<table>
<thead>
<tr>
<th>Title</th>
<th>Real analytic wave interpolation function (Spaces of Analytic and Harmonic Functions and Operator Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Uchiyama, Mitsuru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1996, 946: 174-178</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60234">http://hdl.handle.net/2433/60234</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Real analytic wave interpolation function

福岡教育大学 内山 充 (Mitsuru Uchiyama)

Department of Mathematics, Fukuoka University of Education
Munakata, Fukuoka, 811-41 Japan, e-mail uchiyama@fukuoka-edu.ac.jp

The figure above is a model for the surface of the earth with a fault. $z = F'(t, x)$ has a fault along the line $x = 0$ so that $h_0(x) := F(-\frac{1}{2}\pi, x)$ is discontinuous and $h_1(x) := F'(\pi/2, x)$ is continuous. For each $x$, $F(t, x)$ is a smooth wave function for $t$. We may say that $F'(t, x)$ interpolates $h_0(x)$ and $h_1(x)$.

We show that for given finite functions $h_0(x), \ldots, h_n(x)$ defined on $\mathbb{R}$, there is a surface $z = F'(t, x)$ in $\mathbb{R}^3$ which is real analytic for $t$ and takes the curve $z = h_i(x)$ at an appropriate $t_i$ ($0 \leq i \leq n$); moreover if every $h_i(x)$ is real analytic then $F(t, x)$ is real analytic for $x$ too.
Precisely we show

THEOREM. For $n \geq 1$ there is a $(n + 2)$—real variable function $f_n(t, x_0, x_1, \ldots, x_n)$ which satisfies the following:

i. $f_n$ is real analytic for each variable.

ii. $\text{sign } \frac{\partial f_n}{\partial t} = \text{sign } \sin 2t$.

iii. $f_n(-\frac{\pi}{2}, x) = x_0, f_n((2^{i-1} - \frac{1}{2})\pi, x) = x_i$ ($1 \leq i \leq n$), where $x = (x_0, x_1, \ldots, x_n)$.

iv. $f_n$ is a periodic function for $t$ with period $2^n\pi$.

If we can construct this function $f_n$, then the surface $z = f_n(t, h_0(x), \ldots, h_n(x))$ in $\mathbb{R}^3$ takes $z = h_0(x)$ at $t = -\frac{\pi}{2}$, and $z = h_i(x)$ at $t = (2^{i-1} - \frac{1}{2})\pi$.

From now on we construct $f_n$ with three steps.

STEP 1. We consider the following non-linear differential equation

$$ t^2 f''(t) = f(t)^2, \quad f(-1) = a, \quad f(1) = b. \quad (1) $$

From this we can easily get

$$ f(t) = \frac{1}{c} t - \frac{1}{c^2} \log |1 + ct| + c_1, $$

$$ a - b = -\frac{2}{c} + \frac{1}{c^2} \log \frac{1+c}{1-c}, $$

$$ c_1 = a + \frac{1}{c} + \frac{1}{c^2} \log (1 - c). $$

Set

$$ \phi(\zeta) := \begin{cases} \frac{2}{\zeta} - \frac{1}{\zeta^2} \log \frac{1+\zeta}{1-\zeta}, & (-1 < \zeta < 1, \zeta \neq 0) \\ 0 & (\zeta = 0) \end{cases} $$

Since we can reform it as

$$ \phi(\zeta) = -2 \left( \frac{\zeta}{3} + \frac{\zeta^3}{5} + \frac{\zeta^5}{7} + \ldots \right), $$

$\phi(\zeta)$ is real analytic on $-1 < \zeta < 1$ with the range $\mathbb{R}$. Since $\phi'(\zeta) < 0$, $\phi^{-1}$ is a real analytic function defined on $\mathbb{R}$ too.
We determine two variable function \( h(t, s) \) by

\[
h(t, s) := \begin{cases} 
-\frac{1+t}{\phi^{-1}(s)} - \frac{1}{\phi^{-1}(s)^2} \log \frac{1-t\phi^{-1}(s)}{1+\phi^{-1}(s)} & (s \neq 0) \\
\frac{1}{2}t^2 - \frac{1}{2} & (s = 0)
\end{cases}
\]

\[
= \sum_{n=0}^{\infty} \frac{t^{n+2} + (-1)^{n+1}}{n+2} (\phi^{-1}(s))^n,
\]

whose domain is \( \{(t, s) : |t\phi^{-1}(s)| < 1\} \).

The domain is an open set and includes the closed set \([-1, 1] \times \mathbb{R}\); moreover for any number \( M > 0 \), there is \( r > 0 \) such that \([-1 - r, 1 + r] \times [-M, M]\) is included in the domain. We note that \( h(t, s) \) is real analytic for each variable. Setting \( f(t) = h(t, a-b) + a \), since \( h(-1, s) = 0 \) and \( h(1, s) = -\phi(\phi^{-1}(s)) = -s \), we have \( f(-1) = a, \ f(1) = b \).

Since

\[
\frac{df}{dt} = \frac{t}{1-t\phi^{-1}(a-b)}
\]

we get \( t^2f''(t) = (t')^2 \); therefore \( f(t) \) is a solution of (1).

The above \( f(t) \) depends on the initial values \( a \) and \( b \), so that we denote it by \( f(t, a, b) \), that is \( f(t, a, b) = h(t, a-b) + a \).

We remark that if we consider both of \( a \) and \( b \) as variables then \( f \) is real analytic for every variable.

**STEP 2.** We consider about the Fourier Series

\[
\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{2n+1}
\]

of \( x/|x| \) \((-\pi \leq x \leq \pi)\). The partial sum

\[
S_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin(2k+1)t}{2k+1}
\]

is a periodic function with period \( 2\pi \), and takes the maximum value \( M \) at \( t = \frac{1}{2} \pi \) and \( t = \frac{2n+1}{2(n+1)} \pi \) (Gibbs phenomenon).
Let us set $s_{2n+1}(t) = S_{2n+1}(\frac{1}{n+1}t)/M$. Then $s_{2n+1}(t)$ is an odd function, and a periodic function with period $2(n+1)\pi$. It is clear that

$$\text{sign} s_{2n+1}(t) = \text{sign} \sin \left( \frac{t}{n+1} \right).$$

Since

$$S'_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \cos(2k+1)t = \frac{2}{\pi} \frac{\sin(2n+2)t}{\sin t},$$

we obtain

$$\text{sign} s'_{2n+1}(t) = \text{sign} (\sin 2t/\sin \frac{t}{n+1}).$$

For $f$ gotten at the end of Step 1, since $-1 \leq s_{2n+1}(t) \leq 1$ for $-\infty < t < \infty$, $f(s_{2n+1}(t), a, b)$ is well-defined and periodic with period $2(n+1)\pi$; moreover it takes $a$ at $t = -\frac{\pi}{2}, -(n + \frac{1}{2})\pi$, and $b$ at $t = \frac{\pi}{2}, (n + \frac{1}{2})\pi$. From (2) and the above it follows that

$$\text{sign} \frac{\partial}{\partial t} f(s_{2n+1}(t), a, b) = \text{sign} \sin 2t$$

(3)

STEP 3. Now we construct $f_n$ in Theorem by the mathematical induction.

First, set $f_1(t, x_0, x_1) := f(\sin t, x_0, x_1)$. Since $s_1(t) = \sin t$, by (3) we get the condition ii. It is easy to check that $f_1$ satisfies the rest conditions.

Next, suppose that there is a $(n+1)$-variable function $f_{n-1}$ which satisfies the conditions of Theorem. We denote an arbitrary point in $\mathbb{R}^{n+2}$ by $(t, x_0, x_1, \ldots, x_n)$ and set $x = (x_0, x_1, \ldots, x_n)$.

For $1 \leq i \leq n-1$, we set

$$g_i(x) = x_i - f(s_{2^{i-1}}((2i-1-\frac{1}{2})\pi), x_0, x_n).$$

(4)

Then from the assumption for $f_{n-1}$, we have

$$f_{n-1}((2^{i-1}-\frac{1}{2})\pi, 0, g_1(x), \ldots, g_{n-1}(x)) = g_i(x) \quad 1 \leq i \leq n-1$$

$$f_{n-1}((2^{n-1}-\frac{1}{2})\pi, 0, g_1(x), \ldots, g_{n-1}(x)) = f_{n-1}(-\frac{1}{2}\pi, 0, g_1(x), \ldots, g_n(x)) = 0$$

Now we determine $f_n$ by

$$f_n(t, x) = f(s_{2^{n-1}}(t), x_0, x_n) + f_{n-1}(t, 0, g_1(x), \ldots, g_{n-1}(x)).$$

(5)

We have $f_n(-\frac{\pi}{2}, x) = x_0 + 0 = x_0$, and $f_n(((2^{n-1}-\frac{1}{2})\pi), x) = x_n + 0 = x_n$. Further by (4) we get $f_n((2^{i-1}-\frac{1}{2})\pi, x) = f(s_{2^{i-1}}((2^{i-1}-\frac{1}{2})\pi), x_0, x_n) + g_i(x) = x_i$ for $1 \leq i \leq n-1$. 


Thus we have shown the condition iii. Since the period of $f(s_{2n-1}(t), x_0, x_n)$ is $2^n\pi$ and that of $f_{n-1}$ is $2^{n-1}\pi$, the period of $f_n(t, x)$ is $2^n\pi$. Therefore we get the condition iv.

By (3) and the assumption for $f_{n-1}$, it is easy to show the condition ii. From (4) it follows that $g_i(x)$ is real analytic for each $x_i$; hence by (5) we get the condition i. Thus the proof is complete.

**Problem.** In the condition iii of Theorem can we interpolate $x_i$ at regular intervals?