<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>Real analytic wave interpolation function (Spaces of Analytic and Harmonic Functions and Operator Theory)</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Uchiyama, Mitsuru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1996, 946: 174-178</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60234">http://hdl.handle.net/2433/60234</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
</tbody>
</table>

Kyoto University
Real analytic wave interpolation function

Mitsuru Uchiyama

Department of Mathematics, Fukuoka University of Education
Munakata, Fukuoka, 811-41 Japan, e-mail uchiyama@fukuoka-edu.ac.jp

The figure above is a model for the surface of the earth with a fault. \( z = F(t, x) \) has a fault along the line \( x = 0 \) so that \( h_0(x) := F(-\frac{1}{2}\pi, x) \) is discontinuous and \( h_1(x) := F(\pi/2, x) \) is continuous. For each \( x \), \( F(t, x) \) is a smooth wave function for \( t \). We may say that \( F(t, x) \) interpolates \( h_0(x) \) and \( h_1(x) \).

We show that for given finite functions \( h_0(x), \ldots, h_n(x) \) defined on \( \mathbb{R} \), there is a surface \( z = F(t, x) \) in \( \mathbb{R}^3 \) which is real analytic for \( t \) and takes the curve \( z = h_i(x) \) at an appropriate \( t \) \((0 \leq i \leq n)\); moreover if every \( h_i(x) \) is real analytic then \( F(t, x) \) is real analytic for \( x \) too.
Precisely we show

**THEOREM.** For \( n \geq 1 \) there is a \((n+2)\)-real variable function \( f_n(t, x_0, x_1, \ldots, x_n) \) which satisfies the following:

i. \( f_n \) is real analytic for each variable.

ii. \( \text{sign } \frac{\partial f_n}{\partial t} = \text{sign } \sin 2t. \)

iii. \( f_n(-\frac{\pi}{2}, x) = x_0, f_n((2^{i-1} - \frac{1}{2})\pi, x) = x_i \) (\( 1 \leq i \leq n \)), where \( x = (x_0, x_1, \ldots, x_n) \).

iv. \( f_n \) is a periodic function for \( t \) with period \( 2^n \pi \).

If we can construct this function \( f_n \), then the surface \( z = f_n(t, h_0(x), \ldots, h_n(x)) \) in \( \mathbb{R}^3 \) takes \( z = h_0(x) \) at \( t = -\frac{\pi}{2} \), and \( z = h_i(x) \) at \( t = (2^{i-1} - \frac{1}{2})\pi \).

From now on we construct \( f_n \) with three steps.

**STEP 1.** We consider the following non-linear differential equation

\[
 t^2 f''(t) = f'(t)^2, \quad f(-1) = a, \quad f(1) = b. \tag{1}
\]

From this we can easily get

\[
 f(t) = \frac{1}{c} t - \frac{1}{c^2} \log |1 + ce| + c_1,
\]

\[
 a - b = -\frac{2}{c} + \frac{1}{c^2} \log \frac{1 + c}{1 - c},
\]

\[
 c_1 = a + \frac{1}{c} + \frac{1}{c^2} \log (1 - c).
\]

Set

\[
 \phi(\zeta) := \begin{cases} 
 \frac{2}{\zeta} - \frac{1}{\zeta^2} \log \frac{1 + \zeta}{1 - \zeta}, & (-1 < \zeta < 1, \zeta \neq 0) \\
 0 & (\zeta = 0)
\end{cases}
\]

Since we can reform it as

\[
 \phi(\zeta) = -2 \left( \frac{\zeta}{3} + \frac{\zeta^3}{5} + \frac{\zeta^5}{7} + \cdots \right),
\]

\( \phi(\zeta) \) is real analytic on \(-1 < \zeta < 1\) with the range \( \mathbb{R} \). Since \( \phi'(\zeta) < 0, \phi^{-1} \) is a real analytic function defined on \( \mathbb{R} \) too.
We determine two variable function $h(t, s)$ by

$$h(t, s) := \begin{cases} \\ \frac{-t}{\phi^{-1}(s)} - \frac{1}{\phi^{-1}(s)^2} \log \frac{1-t\phi^{-1}(s)}{1+\phi^{-1}(s)} & (s \neq 0) \\ \frac{1}{2}t^2 - \frac{1}{2} & (s = 0) \end{cases}$$

$$= \sum_{n=0}^\infty \frac{t^{n+2} + (-1)^{n+1}}{n+2} (\phi^{-1}(s))^n,$$

whose domain is $\{(t, s) : |t\phi^{-1}(s)| < 1\}$.

The domain is an open set and includes the closed set $[-1, 1] \times \mathbb{R}$; moreover for any number $M > 0$, there is $r > 0$ such that $[-1 - r, 1 + r] \times [-M, M]$ is included in the domain. We note that $h(t, s)$ is real analytic for each variable.

Setting $f(t) = h(t, a-b) + a$, since $h(-1, s) = 0$ and $h(1, s) = -\phi(\phi^{-1}(s)) = -s$, we have $f(-1) = a, \quad f(1) = b$.

Since $$\frac{df}{dt} = \frac{t}{1-t\phi^{-1}(a-b)},$$
we get $t^2 f''(t) = f'(t)^2$; therefore $f(t)$ is a solution of (1).

The above $f(t)$ depends on the initial values $a$ and $b$, so that we denote it by $f(t, a, b)$, that is $f(t, a, b) = h(t, a - b) + a$.

We remark that if we consider both of $a$ and $b$ as variables then $f$ is real analytic for every variable.

**STEP 2.** We consider about the Fourier Series

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{2n+1}$$

of $x/|x|$ $(-\pi \leq x \leq \pi)$. The partial sum

$$S_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin(2k+1)t}{2k+1}$$

is a periodic function with period $2\pi$, and takes the maximum value $M$ at $t = \frac{1}{2(n+1)}\pi$ and $t = \frac{2n+1}{2(n+1)}\pi$ (Gibbs phenomenon).
Let us set $s_{2n+1}(t) = S_{2n+1}(\frac{1}{n+1}t)/M$. Then $s_{2n+1}(t)$ is an odd function, and a periodic function with period $2(n+1)\pi$. It is clear that

$$\text{sign} s_{2n+1}(t) = \text{sign} \sin(\frac{t}{n+1}).$$

Since

$$S'_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \cos(2k+1)t = \frac{2}{\pi} \frac{\sin(2n+2)t}{\sin t},$$

we obtain

$$\text{sign} s'_{2n+1}(t) = \text{sign}(\sin 2t/\sin \frac{t}{n+1}).$$

For $f$ gotten at the end of Step 1, since $-1 \leq s_{2n+1}(t) \leq 1$ for $-\infty < t < \infty$, $f(s_{2n+1}(t), a, b)$ is well-defined and periodic with period $2(n+1)\pi$; moreover it takes $a$ at $t = -\frac{\pi}{2}, -(n + \frac{1}{2})\pi$, and $b$ at $t = \frac{\pi}{2}, (n + \frac{1}{2})\pi$. From (2) and the above it follows that

$$\text{sign} \frac{\partial}{\partial t} f(s_{2n+1}(t), a, b) = \text{sign} \sin 2t$$

(3)

STEP 3. Now we construct $f_n$ in Theorem by the mathematical induction.
First, set $f_1(t, x_0, x_1) := f(\sin t, x_0, x_1)$. Since $s_1(t) = \sin t$, by (3) we get the condition ii.

It is easy to check that $f_1$ satisfies the rest conditions.

Next, suppose that there is a $(n+1)$-variable function $f_{n-1}$ which satisfies the conditions of Theorem. We denote an arbitrary point in $\mathbb{R}^{n+2}$ by $(t, x_0, x_1, \ldots, x_n)$ and set $\mathbf{x} = (x_0, x_1, \ldots, x_n)$.

For $1 \leq i \leq n-1$, we set

$$g_i(\mathbf{x}) = x_i - f(s_{2^{i-1}}((2i-1-\frac{1}{2})\pi), x_0, x_n).$$

(4)

Then from the assumption for $f_{n-1}$, we have

$$f_{n-1}((2^{i-1} - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = g_i(\mathbf{x}) \quad 1 \leq i \leq n-1$$

$$f_{n-1}((2^{n-1} - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = f_{n-1}(-\frac{1}{2}\pi, 0, g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = 0$$

Now we determine $f_n$ by

$$f_n(t, \mathbf{x}) = f(s_{2^{n-1}}(t), x_0, x_n) + f_{n-1}(t, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})).$$

(5)

We have $f_n(-\frac{\pi}{2}, \mathbf{x}) = x_0 + 0 = x_0$, and $f_n(((2^{n-1} - \frac{1}{2})\pi), \mathbf{x}) = x_n + 0 = x_n$. Further by (4) we get $f_n((2^{i-1} - \frac{1}{2})\pi, \mathbf{x}) = f(s_{2^{n-1}}((2^{i-1} - \frac{1}{2})\pi), x_0, x_n) + g_i(\mathbf{x}) = x_i$ for $1 \leq i \leq n-1$. 


Thus we have shown the condition iii. Since the period of $f(s_{2n-1}(t), x_0, x_n)$ is $2^n \pi$ and that of $f_{n-1}$ is $2^{n-1} \pi$, the period of $f_n(t, x)$ is $2^n \pi$. Therefore we get the condition iv.

By (3) and the assumption for $f_{n-1}$, it is easy to show the condition ii. From (4) it follows that $g_i(x)$ is real analytic for each $x_i$; hence by (5) we get the condition i. Thus the proof is complete.

**Problem.** In the condition iii of Theorem can we interpolate $x_i$ at regular intervals?