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A continuous version of Gale's feasibility theorem

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1. Introduction

There are several approaches to formulate flow problems on continuous networks. In this paper, using a formulation due to Iri (1979) and Strang (1983), we establish a continuous version of Gale's feasibility theorem [1].

The theorem is known as the "Supply-Demand Theorem" in a special case. By means of a cut capacity, this gives a necessary and sufficient condition for an existence of feasible flows.

Let us recall our formulation of continuous network and state a continuous version of the Supply-Demand Theorem. As for a discrete version, one can refer to Ford and Fulderson's book (1962). In this discussion, we assume that all functions and sets are sufficiently smooth. Let $\Omega$ be a bounded domain of $n$-dimensional Euclidean space $\mathbb{R}^n$ and $\partial\Omega$ be the boundary. Let $A, B$ be disjoint subsets of $\partial\Omega$ which are regarded as a source and a sink. In our continuous network, every flow is represented by a vector field and every feasible flow $\sigma$ satisfies the capacity constraint which is written as

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega,$$

where $\Gamma$ is a set-valued mapping from $\Omega$ to $\mathbb{R}^n$. The flow value of $\sigma$ is defined by $\sigma \cdot \nu$ on $\partial\Omega$. We call $\Omega$ with this capacity constraint a continuous network.

Furthermore, every cut is identified with a subset of $\Omega$ in our network. Let $S$ be a cut and $\nu^S$ be the unit outer normal to $S$. Then the cut capacity $C(S)$ is defined by

$$C(S) = \int_{\Omega \cap \partial S} \beta(\nu^S(x), x) ds(x),$$

where

$$\beta(v, x) = \sup_{w \in \Gamma(x)} v \cdot w$$

for $v \in \mathbb{R}^n$ and $ds$ is the surface element. If the capacity constraint is isotropic, that is, $\Gamma(x) = \{w \in \mathbb{R}^n \mid |w| \leq c(x)\}$ with some nonnegative function $c(x)$, then

$$C(S) = \int_{\Omega \cap \partial S} c(x) ds(x).$$
Let \( a, b \) be real-valued functions on \( A, B \) respectively and let \( \nu \) be the unit outer normal to \( \Omega \). Then the problem of supply-demand in a simple case is stated as follows:

\[
\text{(SD)} \quad \begin{align*}
\sigma(x) &\in \Gamma(x) \text{ for all } x \in \Omega, \\
\text{div } \sigma &= 0 \text{ on } \Omega, \quad -\sigma \cdot \nu = 0 \text{ on } \partial \Omega - (A \cap B), \\
-\sigma \cdot \nu &\leq a \text{ on } A, \quad \sigma \cdot \nu \geq b \text{ on } B.
\end{align*}
\]

The Supply-Demand theorem assures that (SD) has a solution if and only if

\[
\text{(G)} \quad C(S) \geq \int_{B \cap \partial S} bds - \int_{A \cap \partial S} ads \quad \text{for each cut } S.
\]

This can be proved by the aid of a continuous version of max-flow min-cut theorem under some assumptions. However, we can not apply the same method to a variant of (SD), which is called a symmetric type by Ford and Fulkerson.

On the other hand, Neumann [5] and Oettli and Yamasaki [8] investigated a problem of feasibility of flows and proved similar results in their own network formulations. Their method is based on a generalized Hahn-Banach Theorem and is applicable even for a symmetric supply-demand problem. In the next section, we give a concrete formulation of our problem in a more general form than (SD), and give a corresponding condition which is equivalent with an existence of solutions for the problem under suitable assumptions. Finally in §3, we consider (SD) as a special case and examine the assumptions.

2. Problem setting and a main theorem

Let \( \Omega \) be a bounded domain in \( n \)-dimensional Euclidean space \( R^n \) with Lipschitz boundary \( \partial \Omega \). One can consider \( n - 1 \)-dimensional surface measure on \( \partial \Omega \) which is equal to \( n - 1 \)-dimensional Hausdorff measure \( H_{n-1} \) on \( \partial \Omega \). We note that the unit outer normal \( \nu \) to \( \Omega \) is defined and essentially bounded measurable on \( \partial \Omega \) with respect to \( H_{n-1} \). Let \( \Gamma \) be a set-valued mapping from \( \Omega \) to \( R^n \) which satisfies the following two conditions:

\[
\text{(H1)} \quad \Gamma(x) \text{ is a compact convex set containing } 0 \text{ for all } x \in \Omega.
\]

\[
\text{(H2)} \quad \text{Let } \varepsilon > 0 \text{ and } \Omega_0 \text{ be a compact subset of } \Omega.
\]

Then there is \( \delta > 0 \) such that

\[
\Gamma(x) \subset \Gamma(y) + B(0, \varepsilon) \text{ if } x, y \in \Omega_0 \text{ and } |x - y| < \delta.
\]
In what follows, we assume that each feasible flow is represented by an essentially bounded vector field $\sigma$ on $\Omega$ satisfying the following capacity constraints:

$$\sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega.$$ 

Furthermore if $\text{div } \sigma \in L^n(\Omega)$, then $\sigma \cdot \nu$ can be defined as a function in $L^\infty(\partial \Omega)$ in a weak sense by Kohn and Temam [2]. Let $F \in L^n(\Omega)$ and $\lambda, \mu \in L^\infty(\partial \Omega)$ with $\lambda \leq \mu$. Then for the quintuple $(\Omega, \Gamma, F, \mu, \lambda)$, our problem is stated as follows:

$$(P) \quad \text{Find } \sigma \in L^\infty(\Omega; \mathbb{R}^n) \text{ such that }$$

$$\sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega,$$

$$\text{div } \sigma = F \quad \text{a.e. on } \Omega \text{ and } \lambda \leq \sigma \cdot \nu \leq \mu \text{ a.e. on } \partial \Omega.$$ 

Problem (SD) considered in §1 can be written in this form with $F = 0$.

To specify the class of cuts, we consider the space $BV(\Omega)$ of functions of bounded variation on $\Omega$:

$$BV(\Omega) = \{u \in L^1(\Omega) \mid \nabla u \text{ is a Radon measure of bounded variation on } \Omega\},$$

where $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ is understood in the sense of distribution. We denote the characteristic function of a subset $S$ of $\Omega$ by $\chi_S$ and set

$$Q = \{S \subset \Omega \mid \chi_S \in BV(\Omega)\}.$$ 

Let $S \in Q$. Then the reduced boundary $\partial^* S$ of $S$ is the set of all $x \in \partial S$ where Federer's normal $\nu = \nu(x)$ to $S$ exists. It is known that $\partial^* S$ is a measurable set with respect to both the measure of total variation of $|\nabla \chi_S|$ and $H_{n-1}$, $|\nabla \chi_S|(R^n - \partial^* S) = 0$ and $|\nabla \chi_S|(E) = H_{n-1}(E)$ for each $|\nabla \chi_S|$-measurable subset $E$ of $\partial^* S$. Furthermore let $\gamma u \in L^1(\partial \Omega)$ be the trace of $u \in BV(\Omega)$. Then [4; Theorem 6.6.2] implies that $\gamma \chi_S = \chi_{\partial^* S \cap \partial \Omega}$ $H_{n-1}$-a.e. on $\partial \Omega$. Accordingly, replacing $ds$ by $H_{n-1}$ and $\partial S$ by $\partial^* S$, we can define the cut capacity as follows:

$$C(S) = \int_{\Omega \cap \partial^* S} \beta(\nu^S(x), x) dH_{n-1},$$

where $\beta(\cdot, x)$ is the support functional of $\Gamma(x)$ as defined in §1. Let $\nabla u/|\nabla u|$ be the Radon-Nikodym derivative of $\nabla u$ with respect to $|\nabla u|$ and set

$$\psi(u) = \int_{\Omega} \beta(\nabla u/|\nabla u|, x) d|\nabla u|(x)$$
for \( u \in BV(\Omega) \). Then \( C(S) = \psi(\chi_S) \). Since \( \beta \) is continuous and nonnegative by (H1) and (H2), \( C(S) \) is finite. We set
\[
\lambda(S) = \int_{\partial^*S} \lambda dH_{n-1}, \quad \mu(S) = \int_{\partial^*S} \mu dH_{n-1}, \quad F(S) = \int_S F dx.
\]
for convenience sake, and consider the condition
\[
(C) \quad C(S) \geq \lambda(S) - F(S) \quad \text{and} \quad C(S) \geq -\mu(\Omega - S) + F(\Omega - S)
\]
hold for all \( S \in Q \).

Now we can state a continuous version of Gale’s feasibility theorem.

**Theorem 2.1.** Assume that (H1) and (H2) hold. If \((P)\) has a solution, then condition (C) holds. Conversely if \( \bigcup_{x \in \Omega} \Gamma(x) \) is bounded and condition (C) holds, then \((P)\) has a solution.

To prove this theorem, we need some lemmas. First applying an isoperimetric inequality due to [4] we have

**Lemma 2.2.** There is \( \sigma_0 \in L^\infty(\Omega; \mathbb{R}^n) \) such that \( \text{div} \sigma = F \) a.e. on \( \Omega \).

**Proof:** First assume that \( \int_{\Omega} F dx = 0 \). We use a max-flow min-cut theorem of Strang’s type (1983):
\[
\sup \{ t \geq 0 \mid \text{div} \sigma = -tF \text{ a.e. on } \Omega, \; \sigma \cdot \nu = 0 \text{ H}_{n-1}\text{-a.e. on } \partial \Omega \}
\]
for some \( \sigma \in L^\infty(\Omega; \mathbb{R}^n) \) with \( \|\sigma\|_\infty \leq 1 \)
\[
= \inf \{ H_{n-1}(\Omega \cap \partial^* S) / \int_S F dx \mid \int_S F dx > 0, \; S \subset \Omega, \chi_S \in BV(\Omega) \}.
\]

(The proof is in [6].) To prove the existence of \( \sigma_0 \), it is sufficient to show that the supremum is positive. We can prove that the infimum is positive as follows. According to [4; p.303] there is a positive constant \( k \) such that
\[
\min(m_n(S), m_n(\Omega - S)) \leq k H_{n-1}(\Omega \cap \partial^* S)^{(n-1)/n},
\]
where \( m_n \) denotes the Lebesgue measure on \( \mathbb{R}^n \). Since
\[
\int_S F dx \leq (\int_S 1 dx)^{(n-1)/n} \cdot \left( \int_S |F|^n dx \right)^{1/n} \leq \|F\|_n (m_n(S))^{(n-1)/n}
\]
and
\[
\int_S F dx = \int_{\Omega - S} -F dx \leq (\int_{\Omega - S} 1 dx)^{(n-1)/n} \cdot \left( \int_{\Omega - S} |F|^n dx \right)^{1/n}
\]
\[
\leq \|F\|_n (m_n(\Omega - S))^{(n-1)/n},
\]
we can conclude that

$$\int_{S} Fdx \leq k_{1}H_{n-1}(\Omega \cap \partial^{*}S)$$

with $k_{1} = \|F\|_{n}k^{(n-1)/n}$ for all $S \in Q$. It follows that the infimum is not less than $1/k_{1}$.

Finally in case of $\int_{\Omega} Fdx \neq 0$, consider $\sigma_{1}$ such that $\text{div} \ \sigma_{1}$ equals constantly $\int_{\Omega} Fdx$, $\sigma_{2}$ such that $\text{div} \ \sigma_{2} = F - \int_{\Omega} Fdx$ and set $\sigma_{0} = \sigma_{1} + \sigma_{2}$. Then $\text{div} \ \sigma_{0} = F$. This completes the proof.

From now on we fix $\sigma_{0}$ in Lemma 2.2. For $\sigma \in L^{\infty}(\Omega; R^{n})$ such that $\text{div} \ \sigma \in L^{n}(\Omega)$ and $u \in BV(\Omega)$, according to [2] we can define the distribution $(\sigma \nabla u)$ by

$$(\sigma \nabla u)(\varphi) = -\int_{\Omega} u\nabla \varphi \cdot \sigma dx - \int_{\Omega} u\varphi \text{div} \ \sigma dx$$

for $\varphi \in C_{0}^{\infty}(\Omega)$. Since $BV(\Omega) \subset L^{n/(n-1)}(\Omega)$, each integral in the definition is finite. Furthermore it is known that $(\sigma \nabla u)$ is regarded as a bounded measure and that

$$(\sigma \nabla u)(\Omega) + \int_{\Omega} u\text{div} \ \sigma dx = \int_{\partial \Omega} \gamma u \sigma \cdot \nu dH_{n-1}$$

holds. This is Green's formula due to Kohn and Temam [2; Proposition 1.1]. (See also [6; Theorem 2.3].) Using this formula, we can prove

**Lemma 2.3.** If (P) has a solution, then (C) holds.

**Proof:** Let $\sigma$ be a solution of (P). Then by Green's formula stated above,

$$C(S) \geq (\sigma \nabla \chi_{S}) (\Omega) = \int_{\partial \Omega \cap \partial^{*}S} \sigma \cdot \nu dH_{n-1} - \int_{S} \text{div} \ \sigma dx$$

$$\geq \lambda(S) - F(S).$$

Another inequality in (C) can be similarly proved.

To prove the converse, we follow the idea in [5] and [8]. Let us consider the Sobolev space

$$W^{1,1}(\Omega) = \{ u \in L^{1}(\Omega) \mid \nabla u \in L^{1}(\Omega; R^{n}) \},$$

which is a linear subspace of $BV(\Omega)$. We set

$$U = L^{1}(\Omega; R^{n}) \times L^{1}(\partial \Omega) \text{ and } V = \{ (\nabla u, \gamma u) \mid u \in W^{1,1}(\Omega) \}.$$
Since $\gamma u \in L^1(\partial \Omega)$ for $u \in W^{1,1}(\Omega)$, $V$ is a linear subspace of $U$. Let $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$. Note that $u^+, u^- \in W^{1,1}(\Omega)$. We define a functional $\Phi$ on $V$ by

$$\Phi(\nabla u, \gamma u) = \int_{\Omega} \sigma_0 \cdot \nabla u \, dx - \int_{\partial \Omega} \sigma_0 \cdot \nu \gamma u \, dH_{n-1} + \int_{\partial \Omega} \lambda \gamma u^+ \, dH_{n-1} - \int_{\partial \Omega} \mu \gamma u^- \, dH_{n-1}$$

and set $K = \{\sigma \in L^\infty(\Omega; \mathbb{R}^n) | \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega\}$.

For $v \in L^1(\Omega; \mathbb{R}^n)$, we define a functional $\rho$ on $U$ by

$$\rho(v, \alpha) = \int_{\Omega} \beta(v(x), x) \, dx = \sup_{\phi \in K} \int_{\Omega} v \cdot \phi \, dx$$

for $(v, \alpha) \in U$. The last equality follows from a measurable selection theorem. (Cf. Castaing and Valadier (1977).) Since $\rho(v, \alpha)$ is independent of $\alpha$, it is sometimes denoted by $\rho(v)$. We note that $\psi(u) = \rho(\nabla u)$ for all $u \in W^{1,1}(\Omega)$. The inequality $\lambda \leq \mu$ implies the next lemma.

**Lemma 2.4.** $\Phi$ is superlinear on $V$, that is, concave and positively homogeneous, and $\rho$ is sublinear on $U$, that is, $-\rho$ is superlinear. Furthermore $\rho$ is continuous at the origin of $U$ if $\cup_{x \in \Omega} \Gamma(x)$ is bounded.

Condition (C) can be replaced by an inequality with $\Phi$ and $\rho$.

**Lemma 2.5.** If (C) holds, then $\Phi \leq \rho$ on $V$.

**Proof:** We use equalities of coarea formula type which are stated in [6]: Let $u \in W^{1,1}(\Omega)$. Set $N_t = \{x \in \Omega | u(x) \geq t\}$ and $M_t = \Omega - N_t$ for any real number $t$. Then $N_t, M_t \in Q$ for a.e. $t$ and

$$\psi(u) = \int_{-\infty}^{\infty} \psi(\chi_{N_t}) \, dt.$$ 

Furthermore by [6; Lemma 4.6]

$$\int_{\Omega} F u \, dx = \int_{0}^{\infty} \left( \int_{\Omega} F \chi_{N_t} \, dx - \int_{\Omega} F \chi_{M_{-t}} \, dx \right) dt,$$

$$\int_{\partial \Omega} \lambda \gamma u^+ \, dH_{n-1} = \int_{0}^{\infty} \int_{\partial \Omega} \lambda \gamma \chi_{N_t} \, dH_{n-1} \, dt,$$

$$\int_{\partial \Omega} \mu \gamma u^- \, dH_{n-1} = \int_{0}^{\infty} \int_{\partial \Omega} \mu \gamma \chi_{M_{-t}} \, dH_{n-1} \, dt.$$
It follows from these equalities and (C) that

\[
\rho(\nabla u) = \psi(u) = \int_{-\infty}^{\infty} \psi(\chi_{N_{t}})dt = \int_{0}^{\infty} \psi(\chi_{N_{t}})dt + \int_{0}^{\infty} \psi(\chi_{-M_{-t}})dt
\]

\[
= \int_{0}^{\infty} C(N_{t})dt + \int_{0}^{\infty} C(\Omega - M_{-t})dt
\]

\[
= \int_{0}^{\infty} (\lambda(N_{t}) - F(N_{t}))dt + \int_{0}^{\infty} (-\mu(M_{-t}) + F(M_{-t}))dt
\]

\[
\geq \int_{0}^{\infty} (\int_{\partial\Omega} \lambda\gamma\chi_{N_{t}}dH_{n-1} - \int_{\Omega} F\chi_{N_{t}}dx)dt
\]

\[
+ \int_{0}^{\infty} (-\int_{\partial\Omega} \mu\gamma\chi_{M_{-t}}dH_{n-1} + \int_{\Omega} F\chi_{M_{-t}}dx)dt
\]

\[
= \int_{\partial\Omega} \lambda\gamma u^{+}dH_{n-1} - \int_{\partial\Omega} \mu\gamma u^{-}dH_{n-1} - \int_{\Omega} u\text{div}{\sigma_{0}}dx
\]

\[
= \int_{\partial\Omega} \lambda\gamma u^{+}dH_{n-1} - \int_{\partial\Omega} \mu\gamma u^{-}dH_{n-1}
\]

\[
- \int_{\partial\Omega} \sigma_{0} \cdot \nu\gamma uH_{n-1} + \int_{\Omega} \sigma_{0} \cdot \nabla udX
\]

\[
\geq \Phi(\nabla u, \gamma u).
\]

Here we have used Green's formula in the last equality. This completes the proof.

By Lemma 2.5 and a version of Hahn-Banach theorem ([3; Corollary 2.2 in p.114]), there is a linear functional \(\xi\) on \(U\) satisfying \(\Phi \leq \xi\) on \(V\) and \(\xi \leq \rho\) on \(U\). The next lemma is directly proved.

**Lemma 2.6.** If \(\cup_{x \in \Omega} \Gamma(x)\) is bounded, then \(\xi\) is continuous on \(U\) with respect to the canonical norm topology.

By Lemma 2.6, there is \(\sigma \in L^\infty(\Omega; R^n)\) and \(\eta \in L^\infty(\partial\Omega)\) such that

\[
\xi(v, \alpha) = \int_{\Omega} \sigma \cdot vdx + \int_{\partial\Omega} \eta\alpha dH_{n-1}
\]

for all \((v, \alpha) \in U\). However, from the inequality \(\xi(v, \alpha) \leq \rho(v)\) for all \(\alpha \in L^\infty(\partial\Omega)\), \(\eta\) must be 0.
LEMMA 2.7. Assume that $\cup_{x \in \Omega} \Gamma(x)$ is bounded. Then the vector field $\sigma$ obtained above is a solution to (P).

PROOF: We set $\Omega_0 = \{x \in \Omega| 0 \notin \Gamma(x) - \sigma(x)\}$. Then $\Omega_0$ is a measurable set. Assume that the measure of $\Omega_0$ is positive. Since $K = \{\phi \in L^{\infty}(\Omega; R^n)| \phi(x) \in \Gamma(x) - \sigma(x)\}$ is a weakly* closed convex set and does not contain 0, there is $\varphi \in L^1(\Omega; R^n)$ such that $\sup_{\phi \in K} \int_{\Omega} \varphi \cdot \phi dx < 0$. Therefore

$$\rho(\varphi) = \sup_{\phi \in K} \int_{\Omega} \varphi \cdot (\phi + \sigma) dx < \int_{\Omega} \varphi \cdot \sigma dx = \xi(\varphi, 0).$$

This is a contradiction since $\xi \leq \rho$ on $U$. Thus $\sigma(x) \in \Gamma(x)$ for almost all $x \in \Omega$.

Next we prove $\text{div} \sigma = F$. If $u \in C_0^\infty(\Omega)$, then $\gamma u = 0$ so that

$$\Phi(\nabla u, \gamma u) = \int_{\Omega} \sigma_0 \cdot \nabla udX \leq \xi(\nabla u, 0) = \int_{\Omega} \sigma \cdot \nabla u dx.$$

It follows that

$$\int_{\Omega} \sigma_0 \cdot \nabla u dx = \int_{\Omega} \sigma \cdot \nabla u dx$$

for all $u \in C_0^\infty(\Omega)$. This implies that $\text{div} \sigma = \text{div} \sigma_0 = F$ in a distribution sense.

Finally we prove that $\lambda \leq \sigma \cdot \nu \leq \mu H_{n-1}$-a.e. on $\partial \Omega$. Since $\text{div} \sigma = F \in L^n(\Omega)$, $\sigma \cdot \nu$ is defined as a function in $L^{\infty}(\partial \Omega)$ and the inequality $\Phi(\nabla u, \gamma u) \leq \int_{\Omega} \sigma \cdot \nabla u dx$ implies that

$$\int_{\partial \Omega} \lambda \gamma u^+ - \mu \gamma u^- dH_{n-1} \leq \int_{\partial \Omega} \gamma u \sigma \cdot \nu dH_{n-1}.$$

For any $\alpha \in L^1(\partial \Omega)$, there is $u \in W^{1,1}(\Omega)$ such that $\alpha = \gamma u$ by Gagliardo (1957). Thus for any nonnegative function $\alpha \in L^1(\partial \Omega)$, we have

$$\int_{\partial \Omega} \lambda \alpha dx \leq \int_{\partial \Omega} \sigma \cdot \nu \alpha dH_{n-1},$$

$$- \int_{\partial \Omega} \mu \alpha dx \leq - \int_{\partial \Omega} \sigma \cdot \nu \alpha dH_{n-1}.$$

Accordingly, $\lambda \leq \sigma \cdot \nu \leq \mu H_{n-1}$-a.e. on $\partial \Omega$. This completes the proof.

PROOF OF THEOREM 2.1: The first statement follows from Lemma 2.3 and the second statement follows from Lemma 2.7.
3. Supply - Demand theorem

Let $A, B$ be disjoint Borel subsets of $\partial \Omega$ and $a, b$ be Borel measurable functions on $A, B$ respectively. Then (SD) in §1 should be written in the following concrete form:

\[(SD) \quad \text{Find } \sigma \in L(\Omega; \mathbb{R}^n) \]
\[\text{such that } \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega, \]
\[\text{div } \sigma = 0 \text{ a.e. on } \Omega, \]
\[\sigma \cdot \nu = 0 \text{ } H_{n-1}\text{-a.e. on } \partial \Omega - (A \cap B), \]
\[- \sigma \cdot \nu \leq a \text{ } H_{n-1}\text{-a.e. on } A, \]
\[\sigma \cdot \nu \geq b \text{ } H_{n-1}\text{-a.e. on } B.\]

By setting $\lambda = -a$ on $A$, $\lambda = b$ on $B$, $\lambda = 0$ elsewhere on $\partial \Omega$ and $\mu = \max(\lambda, 0)$, Theorem 2.1 implies

**Theorem 3.1.** Assume that (H1), (H2) hold and that $\bigcup_{x \in \Omega} \Gamma(x)$ is bounded. Then (SD) has a solution if and only if

\[(G) \quad C(S) \geq \int_{B \cap \partial^* S} b dH_{n-1} - \int_{A \cap \partial^* S} a dH_{n-1} \text{ for all } S \in Q.\]

Finally we refer to a relation between (SD) and a max-flow problem of Strang's type (MFS) which has been used in the proof of Lemma 2.2 with the boundary condition $\sigma \cdot \nu = 0$. Now let $f$ be an arbitrary function in $L^\infty(\partial \Omega)$ which satisfies the conservation law $\int_{\partial \Omega} f dH_{n-1} = 0$. Then for $(\Omega, \Gamma, f)$, (MFS) with $F = 0$ is stated as follows:

\[(MFS) \quad \text{Maximize } \lambda \]
\[\text{subject to } (\lambda, \sigma) \in R \times L^\infty(\Omega; \mathbb{R}^n), \]
\[\sigma(x) \in \Gamma(x) \text{ a.e. } x \in \Omega, \]
\[\text{div } \sigma = 0 \text{ a.e. on } \Omega, \sigma \cdot \nu = \lambda f \text{ a.e. on } \partial \Omega,\]

and the corresponding min-cut problem (MCS) is

\[(MCS) \quad \text{Minimize } C(S)/L(S) \]
\[\text{subject to } S \subset \Omega, \chi_S \in BV(\Omega), L(S) > 0,\]

where $L(S) = \int_{\partial \Omega \cap \partial^* S} f dH_{n-1}$. Then we have
PROPOSITION 3.2. Assume that (H1) and (H2) hold.

(1) Assume that (G) implies the existence of solutions to (SD) for any disjoint Borel subsets $A, B$ of $\partial \Omega$ and $a \in L^\infty(A), b \in L^\infty(B)$. Then $MFS = MCS$ and $(MFS)$ has an optimal solution for any $f \in L^\infty(\partial \Omega)$ satisfying the conservation law.

(2) Conversely if $MFS = MCS$ and $(MFS)$ has an optimal solution for any $f \in L^\infty(\partial \Omega)$ satisfying the conservation law, then (G) implies the existence of solutions to (SD) for any disjoint Borel subsets $A, B$ of $\partial \Omega$ and $a \in L^\infty(A), b \in L^\infty(B)$ such that $\int_A adH_{n-1} = \int_B bdH_{n-1}$.

It is known that there is an example with $MFS < MCS$ if $\Gamma$ is unbounded. (See [7].) Thus Proposition 3.2 (1) shows that there is an example of (SD) such that $\bigcup_{x \in \Omega} \Gamma(x)$ is bounded, condition (G) is satisfied and (SD) has no solution.

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References