$F$-algebra $M$ of holomorphic functions (Spaces of Analytic and Harmonic Functions and Operator Theory)

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\textbf{F-algebra $M$ of holomorphic functions}

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1. Introduction.

Let $U$ be the unit disc $\{|z| < 1\}$ in $\mathbb{C}$. A function $f$ holomorphic in $U$ is said to belong to the class $M$ if

$$\rho(f) \equiv \int_{0}^{2\pi} \log(1 + Mf(\theta)) \, d\theta < \infty$$

where $Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$ and $\log^+ x = \max(\log x, 0)$, $x > 0$. The class $M$ was introduced and studied in [1, 2, 3, 4, 5]. The class $M$ is related to the usual Hardy space $H^p(p > 0)$ and the Nevanlinna class $N^+$ as

$$\bigcup_{p>0} H^p \subsetneq M \subsetneq N^+$$

The class $M$ with the metric $d(f, g) = \rho(f - g)$ is an $F$-algebra, i.e., a topological vector space with a complete translation invariant metric in which multiplication is continuous. The class $M$ has many similarities with $N^+$, but it is not fully studied as $N^+$. In this report we wish to summarize the works on the class $M$ [1, 2, 3, 4, 5] and some open problems. We refer to [7] for the Hardy space and the Smirnov class.

2. $M$ as a class of functions.

For a real-valued function $h$ in $L^1(\partial U)$, we let

$$f(z) = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it}) \, dt \right).$$

We have

2.1 Theorem. If $P[h^+] \in \text{Re } H^1$, then $f \in M$ where $P[h^+]$ is the Poisson integral of $h^+ = \max(h, 0)$. The converse is false.

2.2 Problem. Find a necessary and sufficient condition on $h$ in order that $f \in M$. That is, characterize those outer functions in $M$.

Unlike $N^+$, the inner factor cannot be cancelled in $M$ as in the following theorem.
2.3 Theorem. [1] There exists an \( f \) in \( M \) whose outer factor \( F \) is not in \( M \).
It is easy to see that a finite Blaschke factor of \( f \in M \) can be cancelled in \( M \) but
we do not know whether an infinite Blaschke factor of \( f \) can be cancelled in \( M \) or not.

2.4 For \( \alpha > 1 \), we define
\[
M_{\alpha}f(e^{i\theta}) = \sup\{|f(Z)| : Z \in \Gamma(\alpha)\}
\]
where \( \Gamma(\alpha) \) is the nontangential region at \( e^{i\theta} \) defined as
\[
\Gamma(\alpha) = \{z \in U : |e^{i\theta} - z| < \frac{\alpha}{2}(1 - |z|^2)\}
\]
In the definition of \( M \), the radial maximal function \( Mf(e^{i\theta}) \) can be replaced by the
nontangential maximal function \( M_{\alpha}f(e^{i\theta}) \). Precisely we have

2.5 Theorem. There exists a positive constant \( C_{\alpha} \) such that
\[
\int_{0}^{2\pi} \log(1 + Mf(e^{i\theta})) \, d\theta \leq C_{\alpha} \int_{0}^{2\pi} \log(1 + M_{\alpha}f(e^{i\theta})) \, d\theta
\]

2.6 Corollary. The class \( M \) is invariant under the composition of automorphisms of the unit disc \( U \). More precisely, if \( M \subset M \) then \( f \circ \varphi \in M \) for any \( \varphi \in \text{Aut}(U) \).

2.7 Problem. Is \( M \) invariant under the composition of inner functions? Recall
that \( N^+ \) is invariant under the composition of inner functions.

For the boundary values of functions in \( M \), the following is proved in [5].

2.8 Theorem. [5] A measurable function \( g(e^{i\theta}) \) on \( \partial U \) coincides with the
angular boundary value of some function \( f \) in \( M \) if and only if there exists a sequence of
d polynomials \( p_n \) with properties :

(a) \( p_n(e^{i\theta}) \rightarrow g(e^{i\theta}) \) a.e. on \( \partial U \) and

(b) \( \lim_{n \to \infty} \int_{0}^{2\pi} \log(1 + Mp_n(\theta)) \, d\theta < \infty \).

3. \textit{M as an F-space}

It is proved in [1] that \( M \) with the metric \( d(f, g) = \rho(f - g) \) is a separable
\( F \)-space. The space \( M \) has many similarities as \( N^+ \) as \( F \)-spaces.

3.1 Theorem. \( M \) is not locally bounded.

3.2 Theorem. If \( \Lambda \) is a continuous linear functional on \( M \), then there exists a
\( g \in A^\infty(U) \) (i.e., \( g \) is analytic in \( U \) and \( C^\infty \) on \( \bar{U} \)) such that
\[ \Lambda f = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta, \quad f \in M. \]

Conversely, if \( g \in A^\infty(U) \) and if
\[ \Lambda f = \lim_{r \nearrow 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta \]
exists for all \( f \in M \), then \( \Lambda \) defines a continuous linear functional on \( M \).

3.3 Problem. Describe \( g \in A^\infty(U) \) more precisely in the above theorem.

3.4 Theorem. \( M \) is not locally convex.

4. \( M \) as an \( F \)-algebra

As an \( F \)-algebra \( M \), the invertible elements, multiplicative linear functionals, closed maximal ideals and onto algebra endomorphisms of \( M \) are determined as we see in the following theorems.

4.1 Theorem. The only invertible elements of \( M \) are those outer function \( f \) with \( \log|f| \in \text{Re} H^1 \).

4.2 Theorem. \( \gamma \) is a nontrivial multiplicative linear functional on \( M \) if and only if \( \gamma(f) = f(\lambda), \quad f \in M \), for some \( \lambda \in U \). Therefore, every nontrivial multiplicative linear functional is continuous.

4.3 Theorem. Every closed maximal ideal of \( M \) is the kernel of a multiplicative linear functional.

4.4 Theorem. There exists a maximal ideal \( M \) which is not the kernel of a multiplicative linear functional.

4.5 Theorem. \( \Gamma : M \to M \) is an onto algebra endomorphism if and only if \( \Gamma(f) = f \circ \varphi, f \in M \), for some automorphism \( \varphi \) of \( U \). In particular, \( \Gamma \) is invertible.
References