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Kyoto University
Practical PTAS for Maximum Induced-Subgraph Problems on $K_{3,3}$-free or $K_5$-free Graphs

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Abstract. We show that for an integer $k \geq 2$ and an $n$-vertex graph $G$ without a $K_{3,3}$ (resp., $K_5$) minor, we can compute $k$ induced subgraphs of $G$ with treewidth $\leq 3k-4$ (resp., $\leq 6k-7$) in $O(kn)$ (resp., $O(kn + n^2)$) time such that each vertex of $G$ appears in exactly $k-1$ of these subgraphs. This leads to practical polynomial-time approximation schemes for many maximum induced-subgraph problems on graphs without a $K_{3,3}$ or $K_5$ minor.

1 Introduction

Let $\pi$ be a property on graphs. $\pi$ is hereditary if, whenever a graph $G$ satisfies $\pi$, every induced subgraph of $G$ also satisfies $\pi$. Suppose $\pi$ is a hereditary property. The maximum induced subgraph problem associated with $\pi$ (MISP($\pi$)) is the following: Given a graph $G = (V, E)$, find a maximum subset $U$ of $V$ that induces a subgraph satisfying $\pi$. Yannakakis showed that many natural MISP($\pi$)'s are NP-hard even if the input graph is restricted to a planar graph [13]. Thus, it is of interest to design efficient approximation algorithms for these MISP($\pi$)'s.

An approximation algorithm $A$ for an maximization problem $\Pi$ achieves a performance ratio of $\rho$ if for every instance $I$ of $\Pi$, the ratio of the optimal value for $I$ to the solution value returned by $A$ is at most $\rho$. A polynomial-time approximation scheme (PTAS) for problem $\Pi$ is an approximation algorithm which given an instance $I$ of $\Pi$ and an $\epsilon > 0$, returns a solution $s$ within time polynomial in the size of $I$ such that the ratio of the optimal value for $I$ to the value of $s$ is at most $(1 + \epsilon)$. Much work has been devoted to designing PTASs for MISP($\pi$)'s restricted to certain special instances [1, 5, 11]. Lipton and Tarjan were the first who proved that many MISP($\pi$)'s restricted to planar instances have PTASs [11]. Unfortunately, their schemes are known to be nonpractical [6]. That is, to achieve a reasonable performance ratio (e.g., 2), the number of vertices in the input graph and/or the running time of the schemes has to be enormous ($\approx 2^{2^{400}}$). Later, Baker gave practical PTASs for the same problems [5]. By extending Lipton & Tarjan's approach, Alon et al. [1] showed that many MISP($\pi$)'s restricted to graphs without an excluded minor have polynomial-time approximation schemes. Like Lipton and Tarjan's schemes, Alon et al.'s schemes have the shortage of being very nonpractical.

Since Alon et al.'s schemes are very nonpractical, it is natural to ask whether practical PTASs exist for MISP($\pi$)'s restricted to graphs without an excluded minor. In this paper, we give an affirmative answer...
to this question when the minor is $K_{3,3}$ or $K_5$. Since neither a $K_{3,3}$ minor nor a $K_5$ minor can exist in a planar graph, our result extends Baker's result above. The basic idea behind our practical PTASs is to decompose a graph without a $K_{3,3}$ or $K_5$ minor into subgraphs of bounded treewidth. More precisely, we show that for an integer $k \geq 2$ and an $n$-vertex graph $G$ without a $K_{3,3}$ (resp., $K_5$) minor, we can compute $k$ induced subgraphs of $G$ with treewidth $\leq 3k - 4$ (resp., $\leq 6k - 7$) in $O(kn)$ (resp., $O(kn + n^2)$) time such that each vertex of $G$ appears in exactly $k - 1$ of these subgraphs. Since many MISP($\pi$)'s restricted to graphs of bounded treewidth are solvable optimally in linear time by dynamic programming [4], we obtain practical PTASs for these MISP($\pi$)'s immediately. Moreover, our schemes have the advantage of being easy to parallelize while Alon et al.'s schemes do not. Our results heavily rely on the nice structures of graphs without a $K_{3,3}$ or $K_5$ minor that were developed in [2, 7, 10].

2 Preliminaries

Throughout this paper, a graph is always connected. Unless stated explicitly, a graph is always simple, i.e., has neither multiple edges nor self-loops. Let $G = (V, E)$ be a graph. For convenience, we allow $V = \emptyset$. If $V = \emptyset$, then we call $G$ an empty graph. We sometimes write $V(G)$ instead of $V$ and $E(G)$ instead of $E$.

The neighborhood of a vertex $v$ in $G$ is the set of vertices in $G$ adjacent to $v$. For $U \subseteq V$, the subgraph of $G$ induced by $U$ is the graph $(U, F)$ with $F = \{\{u, v\} \in E : u, v \in U\}$ and is denoted by $G[U]$. When $U \subseteq V$, we sometimes write $G - U$ instead of $G[V - U]$.

A contraction of an edge $\{u, v\}$ in $G$ is made by identifying $u$ and $v$ with a new vertex whose neighborhood is the union of the neighborhoods of $u$ and $v$ (resulting multiple edges and self-loops are deleted). A contraction of $G$ is a graph obtained from $G$ by a sequence of edge contractions. A graph $H$ is a minor of $G$ if $H$ is the contraction of a subgraph of $G$. $G$ is $H$-free if $G$ has no minor isomorphic to $H$. In this paper, we deal with $K_{3,3}$-free and $K_5$-free graphs. Recall that a planar graph must be both $K_{3,3}$-free and $K_5$-free by Kuratowski's Theorem.

A tree-decomposition of $G$ is a pair $\{X_i : i \in I, T = (I, F)\}$, where $\{X_i : i \in I\}$ is a family of subsets of $V$ and $T$ is a tree such that the following hold:

(a) $\cup_{i \in I} X_i = V$.
(b) For every edge $\{v, w\} \in E$, there is a subset $X_i, i \in I$ with $v \in X_i$ and $w \in X_i$.
(c) For all $i, j, k \in I$, if $j$ lies on the path from $i$ to $k$, then $X_i \cap X_k \subseteq X_j$.

The treewidth of a tree-decomposition $\{X_i : i \in I, T\}$ is $\max\{|X_i| - 1 : i \in I\}$. The treewidth of $G$, denoted by $\text{tw}(G)$, is the minimum treewidth of a tree-decomposition of $G$, taken over all possible tree-decompositions of $G$. The treewidth of an empty graph is defined to be 0.

Lemma 1. Let $G = (V, E)$ be a graph, and $R_1$ and $R_2$ be two subsets of $V$ such that (i) $R_1 \cap R_2 = \emptyset$ or $G[R_1 \cap R_2]$ is a clique and (ii) there is no $\{u_1, u_2\} \in E$ with $u_1 \in R_1 - R_2$ and $u_2 \in R_2 - R_1$. Then, $\text{tw}(G[R_1 \cup R_2]) \leq \max\{\text{tw}(G[R_1]), \text{tw}(G[R_2])\}$. 
Proof. We can assume that $G[R_1 \cap R_2]$ is a clique because the lemma trivially holds when $R_1 \cap R_2 = \emptyset$. Let $\mathcal{T}_1 = (\{X_i : i \in I\}, T_1)$ be a tree-decomposition of $G[R_1]$ with treewidth $\text{tw}(G[R_1])$, and $\mathcal{T}_2 = (\{Y_j : j \in J\}, T_2)$ be a tree-decomposition of $G[R_2]$ with treewidth $\text{tw}(G[R_2])$. W.l.o.g., we may assume that $I \cap J = \emptyset$. Since $G[R_1 \cap R_2]$ is a clique in both $G[R_1]$ and $G[R_2]$, there are $k$ and $l$ such that $R_1 \cap R_2 \subseteq X_k$ and $R_1 \cap R_2 \subseteq Y_l$. Let $T$ be the tree obtained from $T_1$ and $T_2$ by adding a new edge $\{k, l\}$. Then, it is easy to verify that $((\{X_i : i \in I\} \cup \{Y_j : j \in J\}, T)$ is a tree-decomposition of $G[R_1 \cup R_2]$ and has treewidth $\max\{\text{tw}(G[R_1]), \text{tw}(G[R_2])\}$.

A set $S \subseteq V$ is a cutset if $G - S$ is disconnected. A cutset $S$ is a $k$-cut if $|S| = k$. A $k$-cut is strong if $G - S$ has at least three connected components. A graph with at least $k$ vertices is $k$-connected if it has no $(k - 1)$-cut. A biconnected component of $G$ is a maximal 2-connected subgraph of $G$.

Let $C$ be a cutset of $G$, and $G_1$, ..., $G_p$ be the connected components of $G - C$. For $1 \leq i \leq p$, let $G_i \cup K(C)$ be the graph obtained from $G[V(G_i)] \cup C$ by adding an edge between every pair of non-adjacent vertices in $C$. The graphs $G_1 \cup K(C)$, ..., $G_p \cup K(C)$ are called the augmented components induced by $C$. Clearly, if $G$ is $k$-connected and $C$ is a $k$-cut of $G$, then all the augmented components induced by $C$ are also $k$-connected.

It is well known that the biconnected components of a graph are unique. Let $\mathcal{C}^1$ be the set of all 1-cuts of $G$, and $\mathcal{B}$ be the set of all biconnected components of $G$. Consider the bipartite graph $H = (\mathcal{C}^1 \cup \mathcal{B}, F)$, where $F = \{\{C, B\} : C \in \mathcal{C}^1, B \in \mathcal{B}, \text{ and } C \subseteq V(B)\}$. It is known that $H$ is a tree. Suppose that $B = \{B_1, ..., B_q\}$. Let $I = \{1, ..., q\}$. Root the tree $H$ at $B_1$ and define $T^1(G)$ to be the tree whose vertex set is $I$ and edge set is $\{\{i, i'\} : B_i$ is the grandparent of $B_{i'}$ in the rooted tree $H\}$. (Note that $T^1(G)$ is undirected.) The following fact is easy to prove.

**Fact 1** ($\{V(B_i) : i \in I\}, T^1(G))$ is a tree-decomposition of $G$ and can be computed from $G$ in $O(|V|)$ time.

Suppose that $G$ is 2-connected. Further suppose that $G$ contains a 2-cut. Replacing $G$ by the augmented components induced by a 2-cut is called splitting $G$. Suppose $G$ is split, the augmented components are split, and so on, until no more splits are possible. The graphs constructed in this way are 3-connected and the set of the graphs are called a 2-decomposition of $G$. Each element of a 2-decomposition of $G$ is called a split component of $G$. It is possible for $G$ to have two or more 2-decompositions. A split component of $G$ must be either a triangle or a 3-connected graph with at least 4 vertices. Let $\mathcal{D}$ be a 2-decomposition of $G$. We use $\mathcal{C}^2(\mathcal{D})$ to denote the set of the 2-cuts used to split $G$ into the split components in $\mathcal{D}$. Consider the bipartite graph $H = (\mathcal{C}^2(\mathcal{D}) \cup \mathcal{D}, F)$, where $F = \{\{C, D\} : C \in \mathcal{C}^2(\mathcal{D}), D \in \mathcal{D}, \text{ and } C \subseteq V(D)\}$. It is known that $H$ is a tree [12]. Suppose that $\mathcal{D} = \{D_1, ..., D_q\}$. Let $I = \{1, ..., q\}$. Root the tree $H$ at $D_1$ and define $T^2(G, \mathcal{D})$ to be the tree whose vertex set is $I$ and edge set is $\{\{i, i'\} : D_i$ is the grandparent of $D_{i'}$ in the rooted tree $H\}$. (Note that $T^2(G, \mathcal{D})$ is undirected.) Construct a supergraph $G^2(\mathcal{D})$ of $G$ as follows: For each $\{u, v\} \in \mathcal{C}^2(\mathcal{D})$ with $\{u, v\} \notin E$, add the edge $\{u, v\}$ to $G$. Then, we have the following fact:
Fact 2 \((\{V(D_i) : i \in I\}, T^2(G, D))\) is a tree-decomposition of \(G^2(D)\).

Proof. It is well known that every edge of \(G\) is contained in some split component in \(D\) and that if some vertex \(u\) of \(G\) is contained in two split components \(D_i\) and \(D_j\) in \(D\), then \(u\) is contained in every split component on the path between \(D_i\) and \(D_j\) in the tree \(H\) [12]. From this, it is easy to see the fact. \(\blacksquare\)

3 A technical lemma

Let \(S\) be a set. For an integer \(k \geq 2\), a \(k\)-cover of \(S\) is a list of \(k\) subsets of \(S\) such that each element of \(S\) is contained in exactly \(k - 1\) subsets in the list.

Lemma 2. Let \(G = (V, E)\) be a graph. Let \(k\) and \(b\) be two integers with \(k \geq 2\), and \(\tau\) be a property on \(k\)-covers of subsets of \(V\). Suppose that \(G\) has a tree-decomposition \((\{X_j : j \in I\}, T)\) and \(T\) has a rooted version such that the following three conditions are satisfied:

1. For every \(j' \in I\) and every child \(j\) of \(j'\) in \(T\), \(G[X_{j'} \cap X_j]\) is a clique.
2. For the root \(r \in I\) of \(T\), we can compute a \(k\)-cover \(\langle R_1, ..., R_k \rangle\) of \(X_r\) in \(f(k, |X_r|)\) time such that
   - (2a) for every \(1 \leq l \leq k\), \(\text{tw}(G[R_l]) \leq b\) and
   - (2b) for every child \(j''\) of \(r\) in \(T\), \(\langle R_1 \cap X_{j''}, ..., R_k \cap X_{j''}\rangle\) is a \(k\)-cover of \(X_r \cap X_{j''}\)
   satisfying \(\tau\).
3. For every \(j' \in I\) and every child \(j\) of \(j'\) in \(T\) and every \(k\)-cover \(\langle Y_1, ..., Y_k \rangle\) of \(X_{j'} \cap X_j\) satisfying \(\tau\), we can compute a \(k\)-cover \(\langle Z_1, ..., Z_k \rangle\) of \(X_j\) in \(f(k, |X_j|)\) time such that
   - (3a) for every \(1 \leq l \leq k\), \(Y_l = Z_l \cap X_{j'}\),
   - (3b) for every \(1 \leq l \leq k\), \(\text{tw}(G[Z_l]) \leq b\), and
   - (3c) for every child \(j''\) of \(j\), \(\langle Z_1 \cap X_{j''}, ..., Z_k \cap X_{j''}\rangle\) is a \(k\)-cover of \(X_j \cap X_{j''}\)
   satisfying \(\tau\).

Then, we can compute a \(k\)-cover \(\langle V_1, ..., V_k \rangle\) of \(V\) in \(O(\sum_{j \in I} f(k, |X_j|))\) time such that for each \(1 \leq l \leq k\), \(\text{tw}(G[V_l]) \leq b\) and \(V_l \cap X_r = R_l\).

Proof. Consider the following algorithm for computing \(\langle V_1, ..., V_k \rangle\):

Algorithm 1

1. Set \(V_1, ..., V_k\) to be the empty set.
2. While traversing \(T\) (starting at \(r\)) in a breadth-first manner, perform the following steps:
   2.1. If the current vertex \(j\) is \(r\), then compute a \(k\)-cover \(\langle R_1, ..., R_k \rangle\) of \(X_r\) satisfying the two conditions (2a) and (2b) above, and further add the vertices in each \(R_l\), \(1 \leq l \leq k\), to \(V_l\).
   2.2. If the current vertex \(j\) is not \(r\), then find the parent \(j'\) of \(j\) in \(T\), set \(\langle Y_1, ..., Y_k \rangle = \langle V_l \cap (X_{j'} \cap X_j), ..., V_k \cap (X_{j'} \cap X_j) \rangle\), compute a \(k\)-cover \(\langle Z_1, ..., Z_k \rangle\) of \(X_j\) satisfying the conditions (3a), (3b), and (3c) above, and add the vertices in each \(Z_l\), \(1 \leq l \leq k\), to \(V_l\).
3. Output \( \{V_1, ..., V_k\} \).

Next, we prove that the output \( \{V_1, ..., V_k\} \) of Algorithm 1 satisfies that \( \text{tw}(G[V_i]) \leq b \) and \( V_i \cap X_r = R_i \) for each \( 1 \leq l \leq k \). First note that the while-loop in Algorithm 1 is executed \(|I|\) times. W.l.o.g., we may assume that \( I = \{1, ..., |I|\} \) and that \( j+1 \) is traversed by Algorithm 1 right after \( j \) for each \( 1 \leq j \leq |I|-1 \). Then, \( r = 1 \). For each \( 1 \leq j \leq |I| \) and each \( 1 \leq l \leq k \), let \( V^j_l \) be the content of the variable \( V_i \) right after the \( j \)th iteration of the while-loop. We claim that for each \( 1 \leq j \leq |I| \), \( \{V^j_1, ..., V^j_k\} \) is a \( k \)-cover of \( U_{1 \leq i \leq j} X_i \) satisfying the following three conditions:

(C1) \( \text{tw}(G[V^j_l]) \leq b \) and \( V^j_l \cap X_1 = R_l \) for each \( 1 \leq l \leq k \).

(C2) For each son \( j' \) of \( j \) in \( T \), \( \langle V^j_l \cap (X_j \cap X_{j'}) \rangle, ..., \langle V^j_k \cap (X_j \cap X_{j'}) \rangle \) is a \( k \)-cover of \( X_j \cap X_{j'} \) satisfying \( \tau \).

(C3) For each \( 1 \leq i \leq j \) and each child \( i' \) of \( i \) in \( T \), \( \langle V^j_1 \cap (X_i \cap X_{i'}) \rangle, ..., \langle V^j_k \cap (X_i \cap X_{i'}) \rangle = \langle V^j_1 \cap (X_i \cap X_{i'}) \rangle, ..., \langle V^j_k \cap (X_i \cap X_{i'}) \rangle \).

The lemma follows from the claim immediately. We prove the claim by induction on \( j \). In case \( j = 1 \), the claim clearly holds. Let \( j \) be some integer with \( 2 \leq j \leq |I| \) and assume that the claim holds for all integers \( i \) with \( i \leq j-1 \). Let \( j' \) be the parent of \( j \) in \( T \), and let \( \langle Y_1, ..., Y_k \rangle = \langle V_{1}^{j-1} \cap (X_j \cap X_{j'}), ..., V_{k}^{j-1} \cap (X_j \cap X_{j'}) \rangle \). Then, since \( j' \leq j-1 \), we have \( \langle Y_1, ..., Y_k \rangle = \langle V_{1}^{j-1} \cap (X_j \cap X_{j'}), ..., V_{k}^{j-1} \cap (X_j \cap X_{j'}) \rangle \) by (C3) in the inductive hypothesis. Combining this with (C2) in the inductive hypothesis, we have that \( \langle Y_1, ..., Y_k \rangle \) is a \( k \)-cover of \( X_{j'} \cap X_j \) satisfying \( \tau \). Thus, in the \( j \)th execution of step 2.2, we can compute a \( k \)-cover \( \langle Z_1, ..., Z_k \rangle \) of \( X_j \) satisfying the conditions (3a), (3b), and (3c) above.

Firstly, we prove that \( \{V^j_1, ..., V^j_k\} \) is a \( k \)-cover of \( U_{1 \leq i \leq j} X_i \). To see this, first observe that \( \{Z_1, ..., Z_k\} \) is a \( k \)-cover of \( X_j \) and that \( \{V^j_1, ..., V^j_k\} = \{V_{1}^{j-1}, ..., V_{1}^{j} \cup Z_1, ..., V_{k}^{j-1}, ..., V_{k}^{j} \cup Z_k\} \). Moreover, by the inductive hypothesis, \( \{V_{1}^{j-1}, ..., V_{l}^{j-1}\} \) is a \( k \)-cover of \( U_{1 \leq i \leq j-1} X_i \). Thus, each \( v \in \bigcap_{1 \leq i \leq j-1} X_i - (\bigcap_{1 \leq i \leq j} X_i) \) appears in exactly \( k-1 \) sets in \( \{V_1^j, ..., V_k^j\} \). It remains to consider the vertices in \( (\bigcap_{1 \leq i \leq j-1} X_i) \cap X_j \). Since the path from \( j \) to each \( i, 1 \leq i \leq j-1 \), in \( T \) must pass \( j' \), we have \( (\bigcap_{1 \leq i \leq j-1} X_i) \cap X_j = X_{j'} \cap X_j \) by the definition of tree-decompositions. Fix a vertex \( v \in X_{j'} \cap X_j \). By the inductive hypothesis, \( v \) appears in exactly \( k-1 \) sets in \( \{V_1^{j-1}, ..., V_k^{j-1}\} \). Also, \( v \) appears in exactly \( k-1 \) sets in \( \{Z_1, ..., Z_k\} \). Moreover, for each \( 1 \leq l \leq k \), \( v \in V_l^{j-1} \) if and only if \( v \in Z_l \) by the condition (3a) above. Thus, \( v \) appears in exactly \( k-1 \) sets in \( \{V_1^j, ..., V_k^j\} \).

Secondly, we prove that for each \( 1 \leq l \leq k \), \( \text{tw}(G[V^j_l]) \leq b \). Fix an integer \( l \) with \( 1 \leq l \leq k \). It suffices to prove that \( \text{tw}(G[V^j_l]) \leq b \). This is done by applying Lemma 1. Let us be more precise. Since \( (\bigcap_{1 \leq i \leq j-1} X_i) \cap X_j = X_{j'} \cap X_j \), we have \( V_l^{j-1} \cap Z_l \subseteq X_{j'} \cap X_j \). On the other hand, \( G[X_{j'} \cap X_j] \) is a clique. Thus, \( G[V_l^{j-1} \cap Z_l] \) is also a clique. Let \( v_1 \in V_l^{j-1} \cap Z_l \) and \( v_2 \in Z_l - V_l^{j-1} \). We want to show that \( \{v_1, v_2\} \notin E \). Assume, on the contrary, that \( \{v_1, v_2\} \in E \). Then, since the path from \( j \) to each \( i, 1 \leq i \leq j-1 \), in \( T \) must pass \( j' \), we have that \( v_1 \in X_{j'} \cap X_j \) or \( v_2 \in X_{j'} \cap X_j \) by the definition of tree-decompositions. If \( v_1 \in X_{j'} \cap X_j \), then \( v_1 \in (V_l^{j-1} \cap (X_{j'} \cap X_j)) - (Z_l \cap X_{j'}) \); otherwise, \( v_2 \in (Z_l \cap X_{j'}) - (V_l^{j-1} \cap (X_{j'} \cap X_j)) \). However, this contradicts that \( V_l^{j-1} \cap (X_{j'} \cap X_j) = Y_l = Z_l \cap X_{j'} \). Therefore, \( \{v_1, v_2\} \notin E \). Recall that
$G[V_{i}^{j-1} \cap Z_{i}]$ is a clique. Hence, if we set $R_{1} = V_{i}^{j-1}$ and $R_{2} = Z_{i}$, then $R_{1}$ and $R_{2}$ satisfy the conditions in Lemma 1. This implies that \( \text{tw}(G[V_{i}^{j}]) \leq \max\{\text{tw}(G[V_{i}^{j-1}]), \text{tw}(G[Z_{i}])\} \). By the inductive hypothesis, \( \text{tw}(G[V_{i}^{j-1}]) \leq b \). By the condition (3b) above, \( \text{tw}(G[Z_{i}]) \leq b \). Thus, we have \( \text{tw}(G[V_{i}^{j}]) \leq b \) by Lemma 1.

Thirdly, we prove that for each $1 \leq l \leq k$, \( V_{l}^{j} \cap X_{1} = R_{l} \). Fix an integer $l$ with $1 \leq l \leq k$. By the inductive hypothesis, \( V_{l}^{j-1} \cap X_{1} = R_{l} \). Thus, to prove that \( V_{l}^{j} \cap X_{1} = R_{l} \), it suffices to prove that \( Z_{l} \cap X_{1} \subseteq V_{l}^{j-1} \cap X_{1} \). Fix a vertex $v \in Z_{l} \cap X_{1}$. Since the path from $j$ to the root 1 in $T$ must pass $j'$, we have $v \in X_{j'}$ by the definition of tree-decompositions. Thus, $v \in Z_{l} \cap X_{j'} \cap X_{1}$. This together with the condition (3a) implies that $v \in Y_{1} \cap X_{1}$. Recall that $Y_{1} = V_{l}^{j-1} \cap (X_{j'} \cap X_{1})$. Therefore, \( v \in V_{l}^{j-1} \cap X_{1} \).

Fourthly, we prove that \( (V_{i}^{1}, \ldots, V_{i}^{k}) \) satisfies the condition (C2) above. Let $j''$ be a son of $j$ in $T$. We want to show that \( (V_{i}^{1} \cap (X_{j} \cap X_{j''}), \ldots, V_{i}^{k} \cap (X_{j} \cap X_{j''})) \) is a $k$-cover of $X_{j} \cap X_{j''}$ satisfying $\tau$ by the condition (3c) above, it suffices to show that for each $1 \leq l \leq k$, $V_{l}^{j} \cap X_{j} = Z_{l}$. Fix an integer $l$ with $1 \leq l \leq k$. Since the path from $j$ to each i, \( 1 \leq i \leq j - 1 \), in $T$ must pass $j'$, we have $V_{l}^{j-1} \cap X_{j} \subseteq X_{j'}$ by the definition of tree-decompositions. Thus, $V_{l}^{j-1} \cap X_{j} \subseteq V_{l}^{j} \cap X_{j'} \cap X_{j}$). On the other hand, $V_{l}^{j-1} \cap X_{j'} \cap X_{j} = Z_{i} \cap X_{j'}$ by the condition (3a) above. Hence, $V_{l}^{j-1} \cap X_{j} \subseteq Z_{i}$. Noting that $V_{l}^{j} = V_{l}^{j-1} \cup Z_{i}$ and $Z_{i} \subseteq X_{j}$, we see that $V_{l}^{j} \cap X_{j} = Z_{i}$ if and only if $V_{l}^{j-1} \cap X_{j} \subseteq Z_{i}$. Therefore, we have $V_{l}^{j} \cap X_{j} = Z_{i}$.

Finally, we prove that \( (V_{i}^{1}, \ldots, V_{i}^{k}) \) satisfies the condition (C3) above. Let $i$ be an integer with $1 \leq i \leq j$, and $i'$ be a child of $i$ in $T$. We want to show that \( (V_{i}^{1} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{k} \cap (X_{i} \cap X_{i'})) = (V_{i}^{1} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{k} \cap (X_{i} \cap X_{i'})) \). This clearly holds if $i' = j$. So, we may assume that $i' \leq j - 1$. Then, \( (V_{i}^{j-1} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{j-1} \cap (X_{i} \cap X_{i'})) = (V_{i}^{j} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{j} \cap (X_{i} \cap X_{i'})) \) by the inductive hypothesis. By this, we only need to show that \( (V_{i}^{j} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{j} \cap (X_{i} \cap X_{i'})) = (V_{i}^{j-1} \cap (X_{i} \cap X_{i'}), \ldots, V_{i}^{j-1} \cap (X_{i} \cap X_{i'})) \).

Fix an integer $l$ with $1 \leq l \leq k$. Since the path from $j$ to each $i$, $1 \leq i \leq j - 1$, in $T$ must pass $j'$, we have $Z_{l} \cap (X_{i} \cap X_{i'}) \subseteq X_{j'}$ by the definition of tree-decompositions. Thus, $Z_{l} \cap (X_{i} \cap X_{i'}) \subseteq Z_{i} \cap X_{j'}$. On the other hand, $Z_{i} \cap X_{j'} \subseteq V_{l}^{j-1} \cap X_{j'}$ by the condition (3a) above. Hence, $Z_{l} \cap (X_{i} \cap X_{i'}) \subseteq V_{l}^{j-1} \cap (X_{i} \cap X_{i'})$. Noting that $V_{l}^{j} = V_{l}^{j-1} \cup Z_{i}$ and $Z_{i} \subseteq X_{j}$, we see that $V_{l}^{j} \cap (X_{i} \cap X_{i'}) = V_{l}^{j-1} \cap (X_{i} \cap X_{i'})$ if and only if $Z_{l} \cap (X_{i} \cap X_{i'}) \subseteq V_{l}^{j-1} \cap (X_{i} \cap X_{i'})$. Therefore, we have $V_{l}^{j} \cap (X_{i} \cap X_{i'}) = V_{l}^{j-1} \cap (X_{i} \cap X_{i'})$.

Let us mention the property $\tau$ that will be used in the remainder of this paper. Let $G = (V, E)$ be a graph, and $U$ be a subset of $V$. A $k$-cover $L$ of $U$ is completely unbalanced if exactly one set in $L$ is empty and the others are equal to $U$. A $k$-cover $L$ of $U$ is weakly unbalanced if there are one vertex $u \in U$ and two sets $U_{1}$ and $U_{2}$ in $L$ such that $U_{1} = \{u\}$, $U_{2} = U - \{u\}$, and all the sets in $L$ except $U_{1}$ and $U_{2}$ are equal to $U$. A $k$-cover of $U$ is unbalanced if it is either completely unbalanced or weakly unbalanced. Note that if $|U| \leq 2$, then every $k$-cover of $U$ must be unbalanced. Hereafter, the property $\tau$ in Lemma 2 means "unbalanced", i.e., a $k$-cover $L$ of $U$ satisfies $\tau$ if and only if $L$ is unbalanced.

4 PTASs for MISP($\pi$)'s on $K_{3,3}$-free graphs

We start by proving a useful lemma.
Lemma 3. Let $G = (V, E)$ be a connected planar graph, and $k$ be an integer $\geq 2$. Suppose that $s_1$ and $s_2$ are two adjacent vertices in $G$ and $(Y_1, ..., Y_k)$ is an unbalanced $k$-cover of $\{s_1, s_2\}$. Then, we can compute a $k$-cover $(Z_1, ..., Z_k)$ of $V$ in $O(k|V|)$ time such that $\text{tw}(G|Z_l|) \leq 3k - 4$ and $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$.

Proof. Let us first suppose that $(Y_1, ..., Y_k)$ is weakly unbalanced. Then, by symmetry, we may assume that $Y_1 = \{s_1\}$, $Y_2 = \{s_2\}$, and $Y_3 = \cdots = Y_k = \{s_1, s_2\}$. We perform a breadth-first-search (BFS) on $G$ starting at $s_2$ to obtain a BFS tree $T$. For each vertex $v$ in $G$, we define $\text{lev}(v)$ to be the length of the path from $s_2$ to $v$ in $T$. Note that $\text{lev}(s_2) = 0$ and $\text{lev}(s_1) = 1$. For each $1 \leq l \leq k$, let $Z_l = V - \{v \in V : \text{lev}(v) \equiv l-1 \pmod{k}\}$. Obviously, $(Z_1, ..., Z_k)$ is a $k$-cover of $V$. Moreover, the subgraph induced by each nonempty $Z_l$, $1 \leq l \leq k$, is $(k-1)$-outerplanar and hence has treewidth $\leq 3k - 4$ [3]. It is also clear that $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$.

Next, suppose that $(Y_1, ..., Y_k)$ is completely unbalanced. Then, by symmetry, we may assume that $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = \{s_1, s_2\}$. Let $H$ be the graph obtained from $G$ by replacing the edge $\{s_1, s_2\}$ with two edges $\{s_1, x\}$ and $\{x, s_2\}$, where $x$ is a new vertex. It is clear that $H$ is still planar. We perform a breadth-first-search (BFS) on $H$ starting at $x$ to obtain a BFS tree $T$. For each vertex $v$ in $H$, we define the level number of $v$ (denoted $\text{lev}(v)$) to be the length of the path from $x$ to $v$ in $T$. Note that only $x$ has level number 0 and only $s_1$ and $s_2$ have level number 1. For each $1 \leq l \leq k$, let $V_l = \{v \in V : \text{lev}(v) \equiv l-1 \pmod{k}\}$. Let $Z_1 = V - V_2$, $Z_2 = V - V_1$, and $Z_l = V - V_l$ for each $3 \leq l \leq k$. Obviously, $(Z_1, ..., Z_k)$ is a $k$-cover of $V$. Moreover, the subgraph induced by each nonempty $Z_l$, $1 \leq l \leq k$, is $(k-1)$-outerplanar and hence has treewidth $\leq 3k - 4$ [3]. It is also clear that $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$.

The following lemma states that a 2-connected $K_{3,3}$-free graph can have very special split components.

Lemma 4. [2, 7]. Each split component of a 2-connected $K_{3,3}$-free graph is either isomorphic to $K_5$ or planar.

Now, we are ready to show the main lemma of this section.

Lemma 5. Let $G = (V, E)$ be a 2-connected $K_{3,3}$-free graph. Then, for any $k \geq 2$, we can compute a $k$-cover $(V_1, ..., V_k)$ of $V$ in $O(k|V|)$ time such that $\text{tw}(G|V_l|) \leq 3k - 4$ for each $1 \leq l \leq k$.

Proof. Let $D = \{D_1, ..., D_q\}$ be a 2-decomposition of $G$, and let $I = \{1, ..., q\}$. It is known that $D$ can be computed in $O(|V|)$ time [8]. Moreover, $\sum_{i \in I} |V(D_i)| = O(|V|)$ [8]. W.l.o.g., we may assume that $G^2(D) = G$ because a $k$-cover $(V_1, ..., V_k)$ of $V$ such that the subgraph of $G^2(D)$ induced by $V_1$ has treewidth $\leq 3k - 4$ for each $1 \leq l \leq k$ is also a $k$-cover $(V_1, ..., V_k)$ of $V$ such that $\text{tw}(G|V_l|) \leq 3k - 4$ for each $1 \leq l \leq k$. Then, by Fact 2, $(\{V(D_j) : j \in I\}, T^2(G, D))$ is a tree-decomposition of $G$. For convenience, let $T = T^2(G, D)$, $b = 3k - 4$, and $X_j = V(D_j)$ and $f(k, |X_j|) = O(k|X_j|)$ for each $j \in I$. We want to apply Lemma 2 to the graph $G$ and the tree-decomposition $(\{X_j : j \in I\}, T)$. To this end, we first (arbitrarily) choose an $r \in I$ and root $T$ at $r$. 

Clearly, the condition (1) in Lemma 2 is satisfied by $G$ and ($\{X_j : j \in I\}, T$). By Lemma 4, $G[X_r] = D_r$ is either isomorphic to $K_5$ or planar. Let us first suppose that $G[X_r]$ is isomorphic to $K_5$. Then, we set $R_1 = \emptyset$ and $R_2 = \cdots = R_k = X_r$ if $k \geq 3$; otherwise ($k = 2$), we arbitrarily choose two vertices $v_1$ and $v_2$ in $X_r$ and set $R_1 = \{v_1, v_2\}$ and $R_2 = X_r - R_1$. Obviously, $(R_1, \ldots, R_k)$ is a $k$-cover of $X_r$ satisfying the condition (2a) in Lemma 2. $(R_1, \ldots, R_k)$ also satisfies the condition (2b) in Lemma 2 since $|X_r \cap X_j''| = 2$ for every child $j''$ of $r$ in $T$. Next, suppose that $G[X_r]$ is a planar graph. Then, we arbitrarily choose an edge $\{s_1, s_2\}$ in $G[X_r]$, set $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = \{s_1, s_2\}$, and use Lemma 3 to compute a $k$-cover $(R_1, \ldots, R_k)$ of $X_r$ in $O(k|X_r|)$ time such that $tw(G[R_l]) \leq 3k - 4$ for each $1 \leq l \leq k$. Clearly, $(R_1, \ldots, R_k)$ satisfies the condition (2a) in Lemma 2. $(R_1, \ldots, R_k)$ also satisfies the condition (2b) in Lemma 2 since $|X_r \cap X_j''| = 2$ for every child $j''$ of $r$ in $T$.

Fix a $j' \in I$ and a child $j$ of $j'$ in $T$. Let $(Y_1, \ldots, Y_k)$ be an unbalanced $k$-cover of $X_{j'} \cap X_j$. W.l.o.g., we may assume that $|Y_l| \leq |Y_{l+1}|$ for each $1 \leq l \leq k - 1$. By Lemma 4, $G[X_j] = D_j$ is either isomorphic to $K_5$ or planar. Let us first suppose that $G[X_j]$ is isomorphic to $K_5$. If $k \geq 3$, then we set $Z_1 = Y_1$ and $Z_l = Y_l \cup (X_j - X_{j'})$ for each $2 \leq l \leq k$. Otherwise ($k = 2$), we arbitrarily choose a vertex $v \in X_j - X_{j'}$, and set $Z_1 = Y_1 \cup (X_j - (X_{j'} \cup \{v\}))$ and $Z_2 = Y_2 \cup \{v\}$. Then, no matter what $k$ is, $(Z_1, \ldots, Z_k)$ is a $k$-cover of $X_j$ satisfying the conditions (3a), (3b), and (3c) in Lemma 2. Next, suppose that $G[X_j]$ is planar. Let $X_{j'} \cap X_j = \{s_1, s_2\}$. Note that $s_1$ and $s_2$ are adjacent in $G$. We use Lemma 3 to compute a $k$-cover $(Z_1, \ldots, Z_k)$ of $X_j$. It should be easy to see that $(Z_1, \ldots, Z_k)$ is a $k$-cover of $X_j$ satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

Now, the lemma follows from Lemma 2.

**Theorem 6.** Let $G = (V, E)$ be a $K_{3,3}$-free graph. Then, for any $k \geq 2$, we can compute a $k$-cover $(V_1, \ldots, V_k)$ of $V$ in $O(k|V|)$ time such that $tw(G[V_l]) \leq 3k - 4$ for $1 \leq l \leq k$.

**Proof.** This follows from Fact 1, Lemma 2, and Lemma 5 immediately.

**Corollary 7.** Let $\pi$ be a hereditary property on graphs. Suppose that $\text{MISP}(\pi)$ restricted to $n$-vertex graphs of treewidth $\leq k$ can be solved in $T_\pi(k, n)$ time. Then, given an integer $k \geq 2$ and a $K_{3,3}$-free graph $G = (V, E)$, we can compute a subset $U$ of $V$ in $O(k|V| + T_\pi(3k - 4, |V|))$ time such that $G[U]$ satisfies $\pi$ and $|U|$ is at least $(k - 1)/k$ optimal.

**Proof.** Given an integer $k \geq 2$ and a $K_{3,3}$-free graph $G = (V, E)$, we first compute a $k$-cover $(V_1, \ldots, V_k)$ of $V$ such that $tw(G[V_l]) \leq 3k - 4$ for each $1 \leq l \leq k$ (cf. Theorem 6). Next, we compute an optimal solution $U_l$ in each $G[V_l]$, $1 \leq l \leq k$. Finally, we set $U$ to be the maximum subset among $U_1, \ldots, U_k$. Obviously, $G[U]$ satisfies $\pi$. Moreover, since $\pi$ is hereditary, reasoning similar to that in [5] can be used to show that $|U|$ is at least $(k - 1)/k$ optimal.

For many $\pi$'s, it is well known that $T_\pi(k, n) = O(c^kn)$ for some small $c$, and hence $\text{MISP}(\pi)$ restricted to $K_{3,3}$-free graphs has a practical PTAS by Corollary 7.
5 PTASs for MISP(\pi)'s on $K_5$-free graphs

We start by giving several definitions. Suppose that $G$ is 3-connected. Further suppose that $G$ contains a strong 3-cut. Replacing $G$ by the augmented components induced by a strong 3-cut is called strongly splitting $G$. Suppose $G$ is strongly split, the augmented components are strongly split, and so on, until no more strong splits are possible. The set of the graphs constructed in this way are called a strong 3-decomposition of $G$.

**Definition 8.** We define $W$ to be the graph obtained from a 8-cycle by adding 4 crossing edges. More precisely, $W = (\{1, ..., 8\}, E_1 \cup E_2)$, where $E_1 = \{(i, i+1) : 1 \leq i \leq 7\} \cup \{8, 1\}$ and $E_2 = \{(i, i+4) : 1 \leq i \leq 4\}$. A $K_5$-free graph $G$ is said to be nice if $G$ is 3-connected, nonplanar, and is not isomorphic to $K_{3,3}$ or $W$.

**Fact 3** [10] Suppose that $G$ is a nice $K_5$-free graph. Let $C$ be a strong 3-cut in $G$. Then, the augmented components induced by $C$ are also nice $K_5$-free graphs. Moreover, $C'$ is a strong 3-cut of $G$ if and only if $C'$ is a strong 3-cut of an augmented component of $G$ induced by $C$.

Based on this fact, Kézdy and McGuinness further proved the following:

**Fact 4** [10] A nice $K_5$-free graph has a unique strong 3-decomposition. Moreover, each graph in the strong 3-decomposition is planar.

Suppose that $G = (V, E)$ is a nice $K_5$-free graph. Let $\mathcal{D}^3(G)$ be the strong 3-decomposition of $G$, and $\mathcal{C}^3(G)$ be the set of all strong 3-cuts in $G$. Define $H(G)$ to be the bipartite graph $(\mathcal{D}^3(G) \cup \mathcal{C}^3(G), F)$, where $F = \{(D, C) : D \in \mathcal{D}^3(G), C \in \mathcal{C}^3(G), C \subseteq V(D)\}$.

**Lemma 9.** The following hold:

1. Every edge of $G$ is contained in some graph in $\mathcal{D}^3(G)$.
2. If a subset $S$ of $V$ induces a triangle but $S \not\in \mathcal{C}^3(G)$, then exactly one graph in $\mathcal{D}^3(G)$ contains the three vertices in $S$.
3. $H(G)$ is a tree. Moreover, if some vertex $u \in V$ is contained in two graphs $D$ and $D'$ in $\mathcal{D}^3(G)$, then $u$ is contained in every graph on the path between $D$ and $D'$ in $H(G)$.

**Proof.** We show the lemma by induction on the number of strong 3-cuts in $G$. The lemma clearly holds when $G$ has no strong 3-cut. Assume that the lemma is true for every graph that has up to $p-1$ strong 3-cuts. Consider a graph $G$ with $p$ strong 3-cuts. Let $C$ be a strong 3-cut in $G$, and $G_1 \cup K(C), ..., G_k \cup K(C)$ be the augmented components induced by $C$. By Fact 3, each $G_i \cup K(C), 1 \leq i \leq k$, is a nice $K_5$-free graph, $\mathcal{C}^3(G) = \cup_{1 \leq i \leq k} \mathcal{C}^3(G_i \cup K(C)) \cup \{C\}$, and $\mathcal{D}^3(G) = \cup_{1 \leq i \leq k} \mathcal{D}^3(G_i \cup K(C))$.

It is clear that every edge of $G$ is contained in at least one of the graphs $G_1 \cup K(C), ..., G_k \cup K(C)$. Moreover, by the inductive hypothesis, every edge of each $G_i \cup K(C), 1 \leq i \leq k$, is contained in some
graph in $\mathcal{D}^3(G_i \cup K(C))$. These together with the fact that $\mathcal{D}^3(G) = \bigcup_{1 \leq i \leq k} \mathcal{D}^3(G_i \cup K(C))$ imply that the statement (1) in the lemma holds for $G$.

Suppose that $S \subseteq V$ induces a triangle but $S \notin \mathcal{C}^3(G)$. Then, it is clear that there is exactly one $G_i \cup K(C)$, $1 \leq i \leq k$, containing $S$. Moreover, $S$ cannot be a 3-cut in the graph $G_i \cup K(C)$ or else $S$ would be a strong 3-cut in $G$ by Fact 3. Thus, by the inductive hypothesis, exactly one graph in $\mathcal{D}^3(G_i \cup K(C))$ contains the three vertices in $S$. Therefore, exactly one graph in $\mathcal{D}^3(G)$ contains the three vertices in $S$. This implies that the statement (2) in the lemma holds for $G$.

By the inductive hypothesis, each $H(G_i \cup K(C))$, $1 \leq i \leq k$, is a tree. Moreover, it is clear that in each graph $G_i \cup K(C)$, $1 \leq i \leq k$, $C$ induces a triangle but is not a 3-cut. Thus, for each $1 \leq i \leq k$, exactly one graph (say, $D_i$) in $\mathcal{D}^3(G_i \cup K(C))$ contains the three vertices in $C$ by the inductive hypothesis. On the other hand, $\mathcal{C}^3(G) = \cup_{1 \leq i \leq k} \mathcal{C}^3(G_i \cup K(C)) \cup \{C\}$ and $\mathcal{D}^3(G) = \cup_{1 \leq i \leq k} \mathcal{D}^3(G_i \cup K(C))$. Therefore, $H(G)$ can be obtained from $C$ and the trees $H(G_1 \cup K(C))$, ..., $H(G_k \cup K(C))$ by adding the edges $\{C, D_1\}, ..., \{C, D_k\}$. This implies that $H(G)$ is a tree. Next, suppose that some vertex $u \in V$ is contained in two graphs $D$ and $D'$ in $\mathcal{D}^3(G)$. If the path between $D$ and $D'$ in $H(G)$ does not pass $C$, then $u$ is contained in every graph on the path between $D$ and $D'$ in $H(G)$ by the inductive hypothesis. So, we may assume that the path between $D$ and $D'$ in $H(G)$ does pass $C$. Then, there are two neighbors $D_i$ and $D_j$, $1 \leq i, j \leq k$, of $C$ in $H(G)$ such that $D_i$ lies on the path between $D$ and $C$ and $D_j$ lies on the path between $D'$ and $C$. Moreover, $u$ must be contained in $C$ since every vertex shared by a pair of two graphs among $G_1 \cup K(C)$, ..., $G_k \cup K(C)$ must be contained in $C$. Hence, $u$ is contained in both $D_i$ and $D_j$. By the inductive hypothesis, $u$ is contained in every graph on both the path between $D$ and $D_i$ in $H(G_i \cup K(C))$ and the path between $D'$ and $D_j$ in $H(G_j \cup K(C))$. This implies that $u$ is contained in every graph on the path between $D$ and $D'$ in $H(G)$. Therefore, the statement (3) in the lemma holds for $G$.

Suppose that $\mathcal{D}^3(G) = \{D_1, ..., D_k\}$. Let $I = \{1, ..., q\}$. Root the tree $H(G)$ at $D_1$ and define $T^3(G)$ to be the tree whose vertex set is $I$ and edge set is $\{\{i, j'\} : D_i$ is the grandparent of $D_{j'}$ in the rooted tree $H(G)\}$. (Note that $T^3(G)$ is undirected.) Construct a supergraph $G^3$ of $G$ as follows: For each strong 3-cut $C$ and each pair of nonadjacent vertices $u$ and $v$ in $C$, add the edge $\{u, v\}$ to $G$.

**Corollary 10.** $(\{V(D_i) : i \in I\}, T^3(G))$ is a tree-decomposition of $G^3$.

Next, we proceed to considering how to decompose a $K_5$-free graph into induced subgraphs of small treewidth. The following lemma is useful:

**Lemma 11.** Let $G = (V, E)$ be a connected planar graph, and $k$ be an integer $\geq 2$. Suppose that $S$ is a subset of $V$ such that $G[S]$ is a triangle, and $(Y_1, ..., Y_k)$ is an unbalanced $k$-cover of $S$. Then, we can compute a $k$-cover $(Z_1, ..., Z_k)$ of $V$ in $O(k|V|)$ time such that $\text{tw}(G[Z_i]) \leq 6k - 7$ and $Z_i \cap S = Y_i$ for each $1 \leq i \leq k$, and $(Z_1 \cap S', ..., Z_k \cap S')$ is an unbalanced $k$-cover of $S'$ for all subsets $S'$ of $V$ with $G[S']$ being a triangle.
Proof. Let $S = \{s_1, s_2, s_3\}$. First, suppose that $Y_1, \ldots, Y_k$ is weakly unbalanced. Then, by symmetry, we may assume that $Y_1 = \{s_1\}$, $Y_2 = \{s_2, s_3\}$, and $Y_3 = \cdots = Y_k = S$. We perform a breadth-first-search (BFS) on $G$ starting at $s_1$ to obtain a BFS tree $T$. For each vertex $v$ in $G$, we define $\text{lev}(v)$ to be the length of the path from $s_1$ to $v$ in $T$. Note that $\text{lev}(s_1) = 0$ and both $\text{lev}(s_2) = \text{lev}(s_3) = 1$. For each $1 \leq l \leq k$, let $V_l = \{v \in V : \text{lev}(v) \equiv l - 1 \pmod{k}\}$. Let $Z_l = V - V_2$, $Z_2 = V - V_1$, and $Z_1 = V - V_l$ for each $3 \leq l \leq k$. Then, it should be clear that $(Z_1, \ldots, Z_k)$ is a $k$-cover of $V$ satisfying the conditions in the lemma.

Next, suppose that $Y_1, \ldots, Y_k$ is completely unbalanced. Then, by symmetry, we may assume that $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = S$. Let $H$ be the graph obtained from $G$ by contracting the three edges in the triangle $G[S]$. That is, $H$ is obtained from $G$ by identifying the three vertices in $S$ with a new vertex $x \not\in V$. It is clear that $H$ is still planar. We perform a breadth-first-search (BFS) on $H$ starting at $x$ to obtain a BFS tree $T$. For each vertex $v$ in $H$, we define $\text{lev}(v)$ to be the length of the path from $x$ to $v$ in $T$. Recall that $s_1, s_2$, and $s_3$ are not in $H$. We define $\text{lev}(s_1) = \text{lev}(s_2) = \text{lev}(s_3) = 0$. For each $1 \leq l \leq k$, let $Z_l = V - \{v \in V : \text{lev}(v) \equiv l - 1 \pmod{k}\}$. Obviously, $(Z_1, \ldots, Z_k)$ is a $k$-cover of $V$, $Z_l \cap S = Y_l$ for each $1 \leq l \leq k$, and $(Z_1 \cap S', \ldots, Z_k \cap S')$ is an unbalanced $k$-cover of $S'$ for all subsets $S'$ of $V$ with $G[S']$ being a triangle. It remains to show that $\text{tw}(G[Z_l]) \leq 6k - 7$ for each $1 \leq l \leq k$. To this end, fix an arbitrary $l$, $1 \leq l \leq k$. Consider a planar embedding of $G$. In the embedding, the triangle $G[S]$ splits the plane into two regions. Exactly one of the regions is infinite and the other is finite. Let $Z_l^{in}$ be the vertices of $Z_l$ falling into the finite region, and $Z_l^{out} = Z_l - Z_l^{in}$. It is not difficult to see that both $G[Z_l^{in}]$ and $G[Z_l^{out}]$ are $(k - 1)$-outerplanar (no matter whether $S \subseteq Z_l$ or not). From this, we observe that $G[Z_l]$ is $(2k - 2)$-outerplanar. Therefore, $\text{tw}(G[Z_l]) \leq 6k - 7$ [3].

Now, we are ready to show two main lemmas in this section.

Lemma 12. Let $G = (V, E)$ be a nice $K_5$-free graph, and $k$ be an integer $\geq 2$. Suppose that $s_1$ and $s_2$ are two adjacent vertices in $G$ and $(U_1, \ldots, U_k)$ is an unbalanced $k$-cover of $\{s_1, s_2\}$. Then, we can compute a $k$-cover $(V_1, \ldots, V_k)$ of $V$ in $O(k|V| + |V|^2)$ time such that $\text{tw}(G[V]) \leq 6k - 7$ and $V_l \cap \{s_1, s_2\} = U_l$ for each $1 \leq l \leq k$.

Proof. Let $\mathcal{D}^3(G) = \{D_1, \ldots, D_q\}$ be the strong 3-decomposition of $G$, and let $I = \{1, \ldots, q\}$. It is known that $\mathcal{D}^3(G)$ can be computed in $O(|V|^2)$ time [9]. W.l.o.g., we may assume that $G^3 = G$ because a $k$-cover $(V_1, \ldots, V_k)$ of $V$ such that the subgraph of $G^3$ induced by $V_i$ has treewidth $\leq 6k - 7$ for each $1 \leq i \leq k$ is also a $k$-cover $(V_1, \ldots, V_k)$ of $V$ such that $\text{tw}(G[V]) \leq 6k - 7$ for each $1 \leq l \leq k$. Then, by Fact 10, $(\{V(D_1) : j \in I\}, T^3(G))$ is a tree-decomposition of $G$. For convenience, let $T = T^3(G)$, $b = 6k - 7$, and $X_j = V(D_j)$ and $f(k, |X_j|) = O(k|X_j|)$ for each $j \in I$. We want to apply Lemma 2 to the graph $G$ and the tree-decomposition $(\{X_j : j \in I\}, T)$. To this end, we first choose an $r \in I$ with $\{s_1, s_2\} \subseteq X_r$ and root $T$ at $r$. Such an $r$ must exist because $\{s_1, s_2\}$ is an edge in $G$. 


Clearly, the condition (1) in Lemma 2 is satisfied by \( G \) and \((\{X_j : j \in I\}, T)\). By Fact 4, \( G[X_r] = D_r \) is planar. So, by Lemma 3, we can compute a \( k \)-cover \( \langle R_1, ..., R_k \rangle \) of \( X_r \) such that \( \text{tw}(G[R_l]) \leq 3k - 4 \) and \( R_l \cap \{s_1, s_2\} = U_l \) for each \( 1 \leq l \leq k \). Moreover, it is clear from the proof of Lemma 3 that for every subset \( S \) of \( X_r \) with \( G[S] \) being a triangle, \( (R_l \cap S, ..., R_k \cap S) \) is an unbalanced \( k \)-cover of \( S \). Now, it should be easy to verify that \( \langle R_1, ..., R_k \rangle \) is a \( k \)-cover of \( X_r \) satisfying the conditions (2a) and (2b) in Lemma 2.

Fix a \( j' \in I \) and a child \( j \) of \( j' \) in \( T \). Let \( (Y_1, ..., Y_k) \) be an unbalanced \( k \)-cover of \( X_{j'} \cap X_j \). Let \( S = X_{j'} \cap X_j \). Recall that \( G[S] \) is a triangle. So, we can compute a \( k \)-cover \( \langle Z_1, ..., Z_k \rangle \) of \( X_j \) satisfying the conditions in Lemma 11. It should be easy to see that \( \langle Z_1, ..., Z_k \rangle \) is a \( k \)-cover of \( X_j \) satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

By the discussions above and Lemma 2, there is a \( k \)-cover \( \langle V_1, ..., V_k \rangle \) of \( V \) such that \( \text{tw}(G[V]) \leq 6k - 7 \) and \( V_l \cap X_r = R_l \) for each \( 1 \leq l \leq k \). Fix an \( l \) with \( 1 \leq l \leq k \). Recall that \( R_l \cap \{s_1, s_2\} = U_l \) and that \( \{s_1, s_2\} \subseteq X_r \). Thus, \( V_l \cap \{s_1, s_2\} = V_l \cap (X_r \cap \{s_1, s_2\}) = R_l \cap \{s_1, s_2\} = U_l \). This establishes the lemma.

**Lemma 13.** Let \( G = (V, E) \) be a 2-connected \( K_5 \)-free graph. Then, for any \( k \geq 2 \), we can compute a \( k \)-cover \( \langle V_1, ..., V_k \rangle \) of \( V \) in \( O(k|V| + |V|^2) \) time such that \( \text{tw}(G[V]) \leq 6k - 7 \) for each \( 1 \leq l \leq k \).

**Proof.** Let \( \mathcal{D} = \{D_1, ..., D_q\} \) be a 2-decomposition of \( G \), and let \( I = \{1, ..., q\} \). It is known that \( \mathcal{D} \) can be computed in \( O(|V|) \) time [8]. W.l.o.g., we may assume that \( G^2(\mathcal{D}) = G \) because a \( k \)-cover \( \langle V_1, ..., V_k \rangle \) of \( V \) such that the subgraph of \( G^2(\mathcal{D}) \) induced by \( V_l \) has treewidth \( \leq 6k - 7 \) for each \( 1 \leq l \leq k \) is also a \( k \)-cover \( \langle V_1, ..., V_k \rangle \) of \( V \) such that \( \text{tw}(G[V]) \leq 6k - 7 \) for each \( 1 \leq l \leq k \). Then, by Fact 2, \( (\{V(D_j) : j \in I\}, T^2(G, \mathcal{D})) \) is a tree-decomposition of \( G \). For convenience, let \( T = T^2(G, \mathcal{D}), b = 6k - 7 \), and \( X_j = V(D_j) \) and \( f(k, |X_j|) = O(k|X_j| + |X_j|^2) \) for each \( j \in I \). We want to apply Lemma 2 to the graph \( G \) and the tree-decomposition \((\{X_j : j \in I\}, T)\). To this end, we first (arbitrarily) choose an \( r \in I \) and root \( T \) at \( r \).

Clearly, the condition (1) in Lemma 2 is satisfied by \( G \) and \((\{X_j : j \in I\}, T)\). To see that the condition (2) in Lemma 2 is also satisfied, we distinguish four cases as follows:

**Case 1:** \( G[X_r] \) is planar. Then, as stated in the proof of Lemma 5, we can compute a \( k \)-cover \( \langle R_1, ..., R_k \rangle \) of \( X_r \) in \( O(k|X_r|) \) time satisfying the conditions (2a) and (2b) in Lemma 2.

**Case 2:** \( G[X_r] \) is isomorphic to \( K_{3,3} \). Then, we set \( R_1 = \emptyset \) and \( R_2 = \cdots = R_k = X_r \). Obviously, \( \langle R_1, ..., R_k \rangle \) is a \( k \)-cover of \( X_r \) satisfying the conditions (2a) and (2b) in Lemma 2.

**Case 3:** \( G[X_r] \) is isomorphic to the graph \( W \) (see Definition 8). Then, we set \( R_1 = \emptyset \) and \( R_2 = \cdots = R_k = X_r \) if \( k \geq 3 \); otherwise \( (k = 2) \), we (arbitrarily) choose four vertices from \( X_r \) and set \( R_1 \) to be the set of the four vertices and \( R_2 \) to be \( X_r - R_1 \). Obviously, \( \langle R_1, ..., R_k \rangle \) is a \( k \)-cover of \( X_r \) satisfying the conditions (2a) and (2b) in Lemma 2.

**Case 4:** \( G[X_r] \) is a nice \( K_5 \)-free graph. Then, we arbitrarily choose an edge \( \{s_1, s_2\} \) in \( G[X_r] \) and set \( U_1 = \emptyset \) and \( U_2 = \cdots = U_k = \{s_1, s_2\} \). By Lemma 12, we can compute a \( k \)-cover \( \langle R_1, ..., R_k \rangle \) of \( X_r \) in \( O(k|X_r| + |X_r|^2) \) time such that \( \text{tw}(G[R_l]) \leq 6k - 7 \) for each \( 1 \leq l \leq k \). Clearly, \( \langle R_1, ..., R_k \rangle \) satisfies the
condition (2a) in Lemma 2. \((R_1, ..., R_k)\) also satisfies the condition (2b) in Lemma 2 since \(|X_r \cap X_j^{;}| = 2\) for every child \(j^{;}\) of \(r\) in \(T\).

Note that one of the above four cases must occur. Thus, the condition (2) in Lemma 2 is satisfied by \(G\) and \(\{(X_j : j \in I), T\}\). To see that the condition (3) in Lemma 2 is also satisfied, fix a \(j' \in I\) and a child \(j\) of \(j'\) in \(T\). Let \((Y_1, ..., Y_k)\) be an unbalanced \(k\)-cover of \(X_{j'} \cap X_j\), and let \(X_{j'} \cap X_j = \{s_1, s_2\}\). Recall that \(\{s_1, s_2\}\) is an edge in both \(G[X_{j'}]\) and \(G[X_j]\). Moreover, by symmetry, we may assume that \(|Y_l| \leq |Y_{l+1}|\) for all \(1 \leq l \leq k - 1\). We distinguish four cases as follows:

**Case 1**: \(G[X_j]\) is planar. Then, as stated in the proof of Lemma 5, we can compute a \(k\)-cover \((Z_1, ..., Z_k)\) of \(X_j\) in \(O(k|X_j|)\) time satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

**Case 2**: \(G[X_j]\) is isomorphic to \(K_{3,3}\). Then, we set \(Z_1 = Y_1\) and \(Z_l = Y_l \cup (X_j - X_{j'})\) for each \(2 \leq l \leq k\). Clearly, \((Z_1, ..., Z_k)\) is a \(k\)-cover of \(X_j\) satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

**Case 3**: \(G[X_j]\) is isomorphic to the graph \(W\) (see Definition 8). If \(k \geq 3\), then we set \(Z_1 = Y_1\) and \(Z_l = Y_l \cup (X_j - X_{j'})\) for each \(2 \leq l \leq k\); otherwise \((k = 2)\), we (arbitrarily) choose a subset \(A\) of \(X_j - X_{j'}\) with \(|A| = 3\) and set \(Z_1 = Y_1 \cup A\) and \(Z_2 = X_j - Z_1\). Then, it is easy to verify that \((Z_1, ..., Z_k)\) is a \(k\)-cover of \(X_j\) satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

**Case 4**: \(G[X_j]\) is a nice \(K_5\)-free graph. Then, by Lemma 12, we can compute a \(k\)-cover \((Z_1, ..., Z_k)\) of \(X_j\) in \(O(k|X_j| + |X_j|^2)\) time such that \(\text{tw}(G[Z_j]) \leq 6k - 7\) and \(Z_l \cap \{s_1, s_2\} = Y_l\) for each \(1 \leq l \leq k\). From this, it should be clear that \((Z_1, ..., Z_k)\) satisfies the conditions (3a), (3b), and (3c) in Lemma 2.

Note that one of the four cases must occur. Thus, by the discussions above and Lemma 2, we have the lemma.

**Theorem 14.** Let \(G = (V, E)\) be a \(K_k\)-free graph. Then, for any \(k \geq 2\), we can compute a \(k\)-cover \((V_1, ..., V_k)\) of \(V\) in \(O(k|V| + |V|^2)\) time such that \(\text{tw}(G[V_l]) \leq 6k - 7\) for each \(1 \leq l \leq k\).

**Proof.** This follows from Fact 1, Lemma 2, and Lemma 13 immediately.

**Corollary 15.** Let \(\pi\) be a hereditary property on graphs. Suppose that \(\text{MISP}(\pi)\) restricted to \(n\)-vertex graphs of treewidth \(\leq \ell\) can be solved in \(T_\pi(k, n)\) time. Then, given an integer \(k \geq 2\) and a \(K_k\)-free graph \(G = (V, E)\), we can compute a subset \(U\) of \(V\) in \(O(k|V| + |V|^2 + T_\pi(6k - 7, |V|))\) time such that \(G[U]\) satisfies \(\pi\) and \(|U|\) is at least \((k - 1)/k\) optimal.

**Proof.** Similar to that of Corollary 7.

For many \(\pi\)'s, it is well known that \(T_\pi(k, n) = O(c^k n)\) for some small \(c\), and hence \(\text{MISP}(\pi)\) restricted to \(K_k\)-free graphs has a practical PTAS by Corollary 15.

### 6 Concluding remarks

We have shown that many \(\text{MISP}(\pi)\)'s restricted to \(K_{3,3}\)-free or \(K_5\)-free graphs have practical PTASs. It is worth mentioning that these PTASs are easy to parallelize. Since the details are almost trivial, we omit
them here.

References