<table>
<thead>
<tr>
<th>Title</th>
<th>HYPERBOLIC BESOV FUNCTIONS (Spaces of Analytic and Harmonic Functions and Operator Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MAKHMUTOV, SHAMIL</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 946: 133-140</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60239">http://hdl.handle.net/2433/60239</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
HYPERBOLIC BESOV FUNCTIONS

SHAMIL MAKMUTOV

Department of Mathematics, Hokkaido University

ABSTRACT. In this paper we study bounded holomorphic functions in the unit disk $D$ with given growth of hyperbolic derivative and application of these functions to composition operators.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in $\mathbb{C}$ with pseudohyperbolic metric

$$\rho(a, b) = \left| \frac{a - b}{1 - \overline{a}b} \right|$$

and with hyperbolic metric

$$\sigma(a, b) = \frac{1}{2} \log \frac{1 + \rho(a, b)}{1 - \rho(a, b)}.$$

Here $dA(z)$ is a normalized area measure on $D$ and $d\lambda(z)$ is hyperbolic area measure

$$d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}.$$

Let $B$ be the family of bounded holomorphic functions $f(z)$, $|f(z)| < 1$, in $D$ and $f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$ be the hyperbolic derivative of $f(z)$. Let $f_a(z) = f\left(\frac{z + a}{1 + \overline{a}z}\right)$, $a \in D$.

**Definition.** For $1 \leq p \leq \infty$ hyperbolic analytic Besov class $B^h_p$ consists of functions $f(z) \in B$ which satisfy the condition

$$\|f\|_{B^h_p} = \left( \int_D \int ((1 - |z|^2)f^*(z))^p \, d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

Classes $B^h_p$, $1 \leq p \leq \infty$, are Möbius invariant. By Schwarz-Pick lemma

$$(1 - |z|^2)f^*(z) \leq 1$$
for any \( f \in B \), i.e. \( B_\infty = B \), but Möbius transforms of \( D \) are not \( p \)-hyperbolic Besov functions for \( 1 \leq p < \infty \). It is easy to see that any function \( f(z), |f(z)| \leq k < 1 \), belongs to all classes \( B_p^h, 1 \leq p \leq \infty \). Class \( B_2^h \) is a class of hyperbolic Dirichlet functions.

Examples.

1. Let \( S_\alpha = \{ z = x + iy : |x|^\alpha + |y|^\alpha < 1 \}, 0 < \alpha \leq 1 \), and \( \varphi_\alpha : D \rightarrow S_\alpha \) then \( \varphi_\alpha(z) \notin B_p^h \) and \( \varphi_\alpha(z) \in B_p^h \) if \( p > 2 \). If \( \alpha < 1 \) then \( \varphi_\alpha(z) \in B_2^h \).

2. It is known [4] that for hyperbolic Lipschitz functions \( \sigma \Lambda_\alpha, 0 < \alpha \leq 1 \), i.e. functions which satisfy the condition \( \sup_{|u-v| \leq \tau} \sigma(f(u), f(v)) \leq Kr^\alpha \), the necessary and sufficient condition that \( f(z) \in \sigma \Lambda_\alpha \) is \( (1 - |z|^2)^{f^*(z)} = O((1 - |z|^2)^\alpha) \) as \( |z| \rightarrow 1 \).

Thus we can see that \( \sigma \Lambda_{\frac{1}{p}} \subset B_p^h \).

In chapter 2 we establish Lipschitz type properties of \( p \)-hyperbolic Besov functions and prove that \( p \)-hyperbolic Besov functions don’t have angular derivatives for finite values \( p \). In chapter 3 we show that composition of Bloch functions and \( p \)-hyperbolic Besov functions are \( p \)-analytic Besov functions.

2. Main properties

Classes \( B_p^h \) satisfy nesting property \( B_p^h \subset B_q^h \) for \( p < q \). It follows from the Schwarz-Pick lemma and inequality

\[
\int_D \int_D (1 - |z|^2)^{q-2}(f^*(z))^q \, dA(z)
\]

\[
= \int_D \int_D ((1 - |z|^2)f^*(z))^{q-2}(1 - |z|^2)^{p-2}(f^*(z))^p \, dA(z)
\]

\[
\leq \int_D \int_D (1 - |z|^2)^{p-2}(f^*(z))^p \, dA(z).
\]

**Theorem 1.** If bounded function \( f(z) \) satisfies the condition

\[
\int_D \int_D \int_D \int_D \frac{\rho(f(z), f(w))^p}{|1-z\overline{w}|^4} \, dA(z) \, dA(w) < \infty
\]

then \( f(z) \in B_p^h, 1 \leq p < \infty \).

**Proof.**

Function \( g(z) = \frac{f(z) - f(0)}{1 - f(z)f(0)} \) is holomorphic in \( D \). Using an expansion of \( g(z) \) on Taylor series we can obtain

\[
f^*(0) = \left| \int_D \bar{z}g(z) \, dA(z) \right|.
\]
HYPERBOLIC BESOV FUNCTIONS

Let \( g = f \circ \varphi_a(z) \), where \( \varphi_a(z) = \frac{z+a}{1+\overline{a}z} \), \( a \in D \). Then

\[
((1-|a|^2)f^*(a))^p \leq \iint_D (\rho(f \circ \varphi_a(z), f(a)))^p dA(z)
\]

and thus

\[
\iint_D ((1-|a|^2)f^*(a))^p d\lambda(a)
\]

\[
\leq \iint d\lambda(a) \iint (\rho(f_a(z), f(a)))^p dA(z)
\]

\[= \iint \iint_D \rho(f(z), f(w))^p \frac{dA(z) dA(w)}{|1-z\overline{w}|^4} \]

\[
< \infty.
\]

Theorem 2. If \( f(z) \in B_p^h, 1 < p < \infty \), then

\[
\iint \iint_D \frac{\sigma(f(z), f(w))^p}{|1-z\overline{w}|^4} dA(z) dA(w) < \infty.
\]

Proof.

At first we estimate hyperbolic distance between \( f(z) \) and \( f(0) \).

\[
\sigma(f(z), f(0)) = \int_0^1 f^*(tz)z dt
\]

\[
= \left| \int_0^1 \frac{f^*(tz)}{1-t^2|z|^2} (1-t^2|z|^2)z dt \right| \leq \left| \int_0^1 \frac{f^*(tz)}{1-t^2|z|^2} dt \right|
\]

\[
\leq \left( \int_0^1 \frac{|z|^q}{(1-t^2|z|^2)^{\frac{1+\frac{q}{2}}{2}}} dt \right)^{\frac{1}{q}} \left( \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{(1-t^2|z|^2)^{\frac{1}{2}}} dt \right)^{\frac{1}{p}}
\]

\[
= \left( \frac{|z|^{q-1}}{q-1} \left( \frac{1}{(1-|z|)^{\frac{q-1}{2}}} - 1 \right) \right)^{\frac{1}{q}} \left( \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{(1-t^2|z|^2)^{\frac{1}{2}}} dt \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \frac{|z|}{\sqrt{1-|z|}} \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-t^2|z|^2}} dt \right)^{\frac{1}{p}}.
\]

Thus, there exists such \( C \) that

\[
\sigma(f(z), f(0))^p \leq \frac{C|z|}{\sqrt{1-|z|}} \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-t^2|z|^2}} dt, \quad p > 1.
\]
Then
\[\int\int_D \sigma(f(z), f(0))^p \, dA(z) \leq \]
\[\leq C \int\int_D dA(z) \int_0^1 \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2} \sqrt{1-t^2|z|^2}} \, dt\]
\[= C \int_0^1 \int_D \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2}} \cdot \frac{dA(z)}{t^3} \frac{dt}{\sqrt{1-t^2|z|^2}} = \text{(by Fubini's theorem)}\]
\[= C \int_D \int |z|^2((1-t^2|z|^2)f^*(z))^p \cdot \frac{dA(z)}{t^3} \frac{dt}{\sqrt{1-t^2}-|z|^2}\]
\[\leq 2C \int_D ((1-|z|^2)^p f^*(z))^p \, dA(z).\]

We obtain that for $1 < p < \infty$ and for some $C_1$
\[\int\int_D \sigma(f(z), f(0))^p \, dA(z) \leq C_1 \int\int_D ((1-|z|^2)^p f^*(z))^p \, dA(z).\]

A change of variables and some properties of Bergman kernel (see e.g. [5]) give
\[\int\int_D \int\int_D \frac{\sigma(f(z), f(w))^p}{|1-z\overline{w}|^4} \, dA(z) \, dA(w)\]
\[= \int\int_D d\lambda(z) \int\int_D \sigma^p(f \circ \varphi_z(w), f(z))dA(w)\]
\[\leq C \int\int_D d\lambda(z) \int\int_D ((1-|w|^2)f \circ \varphi^*_z(w))^p dA(w)\]
\[= \int\int_D d\lambda(z) \int\int_D (1-|w|^2)^p f^*(w)^p |k_z(w)|^2 \, dA(w)\]
\[= \int\int_D (1-|w|^2)^p f^*(w)^p dA(w) \int\int_D \frac{dA(z)}{|1-z\overline{w}|^4}\]
\[= \int\int_D (1-|w|^2)^p f^*(w)^p d\lambda(w).\]
Theorem 3. For any $1 < p \leq \infty$ there exists a constant $C_p$ such that for any function $f(z) \in B^h_p$ and $z, w \in D$, $\sigma(f(z), f(w)) \leq 1$,

$$\sigma(f(z), f(w)) \leq C_p \|f\|_{B^h_p} \sigma(z, w)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$ 

Proof.

Since $\|\cdot\|_{B^h_p}$, $\sigma$, $\rho$ are Möbius invariant it suffices to prove that there is a constant $C_p$ such that

$$\rho(f(z), f(0)) \leq C_p \|f\|_{B^h_p} \sigma(z, 0)^{\frac{1}{q}}$$

for all $f \in B^h_p$ and all $z \in D$.

Let $\varphi_f(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$. Since $\varphi_f(z)$ is holomorphic function in $D$ then for any $f \in B$ and $z \in D$ following Zhu [5]

$$\varphi_f(z) = \iint_D \frac{1 - |w|^2}{\overline{w}(1 - z\overline{w})^2} \varphi'_f(w) \, dA(w).$$

Thus there exists a constant $C_1$ such that (see [2], [3])

$$\rho(f(z), f(0)) = |\varphi_f(z)| \leq \iint_D \frac{1 - |w|^2}{|w||1 - z\overline{w}|^2} |\varphi'_f(w)| \, dA(w)$$

$$\leq C_1 \iint_D \frac{1 - |w|^2}{|1 - z\overline{w}|^2} f^*(w)(1 - |\varphi_f(w)|^2) \, dA(w)$$

$$\leq C_1 \iint_D (1 - |w|^2) f^*(w) \left(\frac{1 - |w|^2}{|1 - z\overline{w}|^2}\right)^2 \, d\lambda(w).$$

By Hölder inequality and 1.4.10 of [3] we get

$$\rho(f(z), f(0)) \leq C_1 \|f\|_{B^h_p} \left( \iint_D \frac{(1 - |w|^2)^{2q-2}}{|1 - z\overline{w}|^{2q}} \, dA(w) \right)^{\frac{1}{q}} \leq C_2 \|f\|_{B^h_p} \sigma(z, 0)^{\frac{1}{q}}.$$ 

When points $a$ and $b$ lie closely one to another the behaviours of hyperbolic distance $\sigma(a, b)$ and pseudohyperbolic distance $\rho(a, b)$ are the same. Thus for sufficiently small $\tau$ there is a constant $C$ such that for any $z, w \in D$, $\sigma(z, w) \leq \tau$,

$$\sigma(f(z), f(w)) \leq C \|f\|_{B^h_p} \sigma(z, w)^{\frac{1}{q}}.$$ 

Let $\varphi : D \to D$ analytic and $\Gamma = \{|z| = 1\}$. 


**Definition.** [1] Bounded function \( \varphi(z) \) has a finite angular derivative at \( \zeta \) on unit circle \( \Gamma \) if there is \( \eta \) on \( \Gamma \) so that \( \frac{\varphi(z) - \eta}{z - \zeta} \) has a finite non-tangential limit as \( z \to \zeta \).

By the Julia-Caratheodory Theorem [1] function \( \varphi(z) \) has a finite angular derivative at \( \zeta \) if and only if

\[
\liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty.
\]

Let \( \zeta \) be a point on the unit circle \( \Gamma \), \( 0 < \delta < 1 \), \( 0 < \epsilon < \frac{\pi}{2} \) and \( \Delta_{\delta, \epsilon} = \{ z : |z - \zeta| < \delta, |\arg(z) - \arg(\zeta)| < \epsilon \} \).

**Theorem 4.** Functions of \( B^{h}_{p} \), \( 1 < p < \infty \), don't have angular derivatives.

**Proof.**

Suppose that \( \varphi(z) \in B^{h}_{p} \), \( 1 < p < \infty \), and \( \varphi(z) \) has an angular derivative at a point \( \zeta \in \Gamma \). Then by the Julia-Caratheodory Theorem \( \frac{1 - |\varphi(z)|}{1 - |z|} \sim |\varphi'(\zeta)| \) for sufficiently small \( \delta \) and \( z \in \Delta_{\delta, \epsilon} \) and moreover, \( |\varphi'(z)| \approx |\varphi'(\zeta)| \) for \( z \in \Delta_{\delta, \epsilon} \). Thus

\[
\int_{\Delta_{\delta, \epsilon}} \left( (1 - |z|^2) \varphi^*(z) \right)^p d\lambda(z) \geq \int_{\Delta_{\delta, \epsilon}} \left( (1 - |z|^2) \varphi^*(z) \right)^p d\lambda(z) \approx \int_{\Delta_{\delta, \epsilon}} d\lambda(z) = \infty.
\]

3. Composition operators

Let \( B \) be Bloch space of holomorphic functions in \( D \). By the definition (e.g. [5]) \( f(z) \in B \) if

\[
\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| < \infty.
\]

Holomorphic in \( D \) function \( f(z) \) belongs to the analytic Besov space \( B_{p}, 1 < p < \infty \), if

\[
\|f\|_{B_{p}} = \left( \int_{D} (1 - |z|^2)^p |f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}} < \infty
\]

and \( B_{\infty} = B \).

Let \( X \) be a normed subspace of \( B \) and \( \varphi(z) \) be a holomorphic self map of the unit disk \( D \). We say that composition operator \( C_{\varphi} : B \to X \) is compact if

\[
\lim_{r \to 1-0} \|f \circ \varphi - f \circ (\chi_{r}\varphi)\|_{X} = 0
\]

where \( \chi_{r}(z) \) is a characteristic function of the disk \( D_{r} = rD \)

\[
\chi_{r}(z) = \begin{cases} 
1, & |z| \leq r \\
0, & |z| > r.
\end{cases}
\]
HYPERBOLIC BESOV FUNCTIONS

**Theorem 5.** For every \( \varphi(z) \in B^h_p \), \( 1 < p < \infty \), and any \( f \in B \) composition \( f \circ \varphi \in B_p \). Composition operator \( C_\varphi \) is a compact.

**Proof.**

Let \( f \in B \) and \( \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| = M_f \). Let \( \varphi(z) \in B^h_p \) and \( \| \varphi(z) \|_{B^h_p} = M_\varphi \). If \( g(z) = f \circ \varphi(z) \) then

\[
\|g(z)\|_{B^h_p}^p = \int_{\mathbb{D}} (1 - |z|^2)^p |g'(z)|^p \, d\lambda(z) = \\
= \int_{\mathbb{D}} (1 - |z|^2)^p |f'(\varphi(z))|^{p} |\varphi'(z)|^p \, d\lambda(z) = \\
= \int_{\mathbb{D}} (1 - |z|^2)^p (\varphi^*(z))^p (1 - |\varphi(z)|^2)^p |f'(z)|^p \, d\lambda(z) \leq M_f^p \cdot M_\varphi^p < \infty.
\]

Now we prove compactness of operator \( C_\varphi \).

\[
\|f(\varphi(z)) - f(\chi_r \cdot \varphi(z))\|_{B_p}^p = \\
= \int_{\mathbb{D}} (1 - |z|^2)^p |(f \circ \varphi(z))'|^p \, d\lambda(z) \leq \\
\leq M_f^p \int_{\mathbb{D} \setminus D_r} (1 - |z|^2)^p \varphi^*(z)^p \, d\lambda(z).
\]

The last integral tends to zero as \( r \to 1 \) because \( \varphi(z) \in B^h_p \).

**Theorem 6.** If for every Bloch function \( f(z) \) the composition \( f \circ \varphi \) is a \( p \)-analytic Besov function, \( 1 < p < \infty \), then \( \varphi(z) \in B^h_p \).

**Proof.**

Ramey and Ullrich [2] constructed such Bloch functions \( f \) and \( g \) that

\[
|f'(z)| + |g'(z)| \geq \frac{1}{1 - |z|^2}.
\]

Then for every \( p > 1 \)

\[
|f'(z)|^p + |g'(z)|^p \geq \frac{K_p}{(1 - |z|^2)^p}.
\]

and thus

\[
K_p \| \varphi \|_{B^h_p}^p \leq \| f \circ \varphi \|_{B^p}^p + \| g \circ \varphi \|_{B^p}^p < \infty.
\]
SHAMIL MAKHMUTOV

REFERENCES


Department of Mathematics
Hokkaido University
Sapporo 060
Japan

*e-mail:* makhm@euler.math.hokudai.ac.jp