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Real analytic wave interpolation function

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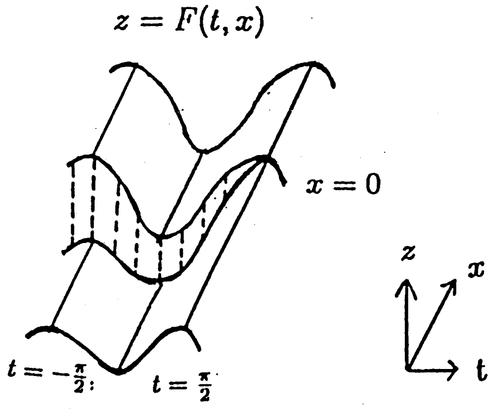

The figure above is a model for the surface of the earth with a fault. $z = F(t,x)$ has a fault along the line $x = 0$ so that $h_0(x) := F(-\frac{1}{2}\pi, x)$ is discontinuous and $h_1(x) := F(\pi/2, x)$ is continuous. For each x, $F(t,x)$ is a smooth wave function for t. We may say that $F(t,x)$ interpolates $h_0(x)$ and $h_1(x)$.

We show that for given finite functions $h_0(x), \ldots, h_n(x)$ defined on $\mathbf{R}$, there is a surface $z = F(t,x)$ in $\mathbf{R}^3$ which is real analytic for t and takes the curve $z = h_i(x)$ at an appropriate t_i $(0 \leq i \leq n)$; moreover if every $h_i(x)$ is real analytic then $F(t,x)$ is real analytic for x too.

Precisely we show

THEOREM. *For $n \geq 1$ there is a $(n+2)$−real variable function $f_n(t, x_0, x_1, \dots, x_n)$ which satisfies the following:*

i. f_n is real analytic for each variable.

ii. sign $\frac{\partial f_n}{\partial t} = sign \sin 2t$.

iii. $f_n(-\frac{\pi}{2}, \mathbf{x}) = x_0, f_n((2^{i-1} - \frac{1}{2})\pi, \mathbf{x}) = x_i \ (1 \leq i \leq n)$, *where* $\mathbf{x} = (x_0, x_1, \dots, x_n)$.

iv. f_n is a periodic function for t with period $2^n\pi$.

If we can construct this function f_n, then the surface $z = f_n(t, h_0(x), \dots, h_n(x))$ in $\mathbf{R}^3$ takes $z = h_0(x)$ at $t = -\frac{\pi}{2}$, and $z = h_i(x)$ at $t = (2^{i-1} - \frac{1}{2})\pi$.

From now on we construct f_n with three steps.

STEP 1. We consider the following non-linear differential equation

$$t^2 f''(t) = f'(t)^2, \quad f(-1) = a, \quad f(1) = b. \tag{1}$$

From this we can easily get

$$\begin{cases} f(t) = \frac{1}{c}t - \frac{1}{c^2}\log|1+ct| + c_1, \\ a - b = -\frac{2}{c} + \frac{1}{c^2}\log\frac{1+c}{1-c}, \\ c_1 = a + \frac{1}{c} + \frac{1}{c^2}\log(1-c). \end{cases}$$

Set

$$\phi(\zeta) := \begin{cases} \frac{2}{\zeta} - \frac{1}{\zeta^2}\log\frac{1+\zeta}{1-\zeta}, & (-1 < \zeta < 1, \zeta \neq 0) \\ 0 & (\zeta = 0) \end{cases}$$

Since we can reform it as

$$\phi(\zeta) = -2\left(\frac{\zeta}{3} + \frac{\zeta^3}{5} + \frac{\zeta^5}{7} + \dots\right),$$

$\phi(\zeta)$ is real analytic on $-1 < \zeta < 1$ with the range $\mathbf{R}$. Since $\phi'(\zeta) < 0, \phi^{-1}$ is a real analytic function defined on $\mathbf{R}$ too.

We determine two variable function $h(t,s)$ by

$$h(t,s) := \begin{cases} -\frac{1+t}{\phi^{-1}(s)} - \frac{1}{\phi^{-1}(s)^2}\log\frac{1-t\phi^{-1}(s)}{1+\phi^{-1}(s)} & (s \neq 0) \\ \frac{1}{2}t^2 - \frac{1}{2} & (s = 0) \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{t^{n+2} + (-1)^{n+1}}{n+2} (\phi^{-1}(s))^n,$$

whose domain is $\{(t,s) : |t\phi^{-1}(s)| < 1|\}$.

The domain is an open set and includes the closed set $[-1,1] \times \mathbf{R}$; moreover for any number $M > 0$, there is $r > 0$ such that $[-1-r, 1+r] \times [-M, M]$ is included in the domain. We note that $h(t,s)$ is real analytic for each variable.
Setting $f(t) = h(t, a-b) + a$, since $h(-1,s) = 0$ and $h(1,s) = -\phi(\phi^{-1}(s)) = -s$, we have $f(-1) = a, \quad f(1) = b$.
Since

$$\frac{df}{dt} = \frac{t}{1 - t\phi^{-1}(a-b)}, \tag{2}$$

we get $t^2 f''(t) = f'(t)^2$; therefore $f(t)$ is a solution of (1).

The above $f(t)$ depends on the initial values a and b, so that we denote it by $f(t,a,b)$, that is $f(t,a,b) = h(t, a-b) + a$.

We remark that if we consider both of a and b as variables then f is real analytic for every variable.

STEP 2. We consider about the Fourier Series

$$\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{2n+1}$$

of $x/|x| \qquad (-\pi \leq x \leq \pi)$. The partial sum

$$S_{2n+1}(t) = \frac{4}{\pi}\sum_{k=0}^{n} \frac{\sin(2k+1)t}{2k+1}$$

is a periodic function with period 2π, and takes the maximum value M at $t = \frac{1}{2(n+1)}\pi$ and $t = \frac{2n+1}{2(n+1)}\pi \qquad$ (Gibbs phenomenon).

Let us set $s_{2n+1}(t) = S_{2n+1}(\frac{1}{n+1}t)/M$. Then $s_{2n+1}(t)$ is an odd function, and a periodic function with period $2(n+1)\pi$. It is clear that

$$sign s_{2n+1}(t) = sign \sin(\frac{t}{n+1}).$$

Since

$$S'_{2n+1}(t) = \frac{4}{\pi}\sum_{k=0}^{n}\cos(2k+1)t = \frac{2}{\pi}\frac{\sin(2n+2)t}{\sin t},$$

we obtain

$$sign s'_{2n+1}(t) = sign(\sin 2t/\sin\frac{t}{n+1}).$$

For f gotten at the end of Step 1, since $-1 \leq s_{2n+}(t) \leq 1$ for $-\infty < t < \infty$, $f(s_{2n+1}(t), a, b)$ is well-defined and periodic with period $2(n+1)\pi$; moreover it takes a at $t = -\frac{\pi}{2}, -(n+\frac{1}{2})\pi$, and b at $t = \frac{\pi}{2}, (n+\frac{1}{2})\pi$. From (2) and the above it follows that

$$sign\frac{\partial}{\partial t}f(s_{2n+1}(t), a, b) = sign \sin 2t \tag{3}$$

STEP 3. Now we construt f_n in Theorem by the mathematical induction.
First, set $f_1(t, x_0, x_1) := f(\sin t, x_0, x_1)$. Since $s_1(t) = \sin t$, by (3) we get the condition ii. It is easy to check that f_1 satisfies the rest conditions.
Next, suppose that there is a $(n+1)-$variable function f_{n-1} which satisfies the conditions of Theorem. We denote an arbitrary point in $\mathbf{R}^{n+2}$ by $(t, x_0, x_1, \ldots, x_n)$ and set $\mathbf{x} = (x_0, x_1, \ldots, x_n)$.
For $1 \leq i \leq n-1$, we set

$$g_i(\mathbf{x}) = x_i - f(s_{2^n-1}((2^{i-1} - \frac{1}{2})\pi), x_0, x_n). \tag{4}$$

Then from the assumption for f_{n-1}, we have
$f_{n-1}((2^{i-1} - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = g_i(\mathbf{x}) \quad 1 \leq i \leq n-1$
$f_{n-1}((2^{n-1} - \frac{1}{2})\pi, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})) = f_{n-1}(-\frac{1}{2}\pi, 0, g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = 0$

Now we determine f_n by

$$f_n(t, \mathbf{x}) = f(s_{2^n-1}(t), x_0, x_n) + f_{n-1}(t, 0, g_1(\mathbf{x}), \ldots, g_{n-1}(\mathbf{x})). \tag{5}$$

We have $f_n(-\frac{\pi}{2}, \mathbf{x}) = x_0 + 0 = x_0$, and $f_n(((2^{n-1} - \frac{1}{2})\pi), \mathbf{x}) = x_n + 0 = x_n$. Further by (4) we get $f_n((2^{i-1} - \frac{1}{2})\pi, \mathbf{x}) = f(s_{2^n-1}((2^{i-1} - \frac{1}{2})\pi), x_0, x_n) + g_i(\mathbf{x}) = x_i$ for $1 \leq i \leq n-1$.

Thus we have shown the condition iii. Since the period of $f(s_{2^n-1}(t), x_0, x_n)$ is $2^n\pi$ and that of f_{n-1} is $2^{n-1}\pi$, the period of $f_n(t, \mathrm{x})$ is $2^n\pi$. Therefore we get the condition iv.

By (3) and the assumption for f_{n-1}, it is easy to show the condition ii. From (4) it follows that $g_i(\mathrm{x})$ is real analytic for each x_i; hence by (5) we get the condition i. Thus the proof is complete.

PROBLEM. In the condition iii of Theorem can we interpolate x_i at regular intervals?