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Kyoto University
A continuous version of Gale’s feasibility theorem

Ryōhei Nozawa

1. Introduction

There are several approaches to formulate flow problems on continuous networks. In this paper, using a formulation due to Iri (1979) and Strang (1983), we establish a continuous version of Gale’s feasibility theorem [1].

The theorem is known as the “Supply-Demand Theorem” in a special case. By means of a cut capacity, this gives a necessary and sufficient condition for an existence of feasible flows.

Let us recall our formulation of continuous network and state a continuous version of the Supply-Demand Theorem. As for a discrete version, one can refer to Ford and Fulderson’s book (1962). In this discussion, we assume that all functions and sets are sufficiently smooth. Let $\Omega$ be a bounded domain of $n$-dimensional Euclidean space $R^n$ and $\partial\Omega$ be the boundary. Let $A, B$ be disjoint subsets of $\partial\Omega$ which are regarded as a source and a sink. In our continuous network, every flow is represented by a vector field and every feasible flow $\sigma$ satisfies the capacity constraint which is written as

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega,$$

where $\Gamma$ is a set-valued mapping from $\Omega$ to $R^n$. The flow value of $\sigma$ is defined by $\sigma \cdot \nu$ on $\partial\Omega$. We call $\Omega$ with this capacity constraint a continuous network.

Furthermore, every cut is identified with a subset of $\Omega$ in our network. Let $S$ be a cut and $\nu^S$ be the unit outer normal to $S$. Then the cut capacity $C(S)$ is defined by

$$C(S) = \int_{\Omega \cap \partial S} \beta(\nu^S(x), x) ds(x),$$

where

$$\beta(v, x) = \sup_{w \in \Gamma(x)} v \cdot w$$

for $v \in R^n$ and $ds$ is the surface element. If the capacity constraint is isotropic, that is, $\Gamma(x) = \{w \in R^n \mid |w| \leq c(x)\}$ with some nonnegative function $c(x)$, then

$$C(S) = \int_{\Omega \cap \partial S} c(x) ds(x).$$
Let $a, b$ be real-valued functions on $A, B$ respectively and let $\nu$ be the unit outer normal to $\Omega$. Then the problem of supply-demand in a simple case is stated as follows:

\[(\text{SD}) \]
Find $\sigma$ such that
\[
\sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega,
\]
\[
\text{div } \sigma = 0 \text{ on } \Omega, \quad -\sigma \cdot \nu = 0 \text{ on } \partial \Omega - (A \cap B),
\]
\[
-\sigma \cdot \nu \leq a \text{ on } A, \quad \sigma \cdot \nu \geq b \text{ on } B.
\]

The Supply-Demand theorem assures that (SD) has a solution if and only if

\[(G) \]
\[
C(S) \geq \int_{B \cap \partial S} b ds - \int_{A \cap \partial S} a ds \quad \text{for each cut } S.
\]

This can be proved by the aid of a continuous version of max-flow min-cut theorem under some assumptions. However, we can not apply the same method to a variant of (SD), which is called a symmetric type by Ford and Fulkerson.

On the other hand, Neumann [5] and Oettli and Yamasaki [8] investigated a problem of feasibility of flows and proved similar results in their own network formulations. Their method is based on a generalized Hahn-Banach Theorem and is applicable even for a symmetric supply-demand problem. In the next section, we give a concrete formulation of our problem in a more general form than (SD), and give a corresponding condition which is equivalent with an existence of solutions for the problem under suitable assumptions. Finally in §3, we consider (SD) as a special case and examine the assumptions.

2. Problem setting and a main theorem

Let $\Omega$ be a bounded domain in $n$-dimensional Euclidean space $R^n$ with Lipschitz boundary $\partial \Omega$. One can consider $n - 1$-dimensional surface measure on $\partial \Omega$ which is equal to $n - 1$-dimensional Hausdorff measure $H_{n-1}$ on $\partial \Omega$. We note that the unit outer normal $\nu$ to $\Omega$ is defined and essentially bounded measurable on $\partial \Omega$ with respect to $H_{n-1}$. Let $\Gamma$ be a set-valued mapping from $\Omega$ to $R^n$ which satisfies the following two conditions:

\[(H1) \]
$\Gamma(x)$ is a compact convex set containing 0 for all $x \in \Omega$.

\[(H2) \]
Let $\epsilon > 0$ and $\Omega_0$ be a compact subset of $\Omega$.

Then there is $\delta > 0$ such that
\[
\Gamma(x) \subset \Gamma(y) + B(0, \epsilon) \text{ if } x, y \in \Omega_0 \text{ and } |x - y| < \delta.
\]
In what follows, we assume that each feasible flow is represented by an essentially bounded vector field $\sigma$ on $\Omega$ satisfying the following capacity constraints:

$$\sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega.$$ 

Furthermore if $\text{div} \sigma \in L^n(\Omega)$, then $\sigma \cdot \nu$ can be defined as a function in $L^\infty(\partial \Omega)$ in a weak sense by Kohn and Temam [2]. Let $F \in L^n(\Omega)$ and $\lambda, \mu \in L^\infty(\partial \Omega)$ with $\lambda \leq \mu$. Then for the quintuple $(\Omega, \Gamma, F, \mu, \lambda)$, our problem is stated as follows:

(P) \quad \text{Find } \sigma \in L^\infty(\Omega; \mathbb{R}^n) \text{ such that } \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega,
\text{div} \sigma = F \text{ a.e. on } \Omega \text{ and } \lambda \leq \sigma \cdot \nu \leq \mu \text{ H}_{n-1}\text{-a.e. on } \partial \Omega.

Problem (SD) considered in §1 can be written in this form with $F = 0$.

To specify the class of cuts, we consider the space $BV(\Omega)$ of functions of bounded variation on $\Omega$:

$$BV(\Omega) = \{ u \in L^1(\Omega) | \nabla u \text{ is a Radon measure of bounded variation on } \Omega \},$$

where $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ is understood in the sense of distribution. We denote the characteristic function of a subset $S$ of $\Omega$ by $\chi_S$ and set

$$Q = \{ S \subset \Omega | \chi_S \in BV(\Omega) \}.$$

Let $S \in Q$. Then the reduced boundary $\partial^* S$ of $S$ is the set of all $x \in \partial S$ where Federer’s normal $\nu = \nu(x)$ to $S$ exists. It is known that $\partial^* S$ is a measurable set with respect to both the measure of total variation of $|\nabla \chi_S|$ and $H_{n-1}$, $|\nabla \chi_S|(R^n - \partial^* S) = 0$ and $|\nabla \chi_S|(E) = H_{n-1}(E)$ for each $|\nabla \chi_S|$-measurable subset $E$ of $\partial^* S$. Furthermore let $\gamma u \in L^1(\partial \Omega)$ be the trace of $u \in BV(\Omega)$. Then [4; Theorem 6.6.2] implies that $\gamma \chi_S = \chi_{\partial^* S \cap \partial \Omega}$ $H_{n-1}$-a.e. on $\partial \Omega$. Accordingly, replacing $ds$ by $H_{n-1}$ and $\partial S$ by $\partial^* S$, we can define the cut capacity as follows:

$$C(S) = \int_{\Omega \cap \partial^* S} \beta(\nu^S(x), x) dH_{n-1},$$

where $\beta(\cdot, x)$ is the support functional of $\Gamma(x)$ as defined in §1. Let $\nabla u / |\nabla u|$ be the Radon-Nikodym derivative of $\nabla u$ with respect to $|\nabla u|$ and set

$$\psi(u) = \int_\Omega \beta(\nabla u / |\nabla u|, x) d|\nabla u|(x)$$
for \( u \in BV(\Omega) \). Then \( C(S) = \psi(\chi_S) \). Since \( \beta \) is continuous and nonnegative by (H1) and (H2), \( C(S) \) is finite. We set

\[
\lambda(S) = \int_{\partial\Omega \cap \partial^* S} \lambda dH_{n-1}, \quad \mu(S) = \int_{\partial\Omega \cap \partial^* S} \mu dH_{n-1}, \quad F(S) = \int_S F dx.
\]

for convenience sake, and consider the condition

\[
(C) \quad C(S) \geq \lambda(S) - F(S) \quad \text{and} \quad C(S) \geq -\mu(\Omega - S) + F(\Omega - S)
\]

hold for all \( S \in Q \).

Now we can state a continuous version of Gale's feasibility theorem.

**Theorem 2.1.** Assume that (H1) and (H2) hold. If (P) has a solution, then condition (C) holds. Conversely if \( \bigcup_{x \in \Omega} \Gamma(x) \) is bounded and condition (C) holds, then (P) has a solution.

To prove this theorem, we need some lemmas. First applying an isoperimetric inequality due to [4] we have

**Lemma 2.2.** There is \( \sigma_0 \in L^\infty(\Omega; R^n) \) such that \( \text{div} \sigma_0 = F \) a.e. on \( \Omega \).

**Proof:** First assume that \( \int_\Omega F dx = 0 \). We use a max-flow min-cut theorem of Strang's type (1983):

\[
\sup \{ t \geq 0 \mid \text{div} \sigma = -tF \text{ a.e. on } \Omega, \sigma \cdot \nu = 0 \text{ H}_{n-1}\text{-a.e. on } \partial \Omega \text{ for some } \sigma \in L^\infty(\Omega; R^n) \text{ with } ||\sigma||_\infty \leq 1 \}
\]

\[
= \inf \left\{ H_{n-1}(\Omega \cap \partial^* S)/\int_S F dx \mid \int_S F dx > 0, \ S \subset \Omega, \chi_S \in BV(\Omega) \right\}.
\]

(The proof is in [6].) To prove the existence of \( \sigma_0 \), it is sufficient to show that the supremum is positive. We can prove that the infimum is positive as follows. According to [4; p.303] there is a positive constant \( k \) such that \( \min(m_n(S), m_n(\Omega - S)) \leq kH_{n-1}(\Omega \cap \partial^* S)^{n/(n-1)} \), where \( m_n \) denotes the Lebesgue measure on \( R^n \). Since

\[
\int_S F dx \leq (\int_S 1 dx)^{(n-1)/n} \cdot (\int_S |F|^n dx)^{1/n} \leq ||F||_n(m_n(S))^{(n-1)/n}
\]

and

\[
\int_S F dx = \int_{\Omega - S} -F dx \leq (\int_{\Omega - S} 1 dx)^{(n-1)/n} \cdot (\int_{\Omega - S} |F|^n dx)^{1/n} \leq ||F||_n(m_n(\Omega - S))^{(n-1)/n},
\]
we can conclude that

$$\int_S F dx \leq k_1 H_{n-1}(\Omega \cap \partial^* S)$$

with $k_1 = \|F\|_n k^{(n-1)/n}$ for all $S \in Q$. It follows that the infimum is not less than $1/k_1$.

Finally in case of $\int_\Omega F dx \neq 0$, consider $\sigma_1$ such that $\text{div} \sigma_1$ equals constantly $\int_\Omega F dx$, $\sigma_2$ such that $\text{div} \sigma_2 = F - \int_\Omega F dx$ and set $\sigma_0 = \sigma_1 + \sigma_2$. Then $\text{div} \sigma_0 = F$. This completes the proof.

From now on we fix $\sigma_0$ in Lemma 2.2. For $\sigma \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\text{div} \sigma \in L^n(\Omega)$ and $u \in BV(\Omega)$, according to [2] we can define the distribution $(\sigma \nabla u)$ by

$$(\sigma \nabla u)(\varphi) = -\int_\Omega u \nabla \varphi \cdot \sigma dx - \int_\Omega u \varphi \text{div} \sigma dx$$

for $\varphi \in C_0^\infty(\Omega)$. Since $BV(\Omega) \subset L^{n/(n-1)}(\Omega)$, each integral in the definition is finite. Furthermore it is known that $(\sigma \nabla u)$ is regarded as a bounded measure and that

$$(\sigma \nabla u)(\Omega) + \int_\Omega u \text{div} \sigma dx = \int_{\partial \Omega} \gamma u \sigma \cdot \nu dH_{n-1}$$

holds. This is Green's formula due to Kohn and Temam [2; Proposition 1.1]. (See also [6; Theorem 2.3].) Using this formula, we can prove

**Lemma 2.3.** If (P) has a solution, then (C) holds.

**Proof:** Let $\sigma$ be a solution of (P). Then by Green's formula stated above,

$$C(S) \geq (\sigma \nabla \chi_S)(\Omega) = \int_{\partial \Omega \cap \partial^* S} \sigma \cdot \nu dH_{n-1} - \int_S \text{div} \sigma dx \geq \lambda(S) - F(S).$$

Another inequality in (C) can be similarly proved.

To prove the converse, we follow the idea in [5] and [8]. Let us consider the Sobolev space

$$W^{1,1}(\Omega) = \{ u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega; \mathbb{R}^n) \},$$

which is a linear subspace of $BV(\Omega)$. We set

$$U = L^1(\Omega; \mathbb{R}^n) \times L^1(\partial \Omega)$$

and

$$V = \{(\nabla u, \gamma u) \mid u \in W^{1,1}(\Omega) \}. $$
Since $\gamma u \in L^1(\partial\Omega)$ for $u \in W^{1,1}(\Omega)$, $V$ is a linear subspace of $U$. Let $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$. Note that $u^+, u^- \in W^{1,1}(\Omega)$. We define a functional $\Phi$ on $V$ by

$$\Phi(\nabla u, \gamma u) = \int_{\Omega} \sigma_0 \cdot \nabla u \, dx - \int_{\partial\Omega} \sigma_0 \cdot \nu \gamma u \, H_{n-1} + \int_{\partial\Omega} \lambda \gamma u^+ \, dH_{n-1} - \int_{\partial\Omega} \mu \gamma u^- \, dH_{n-1}$$

and set

$$K = \{ \sigma \in L^\infty(\Omega; R^n) | \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega \}.$$

For $v \in L^1(\Omega; R^n)$, we define a functional $\rho$ on $U$ by

$$\rho(v, \alpha) = \int_{\Omega} \beta(v(x), x) \, dx = \sup_{\phi \in K} \int_{\Omega} v \cdot \phi \, dx$$

for $(v, \alpha) \in U$. The last equality follows from a measurable selection theorem. (Cf. Castaing and Valadier (1977).) Since $\rho(v, \alpha)$ is independent of $\alpha$, it is sometimes denoted by $\rho(v)$. We note that $\psi(u) = \rho(\nabla u)$ for all $u \in W^{1,1}(\Omega)$. The inequality $\lambda \leq \mu$ implies the next lemma.

**Lemma 2.4.** $\Phi$ is superlinear on $V$, that is, concave and positively homogeneous, and $\rho$ is sublinear on $U$, that is, $-\rho$ is superlinear. Furthermore $\rho$ is continuous at the origin of $U$ if $\cup_{x \in \Omega} \Gamma(x)$ is bounded.

Condition (C) can be replaced by an inequality with $\Phi$ and $\rho$.

**Lemma 2.5.** If (C) holds, then $\Phi \leq \rho$ on $V$.

**Proof:** We use equalities of coarea formula type which are stated in [6]: Let $u \in W^{1,1}(\Omega)$. Set $N_t = \{ x \in \Omega \mid u(x) \geq t \}$ and $M_t = \Omega - N_t$ for any real number $t$. Then $N_t, M_t \in Q$ for a.e. $t$ and

$$\psi(u) = \int_{-\infty}^{\infty} \psi(\chi_{N_t}) \, dt.$$  

Furthermore by [6; Lemma 4.6]

$$\int_{\Omega} F \, u \, dx = \int_0^\infty \left( \int_{\Omega} F \chi_{N_t} \, dx - \int_{\Omega} F \chi_{M_t} \, dx \right) \, dt,$$

$$\int_{\partial\Omega} \lambda \gamma u^+ \, dH_{n-1} = \int_0^\infty \int_{\partial\Omega} \lambda \gamma \chi_{N_t} \, dH_{n-1} \, dt,$$

$$\int_{\partial\Omega} \mu \gamma u^- \, dH_{n-1} = \int_0^\infty \int_{\partial\Omega} \mu \gamma \chi_{M_t} \, dH_{n-1} \, dt.$$
It follows from these equalities and (C) that
\[
\rho(\nabla u) = \psi(u) = \int_{-\infty}^{\infty} \psi(\chi_{N_t})dt + \int_{0}^{\infty} \psi(\chi_{M_t})dt
\]
\[
= \int_{0}^{\infty} C(N_t)dt + \int_{0}^{\infty} C(\Omega - M_t)dt
\]
\[
= \int_{0}^{\infty} (\lambda(N_t) - F(N_t))dt + \int_{0}^{\infty} (-\mu(M_t) + F(M_t))dt
\]
\[
\geq \int_{0}^{\infty} \left( \int_{\partial\Omega} \lambda \gamma \chi_{N_t} dH_{n-1} - \int_{\Omega} F\chi_{N_t} dx \right) dt
\]
\[
+ \int_{0}^{\infty} \left( -\int_{\partial\Omega} \mu \gamma \chi_{M_t} dH_{n-1} + \int_{\Omega} F\chi_{M_t} dx \right) dt
\]
\[
= \int_{\partial\Omega} \lambda \gamma u^+ dH_{n-1} - \int_{\partial\Omega} \mu \gamma u^- dH_{n-1} - \int_{\Omega} u \text{div} \sigma_0 dx
\]
\[
= \int_{\partial\Omega} \lambda \gamma u^+ dH_{n-1} - \int_{\partial\Omega} \mu \gamma u^- dH_{n-1}
\]
\[
- \int_{\partial\Omega} \sigma_0 \cdot \nu \gamma u H_{n-1} + \int_{\Omega} \sigma_0 \cdot \nabla u dx
\]
\[
\geq \Phi(\nabla u, \gamma u).
\]

Here we have used Green's formula in the last equality. This completes the proof.

By Lemma 2.5 and a version of Hahn-Banach theorem ([3; Corollary 2.2 in p.114]), there is a linear functional $\xi$ on $U$ satisfying $\Phi \leq \xi$ on $V$ and $\xi \leq \rho$ on $U$. The next lemma is directly proved.

**Lemma 2.6.** If $\bigcup_{x \in \Omega} \Gamma(x)$ is bounded, then $\xi$ is continuous on $U$ with respect to the canonical norm topology.

By Lemma 2.6, there is $\sigma \in L^\infty(\Omega; R^n)$ and $\eta \in L^\infty(\partial\Omega)$ such that
\[
\xi(v, \alpha) = \int_{\Omega} \sigma \cdot v dx + \int_{\partial\Omega} \eta \alpha dH_{n-1}
\]
for all $(v, \alpha) \in U$. However, from the inequality $\xi(v, \alpha) \leq \rho(v)$ for all $\alpha \in L^\infty(\partial\Omega)$, $\eta$ must be 0.
Lemma 2.7. Assume that $\cup_{x \in \Omega} \Gamma(x)$ is bounded. Then the vector field $\sigma$ obtained above is a solution to $(P)$.

Proof: We set $\Omega_0 = \{x \in \Omega| 0 \notin \Gamma(x) - \sigma(x)\}$. Then $\Omega_0$ is a measurable set. Assume that the measure of $\Omega_0$ is positive. Since $\hat{K} = \{\phi \in L^\infty(\Omega; \mathbb{R}^n)| \phi(x) \in \Gamma(x) - \sigma(x)\}$ is a weakly* closed convex set and does not contain 0, there is $\varphi \in L^1(\Omega; \mathbb{R}^n)$ such that $\sup_{\phi \in \hat{K}} \int_\Omega \varphi \cdot \phi dx < 0$. Therefore

$$\rho(\varphi) = \sup_{\phi \in \hat{K}} \int_\Omega \varphi \cdot (\phi + \sigma) dx < \int_\Omega \varphi \cdot \sigma dx = \xi(\varphi, 0).$$

This is a contradiction since $\xi \leq \rho$ on $U$. Thus $\sigma(x) \in \Gamma(x)$ for almost all $x \in \Omega$.

Next we prove $\text{div} \sigma = F$. If $u \in C_0^\infty(\Omega)$, then $\gamma u = 0$ so that

$$\Phi(\nabla u, \gamma u) = \int_\Omega \sigma_0 \cdot \nabla u dx \leq \xi(\nabla u, 0) = \int_\Omega \sigma \cdot \nabla u dx.$$

It follows that

$$\int_\Omega \sigma_0 \cdot \nabla u dx = \int_\Omega \sigma \cdot \nabla u dx$$

for all $u \in C_0^\infty(\Omega)$. This implies that $\text{div} \sigma = \text{div} \sigma_0 = F$ in a distribution sense.

Finally we prove that $\lambda \leq \sigma \cdot \nu \leq \mu H_{n-1}$-a.e. on $\partial \Omega$. Since $\text{div} \sigma = F \in L^n(\Omega)$, $\sigma \cdot \nu$ is defined as a function in $L^\infty(\partial \Omega)$ and the inequality $\Phi(\nabla u, \gamma u) \leq \int_{\partial \Omega} \sigma \cdot \nabla u dx$ implies that

$$\int_{\partial \Omega} \lambda \gamma u^+ - \mu \gamma u^- dH_{n-1} \leq \int_{\partial \Omega} \gamma u \sigma \cdot \nu dH_{n-1}.$$

For any $\alpha \in L^1(\partial \Omega)$, there is $u \in W^{1,1}(\Omega)$ such that $\alpha = \gamma u$ by Gagliardo (1957). Thus for any nonnegative function $\alpha \in L^1(\partial \Omega)$, we have

$$\int_{\partial \Omega} \lambda \alpha dx \leq \int_{\partial \Omega} \sigma \cdot \nu dH_{n-1},$$

$$- \int_{\partial \Omega} \mu \alpha dx \leq - \int_{\partial \Omega} \sigma \cdot \nu dH_{n-1}.$$
3. Supply - Demand theorem

Let $A, B$ be disjoint Borel subsets of $\partial \Omega$ and $a, b$ be Borel measurable functions on $A, B$ respectively. Then (SD) in §1 should be written in the following concrete form:

\begin{align*}
\text{(SD)} \quad & \text{Find } \sigma \in L(\Omega; R^n) \\
& \text{such that } \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega, \\
& \text{div } \sigma = 0 \text{ a.e. on } \Omega, \\
& \sigma \cdot \nu = 0 \text{ } H_{n-1}\text{-a.e. on } \partial \Omega - (A \cap B), \\
& -\sigma \cdot \nu \leq a \text{ } H_{n-1}\text{-a.e. on } A, \\
& \sigma \cdot \nu \geq b \text{ } H_{n-1}\text{-a.e. on } B.
\end{align*}

By setting $\lambda = -a$ on $A$, $\lambda = b$ on $B$, $\lambda = 0$ elsewhere on $\partial \Omega$ and $\mu = \max(\lambda, 0)$, Theorem 2.1 implies

\text{THEOREM 3.1.} \quad \text{Assume that (H1), (H2) hold and that } \bigcup_{x \in \Omega} \Gamma(x) \text{ is bounded. Then (SD) has a solution if and only if }

\begin{equation}
\text{(G) } \quad C(S) \geq \int_{B \cap \partial^* S} bdH_{n-1} - \int_{A \cap \partial^* S} adH_{n-1} \text{ for all } S \in Q.
\end{equation}

Finally we refer to a relation between (SD) and a max-flow problem of Strang's type (MFS) which has been used in the proof of Lemma 2.2 with the boundary condition $\sigma \cdot \nu = 0$. Now let $f$ be an arbitrary function in $L^\infty(\partial \Omega)$ which satisfies the conservation law $\int_{\partial \Omega} f dH_{n-1} = 0$. Then for $(\Omega, \Gamma, f)$, (MFS) with $F = 0$ is stated as follows:

\begin{align*}
\text{(MFS)} \quad & \text{Maximize } \lambda \\
& \text{subject to } (\lambda, \sigma) \in R \times L^\infty(\Omega; R^n), \\
& \sigma(x) \in \Gamma(x) \text{ a.e. } x \in \Omega, \\
& \text{div } \sigma = 0 \text{ a.e. on } \Omega, \sigma \cdot \nu = \lambda f \text{ a.e. on } \partial \Omega,
\end{align*}

and the corresponding min-cut problem (MCS) is

\begin{align*}
\text{(MCS)} \quad & \text{Minimize } C(S)/L(S) \\
& \text{subject to } S \subset \Omega, \chi_S \in BV(\Omega), L(S) > 0,
\end{align*}

where $L(S) = \int_{\partial \Omega \cap \partial^* S} f dH_{n-1}$. Then we have
PROPOSITION 3.2. Assume that (H1) and (H2) hold.

(1) Assume that (G) implies the existence of solutions to (SD) for any disjoint Borel subsets $A, B$ of $\partial \Omega$ and $a \in L^\infty(A), b \in L^\infty(B)$. Then $MFS = MCS$ and (MFS) has an optimal solution for any $f \in L^\infty(\partial \Omega)$ satisfying the conservation law.

(2) Conversely if $MFS = MCS$ and (MFS) has an optimal solution for any $f \in L^\infty(\partial \Omega)$ satisfying the conservation law, then (G) implies the existence of solutions to (SD) for any disjoint Borel subsets $A, B$ of $\partial \Omega$ and $a \in L^\infty(A), b \in L^\infty(B)$ such that $\int_{A} adH_{n-1} = \int_{B} bdH_{n-1}$.

It is known that there is an example with $MFS < MCS$ if $\Gamma$ is unbounded. (See [7].) Thus Proposition 3.2 (1) shows that there is an example of (SD) such that $\bigcup_{x \in \Omega} \Gamma(x)$ is bounded, condition (G) is satisfied and (SD) has no solution.

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References