[\mathcal{O}^\times : (1 + N\mathcal{D})^\times]N^3 \). Hence we get
\[
P_m(Y(N)) \geq \dim S_{3m}(\Gamma(N)) - \frac{[\Gamma(1) : \Gamma(N)]}{[\mathcal{O}^\times : (1 + N\mathcal{D})^\times]N^3} \cdot d_m(\infty).
\]
If \( \text{tr}(yx) \leq m \), then we have \( N(y)N(x) \leq m^2 \). Now we evaluate the cardinality of
\[
\{ y \in L^*_+ \mid N(y) \leq \frac{m^2}{N(L^*_+)} \} / \sim.
\]
Here \( \sim \) is defined as above. Then we can show that the cardinality is not bigger than \( \frac{\pi}{3\sqrt{2}d(B)} \prod_{p|d(B)}(p - 1)m^3 + \epsilon m^3 \) for fully big \( m \) and fully small \( \epsilon \). By using this evaluation and the dimensional formula of Arakawa, we can prove the above theorem.

**References**


5. H. Yamaguchi: The parabolic contribution to the dimension of the space of cusp forms on Siegel space of degree two, Preprint, (1976).