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$F$-algebra $M$ of holomorphic functions (Spaces of Analytic and Harmonic Functions and Operator Theory)

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$F$-algebra $M$ of holomorphic functions

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1. Introduction.

Let $U$ be the unit disc $\{|z|<1\}$ in $\mathbb{C}$. A function $f$ holomorphic in $U$ is said to belong to the class $M$ if

$$\rho(f) \equiv \int_0^{2\pi} \log(1 + Mf(\theta)) \, d\theta < \infty$$

where $Mf(\theta) = \sup_{0 \leq \rho < 1} |f(re^{i\theta})|$, and $\log^+ x = \max(\log x, 0)$, $x > 0$. The class $M$ was introduced and studied in [1, 2, 3, 4, 5]. The class $M$ is related to the usual Hardy space $H^p(p > 0)$ and the Nevanlinna class $N^+$ as

$$\bigcup_{p>0} H^p \subsetneqq M \subsetneqq N^+$$

The class $M$ with the metric $d(f, g) = \rho(f - g)$ is an $F$-algebra, i.e., a topological vector space with a complete translation invariant metric in which multiplication is continuous. The class $M$ has many similarities with $N^+$, but it is not fully studied as $N^+$. In this report we wish to summarize the works on the class $M$ [1, 2, 3, 4, 5] and some open problems. We refer to [7] for the Hardy space and the Smirnov class.

2. $M$ as a class of functions.

For a real-valued function $h$ in $L^1(\partial U)$, we let

$$f(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} + z \frac{e^{i\theta} + h(e^{i\theta})}{e^{i\theta} - z} \, d\theta \right).$$

We have

2.1 Theorem. If $P[h^+] \in \text{Re } H^1$, then $f \in M$ where $P[h^+]$ is the Poisson integral of $h^+ = \max(h, 0)$. The converse is false.

2.2 Problem. Find a necessary and sufficient condition on $h$ in order that $f \in M$. That is, characterize those outer functions in $M$.

Unlike $N^+$, the inner factor cannot be cancelled in $M$ as in the following theorem.
2.3 **Theorem.** [1] There exists an $f$ in $M$ whose outer factor $F$ is not in $M$. It is easy to see that a finite Blaschke factor of $f \in M$ can be cancelled in $M$ but we do not know whether an infinite Blaschke factor of $f$ can be cancelled in $M$ or not.

2.4 For $\alpha > 1$, we define

$$M_\alpha f(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma_\alpha(e^{i\theta})\}$$

where $\Gamma_\alpha(e^{i\theta})$ is the nontangential region at $e^{i\theta}$ defined as

$$\Gamma_\alpha(e^{i\theta}) = \{z \in U : |e^{i\theta} - z| < \frac{\alpha}{2}(1 - |z|^2)\}$$

In the definition of $M$, the radial maximal function $Mf(e^{i\theta})$ can be replaced by the nontangential maximal function $M_\alpha f(e^{i\theta})$. Precisely we have

2.5 **Theorem.** There exists a positive constant $C_\alpha$ such that

$$\int_0^{2\pi} \log(1 + Mf(e^{i\theta}))\,d\theta \leq C_\alpha \int_0^{2\pi} \log(1 + Mf(e^{i\theta}))\,d\theta$$

2.6 **Corollary.** The class $M$ is invariant under the composition of automorphisms of the unit disc $U$. More precisely, if $M \in M$ then $f \circ \varphi \in M$ for any $\varphi \in \text{Aut}(U)$.

2.7 **Problem.** Is $M$ invariant under the composition of inner functions? Recall that $N^+$ is invariant under the composition of inner functions.

For the boundary values of functions in $M$, the following is proved in [5].

2.8 **Theorem.** [5] A measurable function $g(e^{i\theta})$ on $\partial U$ coincides with the angular boundary value of some function $f$ in $M$ if and only if there exists a sequence of polynomials $p_n$ with properties:

(a) $p_n(e^{i\theta}) \to g(e^{i\theta})$ a.e. on $\partial U$ and

(b) $\lim_{n \to \infty} \int_0^{2\pi} \log(1 + Mp_n(\theta))\,d\theta < \infty$.

3. **$M$ as an $F$-space**

It is proved in [1] that $M$ with the metric $d(f, g) = \rho(f - g)$ is a separable $F$-space. The space $M$ has many similarities as $N^+$ as $F$-spaces.

3.1 **Theorem.** $M$ is not locally bounded.

3.2 **Theorem.** If $\Lambda$ is a continuous linear functional on $M$, then there exists a $g \in A^\infty(U)$ (i.e., $g$ is analytic in $U$ and $C^\infty$ on $\bar{U}$) such that
\[ \Lambda f = \lim_{r \to 1} \int_{0}^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta, \quad f \in M. \]

Conversely, if \( g \in A^\infty(U) \) and if

\[ \Lambda f = \lim_{r \nearrow 1} \int_{0}^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta \]

exists for all \( f \in M \), then \( \Lambda \) defines a continuous linear functional on \( M \).

3.3 Problem. Describe \( g \in A^\infty(U) \) more precisely in the above theorem.

3.4 Theorem. \( M \) is not locally convex.

4. \( M \) as an \( F \)-algebra

As an \( F \)-algebra \( M \), the invertible elements, multiplicative linear functionals, closed maximal ideals and onto algebra endomorphisms of \( M \) are determined as we see in the following theorems.

4.1 Theorem. The only invertible elements of \( M \) are those outer function \( f \) with \( \log |f| \in \text{Re } H^1 \).

4.2 Theorem. \( \gamma \) is a nontrivial multiplicative linear functional on \( M \) if and only if \( \gamma(f) = f(\lambda), \quad f \in M \), for some \( \lambda \in U \). Therefore, every nontrivial multiplicative linear functional is continuous.

4.3 Theorem. Every closed maximal ideal of \( M \) is the kernel of a multiplicative linear functional.

4.4 Theorem. There exists a maximal ideal \( M \) which is not the kernel of a multiplicative linear functional.

4.5 Theorem. \( \Gamma : M \to M \) is an onto algebra endomorphism if and only if \( \Gamma(f) = f \circ \varphi, \quad f \in M \), for some automorphism \( \varphi \) of \( U \). In particular, \( \Gamma \) is invertible.
References