HYPERBOLIC BESOV FUNCTIONS

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ABSTRACT. In this paper we study bounded holomorphic functions in the unit disk $D$ with given growth of hyperbolic derivative and application of these functions to composition operators.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in $\mathbb{C}$ with pseudohyperbolic metric

$$\rho(a, b) = \left| \frac{a - b}{1 - \overline{a}b} \right|$$

and with hyperbolic metric

$$\sigma(a, b) = \frac{1}{2} \log \frac{1 + \rho(a, b)}{1 - \rho(a, b)}.$$

Here $dA(z)$ is a normalized area measure on $D$ and $d\lambda(z)$ is hyperbolic area measure

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}.$$

Let $B$ be the family of bounded holomorphic functions $f(z), |f(z)| < 1,$ in $D$ and

$$f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$$

be the hyperbolic derivative of $f(z).$ Let $f_a(z) = f\left(\frac{z + a}{1 + \overline{a}z}\right), \ a \in D.$

**Definition.** For $1 \leq p \leq \infty$ hyperbolic analytic Besov class $B^h_p$ consists of functions $f(z) \in B$ which satisfy the condition

$$\|f\|_{B^h_p} = \left( \iint_D ((1 - |z|^2)f^*(z))^p \ d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

Classes $B^h_p, 1 \leq p \leq \infty,$ are Möbius invariant. By Schwarz-Pick lemma

$$(1 - |z|^2)f^*(z) \leq 1$$

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for any $f \in B$, i.e. $B^h = B$, but Möbius transforms of $D$ are not $p$-hyperbolic Besov functions for $1 \leq p < \infty$. It is easy to see that any function $f(z)$, $|f(z)| \leq k < 1$, belongs to all classes $B^h_p$, $1 \leq p \leq \infty$. Class $B^h_2$ is a class of hyperbolic Dirichlet functions.

Examples.

1. Let $S_{\alpha} = \{z = x + iy : |x|^\alpha + |y|^\alpha < 1\}$, $0 < \alpha \leq 1$, and $\varphi_{\alpha} : D \to S_{\alpha}$ then $\varphi_1(z) \notin B^h_2$ and $\varphi_1(z) \in B^h_p$ if $p > 2$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B^h_2$.

2. It is known [4] that for hyperbolic Lipschitz functions $\sigma\Lambda_\alpha$, $0 < \alpha \leq 1$, i.e. functions which satisfy the condition $\sup_{|u-v| \leq \tau} \sigma(f(u), f(v)) \leq K\tau^\alpha$, the necessary and sufficient condition that $f(z) \in \sigma\Lambda_\alpha$ is $(1 - |z|^2)^{f^*(z)} = O((1 - |z|^2)^\alpha)$ as $|z| \to 1$.

Thus we can see that $\sigma\Lambda_\frac{1}{p} \subset B^h_p$.

In chapter 2 we establish Lipschitz type properties of $p$-hyperbolic Besov functions and prove that $p$-hyperbolic Besov functions don’t have angular derivatives for finite values $p$. In chapter 3 we show that composition of Bloch functions and $p$-hyperbolic Besov functions are $p$-analytic Besov functions.

2. Main properties

Classes $B^h_p$ satisfy nesting property $B^h_p \subset B^h_q$ for $p < q$. It follows from the Schwarz-Pick lemma and inequality

$$\int_D \int (1 - |z|^2)^{q-2} (f^*(z))^q dA(z)$$

$$= \int_D \int ((1 - |z|^2)^{q-2}(1 - |z|^2)^{p-2} (f^*(z))^p dA(z)$$

$$\leq \int_D \int (1 - |z|^2)^{q-2} (f^*(z))^p dA(z).$$

**Theorem 1.** If bounded function $f(z)$ satisfies the condition

$$\int_D \int \frac{\rho(f(z), f(w))^p}{|1-z\overline{w}|^4} dA(z) dA(w) < \infty$$

then $f(z) \in B^h_p$, $1 \leq p < \infty$.

**Proof.**

Function $g(z) = \frac{f(z) - f(0)}{1 - f(z)\overline{f(0)}}$ is holomorphic in $D$. Using an expansion of $g(z)$ on Taylor series we can obtain

$$f^*(0) = \left| \int_D \overline{\tilde{z}} g(z) dA(z) \right|.$$
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Let \( g = f \circ \varphi_a(z) \), where \( \varphi_a(z) = \frac{z+a}{1+\overline{a}z} \), \( a \in \mathcal{D} \). Then

\[
((1 - |a|^2)f^*(a))^p \leq \int_{\mathcal{D}} \left( \rho(f \circ \varphi_a(z), f(a)) \right)^p dA(z)
\]

and thus

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} ((1 - |a|^2)f^*(a))^p \, d\lambda(a) \\
\leq \int_{\mathcal{D}} d\lambda(a) \int_{\mathcal{D}} \left( \rho(f_a(z), f(a)) \right)^p dA(z) \\
= \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{\rho(f(z), f(w))^p}{|1 - z\overline{w}|^4} dA(z) \, dA(w) < \infty.
\]

**Theorem 2.** If \( f(z) \in B_p^h, 1 < p < \infty \), then

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{\sigma(f(z), f(w))^p}{|1 - z\overline{w}|^4} dA(z) \, dA(w) < \infty.
\]

**Proof.**

At first we estimate hyperbolic distance between \( f(z) \) and \( f(0) \).

\[
\sigma(f(z), f(0)) = |\int_0^1 f^*(tz)z \, dt|
\]

\[
= \left| \int_0^1 \frac{f^*(tz)}{1 - t^2|z|^2} (1 - t^2|z|^2)z \, dt \right| \\
\leq \left( \int_0^1 \frac{|z|^q}{(1 - t^2|z|^2)^{\frac{1}{2}} + 1} \right)^\frac{1}{q} \left( \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{(1 - t^2|z|^2)^{\frac{1}{2}}} \, dt \right)^\frac{1}{p}
\]

\[
= \left( \frac{2|z|^{q-1}}{q - 1} \left( \frac{1}{(1 - |z|)^{\frac{q-1}{2}}} - 1 \right) \right)^\frac{1}{q} \left( \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{(1 - t^2|z|^2)^{\frac{1}{2}}} \, dt \right)^\frac{1}{p}
\]

\[
\leq C \left( \frac{|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - t^2|z|^2}} \, dt \right)^\frac{1}{p}.
\]

Thus, there exists such \( C \) that

\[
\sigma(f(z), f(0))^p \leq \frac{C|z|}{\sqrt{1 - |z|}} \int_0^1 \frac{((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - t^2|z|^2}} \, dt, \quad p > 1.
\]
Then
\[\int_D \int \sigma(f(z), f(0))^p \, dA(z) \leq \]
\[\leq C \int_D dA(z) \int_0^1 \frac{|z|((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - |z|^2} \sqrt{1 - t^2|z|^2}} \, dt \]
\[= C \int_0^1 \int_{tD} \frac{|z|((1 - t^2|z|^2)f^*(tz))^p}{\sqrt{1 - |z|^2}} \frac{dA(z)}{t^3} = \text{(by Fubini's theorem)} \]
\[= C \int_D \int \frac{|z|((1 - t^2|z|^2)f^*(z))^p}{\sqrt{1 - |z|^2}} dA(z) \int_0^1 \frac{dt}{t^2 \sqrt{t^2 - |z|^2}} \]
\[\leq 2C \int_D \int ((1 - |z|^2)f^*(z))^p \, dA(z).\]

We obtain that for \(1 < p < \infty\) and for some \(C_1\)
\[\int_D \int \sigma(f(z), f(0))^p \, dA(z) \leq C_1 \int_D \int ((1 - |z|^2)f^*(z))^p \, dA(z).\]

A change of variables and some properties of Bergman kernel (see e.g. [5]) give
\[\int_D \int \int \int \frac{\sigma(f(z), f(w))^p}{|1 - z\bar{w}|^4} \, dA(z) \, dA(w) \]
\[= \int_D d\lambda(z) \int_D \sigma^p(f \circ \varphi_z(w), f(z)) \, dA(w) \]
\[\leq C \int_D d\lambda(z) \int_D ((1 - |w|^2)f \circ \varphi_z^*(w))^p \, dA(w) \]
\[= \int_D d\lambda(z) \int_D (1 - |w|^2)^p f^*(w)^p |k_z(w)|^2 \, dA(w) \]
\[= \int_D (1 - |w|^2)^p f^*(w)^p \, dA(w) \int_D \frac{dA(z)}{|1 - z\bar{w}|^4} \]
\[= \int_D (1 - |w|^2)^p f^*(w)^p d\lambda(w).\]
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**Theorem 3.** For any $1 < p \leq \infty$ there exists a constant $C_p$ such that for any function $f(z) \in B^h_p$ and $z, w \in D$, $\sigma(f(z), f(w)) \leq 1$,

$$\sigma(f(z), f(w)) \leq C_p \|f\|_{B^h_p} \sigma(z, w)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

**Proof.**

Since $\|\cdot\|_{B^h_p}$, $\sigma$, $\rho$ are Möbius invariant it suffices to prove that there is a constant $C_p$ such that

$$\rho(f(z), f(0)) \leq C_p \|f\|_{B^h_p} \sigma(z, 0)^{\frac{1}{q}}$$

for all $f \in B^h_p$ and all $z \in D$.

Let $\varphi_f(z) = \frac{f(z) - f(0)}{1 - f(0)f(z)}$. Since $\varphi_f(z)$ is holomorphic function in $D$ then for any $f \in B$ and $z \in D$ following Zhu [5]

$$\varphi_f(z) = \iint_D \frac{1 - |w|^2}{\overline{w}(1 - z\overline{w})^2} \varphi_f'(w) dA(w).$$

Thus there exists a constant $C_1$ such that (see [2], [3])

$$\rho(f(z), f(0)) = |\varphi_f(z)| \leq \iint \frac{1 - |w|^2}{|w||1 - z\overline{w}|^2} |\varphi_f'(w)| dA(w)$$

$$\leq C_1 \iint_D \frac{1 - |w|^2}{|1 - z\overline{w}|^2} f^*(w)(1 - |\varphi_f(w)|^2) dA(w)$$

$$\leq C_1 \iint (1 - |w|^2) f^*(w) \left(\frac{1 - |w|^2}{|1 - z\overline{w}|^2}\right)^2 d\lambda(w).$$

By Hölder inequality and 1.4.10 of [3] we get

$$\rho(f(z), f(0)) \leq C_1 \|f\|_{B^h_p} \left(\iint_D \frac{(1 - |w|^2)^{2q-2}}{|1 - z\overline{w}|^{2q}} dA(w)\right)^{\frac{1}{q}} \leq C_2 \|f\|_{B^h_p} \sigma(z, 0)^{\frac{1}{q}}.$$

When points $a$ and $b$ lie closely one to another the behaviours of hyperbolic distance $\sigma(a, b)$ and pseudohyperbolic distance $\rho(a, b)$ are the same. Thus for sufficiently small $\tau$ there is a constant $C$ such that for any $z, w \in D$, $\sigma(z, w) \leq \tau$,

$$\sigma(f(z), f(w)) \leq C \|f\|_{B^h_p} \sigma(z, w)^{\frac{1}{q}}.$$

Let $\varphi : D \rightarrow D$ analytic and $\Gamma = \{|z| = 1\}$. 
**Definition.** [1] Bounded function $\varphi(z)$ has a finite angular derivative at $\zeta$ on unit circle $\Gamma$ if there is $\eta$ on $\Gamma$ so that $\frac{\varphi(z)-\eta}{z-\zeta}$ has a finite nontangential limit as $z \to \zeta$.

By the Julia-Caratheodory Theorem [1] function $\varphi(z)$ has a finite angular derivative at $\zeta$ if and only if

$$\liminf_{z \to \zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty.$$ 

Let $\zeta$ be a point on the unit circle $\Gamma$, $0 < \delta < 1$, $0 < \epsilon < \frac{\pi}{2}$ and $\Delta_\zeta(\delta, \epsilon) = \{z : |z - \zeta| < \delta, |\text{arg}(\zeta) - \text{arg}(z)| < \epsilon\}$.

**Theorem 4.** Functions of $B_p^h$, $1 < p < \infty$, don't have angular derivatives.

**Proof.**

Suppose that $\varphi(z) \in B_p^h$, $1 < p < \infty$, and $\varphi(z)$ has an angular derivative at a point $\zeta \in \Gamma$. Then by the Julia-Caratheodory Theorem $\frac{1-|\varphi(z)|}{1-|z|} \sim |\varphi'(\zeta)|$ for sufficiently small $\delta$ and $z \in \Delta_\zeta(\delta, \epsilon)$ and moreover, $|\varphi'(z)| \approx |\varphi'(\zeta)|$ for $z \in \Delta_\zeta(\delta, \epsilon)$. Thus

$$\int_{D} \int_{\Delta_\zeta(\delta, \epsilon)} ((1-|z|^2)\varphi'(z))^p d\lambda(z) \geq \int_{\Delta_\zeta(\delta, \epsilon)} (1-|z|^2)\varphi'(z)^p d\lambda(z) \approx \int_{\Delta_\zeta(\delta, \epsilon)} d\lambda(z) = \infty.$$

### 3. Composition operators

Let $B$ be Bloch space of holomorphic functions in $D$. By the definition (e.g. [5]) $f(z) \in B$ if

$$\lim_{|z| \to 1} (1-|z|^2)|f'(z)| < \infty.$$ 

Holomorphic in $D$ function $f(z)$ belongs to the analytic Besov space $B_p$, $1 < p < \infty$, if

$$\|f\|_{B_p} = \left( \int_{D} (1-|z|^2)^p |f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}} < \infty$$

and $B_\infty = B$.

Let $X$ be a normed subspace of $B$ and $\varphi(z)$ be a holomorphic self map of the unit disk $D$. We say that composition operator $C_\varphi : B \to X$ is compact if

$$\lim_{r \to 1^-} \|f \circ \varphi - f \circ (\chi_r \varphi)\|_X = 0$$

where $\chi_r(z)$ is a characteristic function of the disk $D_r = rD$

$$\chi_r(z) = \begin{cases} 1, & |z| \leq r \\ 0, & |z| > r. \end{cases}$$
**Theorem 5.** For every $\varphi(z) \in B^h_p$, $1 < p < \infty$, and any $f \in B^h$ composition $f \circ \varphi \in B^p_p$. Composition operator $C_\varphi$ is a compact.

**Proof.**
Let $f \in B$ and $\sup_{z \in D}(1 - |z|^2)|f'(z)| = M_f$. Let $\varphi(z) \in B^h_p$ and $\|\varphi\|_{B^h_p} = M_\varphi$. If $g(z) = f \circ \varphi(z)$ then

$$\|g(z)\|_{B^h_p} = \int_D \int (1 - |z|^2)^p |g'(z)|^p d\lambda(z) =$$

$$= \int_D \int (1 - |z|^2)^p |f'(\varphi(z))|^p |\varphi'(z)|^p d\lambda(z) =$$

$$= \int_D \int (1 - |z|^2)^p (\varphi^*(z))^p (1 - |\varphi(z)|^2)^p |f'(\varphi(z))|^p d\lambda(z) \leq M_f^p \cdot M_\varphi^p < \infty.$$

Now we prove compactness of operator $C_\varphi$.

$$\|f(\varphi(z)) - f(\chi_r \cdot \varphi(z))\|_{B^h_p}^p =$$

$$= \int_{D \setminus D_r} \int (1 - |z|^2)^p |(f \circ \varphi(z))'|^p d\lambda(z) \leq$$

$$\leq M_f^p \int_{D \setminus D_r} \int (1 - |z|^2)^p (\varphi^*(z))^p d\lambda(z).$$

The last integral tends to zero as $r \to 1$ because $\varphi(z) \in B^h_p$.

**Theorem 6.** If for every Bloch function $f(z)$ the composition $f \circ \varphi$ is a $p$-analytic Besov function, $1 < p < \infty$, then $\varphi(z) \in B^h_p$.

**Proof.**
Ramey and Ullrich [2] constructed such Bloch functions $f$ and $g$ that

$$|f'(z)| + |g'(z)| \geq \frac{1}{1 - |z|^2}.$$

Then for every $p > 1$

$$|f'(z)|^p + |g'(z)|^p \geq \frac{K_p}{(1 - |z|^2)^p}.$$

and thus

$$K_p \|\varphi\|_{B^h_p}^p \leq \|f \circ \varphi\|_{B^p_p}^p + \|g \circ \varphi\|_{B^p_p}^p < \infty.$$
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REFERENCES


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