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Real analytic wave interpolation function

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The figure above is a model for the surface of the earth with a fault. \( z = F(t, x) \) has a fault along the line \( x = 0 \) so that \( h_0(x) := F(-\frac{1}{2}\pi, x) \) is discontinuous and \( h_1(x) := F(\pi/2, x) \) is continuous. For each \( x \), \( F(t, x) \) is a smooth wave function for \( t \). We may say that \( F(t, x) \) interpolates \( h_0(x) \) and \( h_1(x) \).

We show that for given finite functions \( h_0(x), \ldots, h_n(x) \) defined on \( \mathbb{R} \), there is a surface \( z = F(t, x) \) in \( \mathbb{R}^3 \) which is real analytic for \( t \) and takes the curve \( z = h_i(x) \) at an appropriate \( t_i \) \( (0 \leq i \leq n) \); moreover if every \( h_i(x) \) is real analytic then \( F(t, x) \) is real analytic for \( x \) too.
Precisely we show

**THEOREM.** For $n \geq 1$ there is a $(n + 2)$-real variable function $f_n(t, x_0, x_1, \ldots, x_n)$ which satisfies the following:

i. $f_n$ is real analytic for each variable.

ii. $\text{sign } \frac{\partial f_n}{\partial t} = \text{sign } \sin 2t$.

iii. $f_n(-\frac{\pi}{2}, x) = x_0, f_n((2^{i-1} - \frac{1}{2})\pi, x) = x_i$ ($1 \leq i \leq n$), where $x = (x_0, x_1, \ldots, x_n)$.

iv. $f_n$ is a periodic function for $t$ with period $2^n\pi$.

If we can construct this function $f_n$, then the surface $z = f_n(t, h_0(x), \ldots, h_n(x))$ in $\mathbb{R}^3$ takes $z = h_0(x)$ at $t = -\frac{\pi}{2}$, and $z = h_i(x)$ at $t = (2^{i-1} - \frac{1}{2})\pi$.

From now on we construct $f_n$ with three steps.

**STEP 1.** We consider the following non-linear differential equation

$$i^2 f''(t) = f(t)^2, \quad f(-1) = a, \quad f(1) = b. \quad (1)$$

From this we can easily get

$$f(t) = \frac{1}{c}t - \frac{1}{c^2}\log|1+ct| + c_1,$$

$$a - b = -\frac{2}{c} + \frac{1}{c^2}\log\frac{1+c}{1-c},$$

$$c_1 = a + \frac{1}{c} + \frac{1}{c^2}\log(1-c).$$

Set

$$\phi(\zeta) := \begin{cases} \frac{2}{\zeta} - \frac{1}{\zeta^2}\log\frac{1+\zeta}{1-\zeta}, & (-1 < \zeta < 1, \zeta \neq 0) \\
0 & (\zeta = 0) \end{cases}$$

Since we can reform it as

$$\phi(\zeta) = -2\left(\frac{\zeta}{3} + \frac{\zeta^3}{5} + \frac{\zeta^5}{7} + \ldots\right),$$

$\phi(\zeta)$ is real analytic on $-1 < \zeta < 1$ with the range $\mathbb{R}$. Since $\phi'(\zeta) < 0, \phi^{-1}$ is a real analytic function defined on $\mathbb{R}$ too.
We determine two variable function $h(t, s)$ by

$$h(t, s) := \begin{cases} \frac{-1+t}{\phi^{-1}(s)} - \frac{1}{\phi^{-1}(s)^2} \log \frac{1-t\phi^{-1}(s)}{1+\phi^{-1}(s)} & (s \neq 0) \\ \frac{1}{2}t^2 - \frac{1}{2} & (s = 0) \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{t^{n+2}+(-1)^{n+1}}{n+2} (\phi^{-1}(s))^n,$$

whose domain is $\{(t, s) : |t\phi^{-1}(s)|<1\}$.

The domain is an open set and includes the closed set $[-1, 1] \times \mathbb{R}$; moreover for any number $M > 0$, there is $r > 0$ such that $[-1 - r, 1 + r] \times [-M, M]$ is included in the domain. We note that $h(t, s)$ is real analytic for each variable.

Setting $f(t) = h(t, a - b) + a$, since $h(-1, s) = 0$ and $h(1, S) = -\phi(\phi^{-1}(s)) = -s$, we have $f(-1) = a$, $f(1) = b$.

Since

$$\frac{df}{dt} = \frac{t}{1-t\phi^{-1}(a-b)} \text{,}$$

we get $t^2 f''(t) = f'(t)^2$; therefore $f(t)$ is a solution of (1).

The above $f(t)$ depends on the initial values $a$ and $b$, so that we denote it by $f(t, a, b)$, that is $f(t, a, b) = h(t, a - b) + a$.

We remark that if we consider both of $a$ and $b$ as variables then $f$ is real analytic for every variable.

**STEP 2.** We consider about the Fourier Series

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \sin(2n+1)t \frac{t}{2n+1}$$

of $x/|x|$ ($-\pi \leq x \leq \pi$). The partial sum

$$S_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \sin(2k+1)t \frac{t}{2k+1}$$

is a periodic function with period $2\pi$, and takes the maximum value $M$ at $t = \frac{1}{2(n+1)}\pi$ and $t = \frac{2n+1}{2(n+1)}\pi$ (Gibbs phenomenon).
Let us set \( s_{2n+1}(t) = S_{2n+1}(\frac{1}{n+1}t)/M \). Then \( s_{2n+1}(t) \) is an odd function, and a periodic function with period \( 2(n+1)\pi \). It is clear that

\[
sign s_{2n+1}(t) = sign \sin \left( \frac{t}{n+1} \right).
\]

Since

\[
S'_{2n+1}(t) = \frac{4}{\pi} \sum_{k=0}^{n} \cos(2k+1)t = \frac{2}{\pi} \frac{\sin(2n+2)t}{\sin t},
\]

we obtain

\[
sign s'_{2n+1}(t) = sign \left( \frac{\sin 2t}{\sin \frac{t}{n+1}} \right).
\]

For \( f \) gotten at the end of Step 1, since \(-1 \leq s_{2n+1}(t) \leq 1 \) for \(-\infty < t < \infty \), \( f(s_{2n+1}(t), a, b) \) is well-defined and periodic with period \( 2(n+1)\pi \); moreover it takes \( a \) at \( t = -\frac{\pi}{2} - \left( n + \frac{1}{2} \right)\pi \), and \( b \) at \( t = \frac{\pi}{2} + \left( n + \frac{1}{2} \right)\pi \). From (2) and the above it follows that

\[
sign \frac{\partial}{\partial t} f(s_{2n+1}(t), a, b) = sign \sin 2t \tag{3}
\]

STEP 3. Now we construct \( f_n \) in Theorem by the mathematical induction.

First, set \( f_1(t, x_0, x_1) := f(\sin t, x_0, x_1) \). Since \( s_1(t) = \sin t \), by (3) we get the condition ii.

It is easy to check that \( f_1 \) satisfies the rest conditions.

Next, suppose that there is a \((n+1)\)-variable function \( f_{n-1} \) which satisfies the conditions of Theorem. We denote an arbitrary point in \( \mathbb{R}^{n+2} \) by \((t, x_0, x_1, \ldots, x_n)\) and set \( \mathbf{x} = (x_0, x_1, \ldots, x_n) \).

For \( 1 \leq i \leq n-1 \), we set

\[
g_i(\mathbf{x}) = x_i - f(s_{2^{n-1}}((2i-1 - \frac{1}{2})\pi), x_0, x_n).
\]

Then from the assumption for \( f_{n-1} \), we have

\[
f_{n-1}((2i-1 - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = g_i(\mathbf{x}) \quad 1 \leq i \leq n-1
\]

\[
f_{n-1}((2^{n-1} - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = f_{n-1}(-\frac{1}{2}\pi, 0, g_1(\mathbf{x}), \ldots, g_{n}(\mathbf{x})) = 0
\]

Now we determine \( f_n \) by

\[
f_n(t, \mathbf{x}) = f(s_{2^{n-1}}(t), x_0, x_n) + f_{n-1}(t, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})). \tag{5}
\]

We have \( f_n(-\frac{\pi}{2}, \mathbf{x}) = x_0 + 0 = x_0 \), and \( f_n((2^{n-1} - \frac{1}{2})\pi, \mathbf{x}) = x_n + 0 = x_n \). Further by (4) we get \( f_n((2i-1 - \frac{1}{2})\pi, \mathbf{x}) = f(s_{2^{n-1}}((2i-1 - \frac{1}{2})\pi), x_0, x_n) + g_i(\mathbf{x}) = x_i \) for \( 1 \leq i \leq n-1 \).
Thus we have shown the condition iii. Since the period of \( f(s_{2n-1}(t), x_0, x_n) \) is \( 2^n \pi \) and that of \( f_{n-1} \) is \( 2^{n-1} \pi \), the period of \( f_n(t, x) \) is \( 2^n \pi \). Therefore we get the condition iv.

By (3) and the assumption for \( f_{n-1} \), it is easy to show the condition ii. From (4) it follows that \( g_i(x) \) is real analytic for each \( x_i \); hence by (5) we get the condition i. Thus the proof is complete.

**Problem.** In the condition iii of Theorem can we interpolate \( x_i \) at regular intervals?