Ergodic control in a single product manufacturing system

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Abstract

We study the ergodic control problem related to stochastic production planning in a single product manufacturing system with production constraints. The existence of a solution to the corresponding Bellman equation and the optimal control are shown.

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1 Introduction

This paper deals with the following 1st order differential equation:

$$\lambda = F' \left( \frac{\partial v}{\partial x} (x, i) \right) - i \frac{\partial v}{\partial x} (x, i) + Av(x, i) + h(x), \quad x \in \mathbb{R}^1, \; i = 1, 2, \ldots, d.$$  (1)

Here $\lambda$ is a constant, $F(x) = kx$ if $x < 0$, $= 0$ if $x \geq 0$ for some positive constant $k > 0$, $h$ is convex function, and $A$ denotes the infinitesimal generator of an irreducible Markov chain $(z(t), P)$ with state space $Z = \{1, 2, \ldots, d\}$:

$$Av(x, i) = \sum_{j \neq i} q_{ij}[v(x, j) - v(x, i)],$$  (2)

where $q_{ij}$ is the jump rate of $z(t)$ from $i$ to $j$. The unknown are the pair $(v, \lambda)$, where $v(\cdot, i) \in C^1(\mathbb{R}^1)$ for every $i \in Z$.

Equation (1) arises in the ergodic control problem of stochastic production planning in a single product manufacturing system and is called the Bellman equation. The inventory level $x(t)$ of stochastic production planning modeled by Sethi and Zhang [11] is governed by the differential equation

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \; z(0) = i, \; P\text{-a.s.},$$  (3)
for production rate $0 \leq p(t) \leq k$, in which $z(t)$ is interpreted as the demand rate. For ergodic control, the cost $J(p(\cdot) : x, i)$ associated with $p(\cdot)$ is given by

$$J(p(\cdot) : x, i) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T h(x(t)) dt \mid x(0) = x, z(0) = i \right], \tag{4}$$

where $h(x)$ represents the convex inventory cost.

The purpose of this paper is to show the existence of a solution of Bellman equation (1) and to present an optimal control minimizing the cost $J(p(\cdot) : x, i)$ subject to (3). In the control problem of manufacturing systems [5], [12] with discounted rate $\alpha > 0$, many authors have investigated the Bellman equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i \frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x). \tag{5}$$

Our method consists in studying the limit of (5) as $\alpha$ tends to 0. This approach develops the technique of Bensoussan-Frehse [2] concerning non-degenerate 2nd order partial differential equations to our degenerate case. We also refer to Ghosh et al. [7], [8] in the case that the Brownian motion is added to (3) as sales returns and a bounded restriction on production rate $p$ is made.

Section 2 is devoted to the existence problem of (1) under the convexity assumption and others on $h$, and properties of the solution are shown in § 3. In § 4 an optimal control for the ergodic control problem and the value are given. In § 5 we present an example of the solution to (1).

## 2 Existence

We are concerned with the equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i \frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x) \quad x \in \mathbb{R}^1, \ i \in \mathbb{Z}, \tag{6}$$

and make the following assumptions:

- $h(x)$ is nonnegative and convex on $\mathbb{R}^1$,
- $\exists C > 0; 0 \leq h(x) \leq C(1 + |x|^\kappa)$ for some positive integer $\kappa$, \tag{7}
- $k - d > 0$. \tag{8}

**Theorem 2.1** We assume (7), (8) and (9). Then there exists a unique convex solution $u_\alpha(\cdot, i) \in C^1(\mathbb{R}^1), i \in \mathbb{Z}$ of equation (6) such that

$$\alpha \|u_\alpha(\cdot, i)\|_{L^\infty(\mathbb{R})} \leq K_r, \tag{10}$$

$$\|\frac{\partial u_\alpha}{\partial x}(\cdot, i)\|_{L^\infty(\mathbb{R})} \leq K_r, \tag{11}$$

$$\|A u_\alpha(\cdot, i)\|_{L^\infty(\mathbb{R})} \leq K_r, \quad i \in \mathbb{Z}. \tag{12}$$
where $K_r$ is a positive constant depending only on $r$ of $I_r = (-r, r)$.

Proof. According to [11, Theorem 3.1], equation (6) has a viscosity solution [6] given by

$$u_{\alpha}(x, i) = \inf_{p(\cdot) \in \mathcal{P}(x, i)} \left\{ E\left[ \int_0^{\infty} e^{-\alpha t} h(x(t)) dt \mid x(0) = x, Z(0) = i \right] \right\},$$

where $x(t)$ is as in (3), and the infimum is taken over the class $\mathcal{P}(x, i)$ of control processes $p(\cdot)$ such that $0 \leq p(t) \leq k$ and $p(t)$ is adapted to $\mathcal{F}_t = \sigma(z(s), s \leq t)$. Moreover, $u_{\alpha}(x, i)$ is convex and hence a classical solution of (6) in $C^1(R^1)$. As is well-known [9], for the irreducible Markov chain $(z(t), P)$ there exists a unique equilibrium distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_d) > 0$ such that

$$\pi A = 0 \quad \text{and} \quad \sum_{i \in \mathbb{Z}} \pi_i = 1. \quad (13)$$

Now, multiplying (6) by $\pi_i$ and summing up, we have

$$\alpha \sum_i \pi_i u_{\alpha}(x, i) = \sum_i \pi_i \{ F(\frac{\partial u_{\alpha}}{\partial x}(x, i)) - \frac{i}{\alpha} \frac{\partial u_{\alpha}}{\partial x}(x, i) \} + h(x). \quad (14)$$

Since $F(x) - ix \leq 0$ under (9), we have

$$\alpha \sum_i \pi_i u_{\alpha}(x, i) \leq h(x) \leq K_r \quad \text{on} \quad I_r.$$

Thus we can obtain (10) by $u_{\alpha}(x, i) \geq 0$.

Next, note that

$$F(x) - ix \leq -a|x|, \quad (15)$$

where $a = \min\{k - d, 1\} > 0$. Hence, we have by (14)

$$a \sum_i \pi_i \frac{\partial u_{\alpha}}{\partial x}(x, i) \leq h(x) - \alpha \sum_i \pi_i u_{\alpha}(x, i).$$

Thus we deduce $|\frac{\partial u_{\alpha}}{\partial x}(x, i)| \leq K_r$ on $I_r$ by (10) and then (11). Finally, (12) follows from (6), (10) and (11) immediately.

Next we show the behavior of a solution to equation (6) as $\alpha \to 0$.

**Theorem 2.2** Under the assumptions of Theorem 2.1, there exists a subsequence $\alpha \to 0$ such that

$$v_{\alpha}(x, i) := u_{\alpha}(x, i) - u_{\alpha}(0, i) \to v_0(x, i) \in C^1(R^1),$$

$$\mu(\alpha) := \alpha u_{\alpha}(0, i) - Au_{\alpha}(0, i) \to \mu_\mu \in R^1,$$
uniformly on each $I_r$. The limit $(v_0(\cdot, i), \mu_i), \ i \in Z$, satisfies

$$\mu_i = F\left(\frac{\partial v_0}{\partial x}(x, i)\right) - i \frac{\partial v_0}{\partial x}(x, i) + Av_0(x, i) + h(x), \ x \in R^1. \ (16)$$

Proof. Let us note that $(v_\alpha(\cdot, i), \mu(\alpha))$ satisfies

$$\alpha v_\alpha(x, i) + \mu(\alpha) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \frac{\partial v_\alpha}{\partial x}(x, i) + Av_\alpha(x, i) + h(x). \ (17)$$

By (11) it is obvious that

$$\|v_\alpha(\cdot, i)\|_{L^\infty(I_r)} + \|\frac{\partial v_\alpha}{\partial x}(\cdot, i)\|_{L^\infty(I_r)} \leq K', \ i \in Z. \ (18)$$

Hence $\{v_\alpha(\cdot, i)\}$ is equicontinuous on $I_r$.

Let us define

$$B_\alpha(x) = \alpha v_\alpha(x, i) + \mu(\alpha) - Av_\alpha(x, i) - h(x).$$

We recall that, by assumption, $h(x)$ is Lipschitz continuous on $I_r$. Then, by (18)

$$|B_\alpha(x) - B_\alpha(y)| \leq C|x - y|, \quad (C > 0 : \text{indep. of } \alpha),$$

From (17) it follows that

$$B_\alpha(x) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \frac{\partial v_\alpha}{\partial x}(x, i),$$

Then we have

$$\frac{\partial v_\alpha}{\partial x} = \begin{cases} \frac{B_\alpha(x)}{k-i} & \text{if } \frac{\partial v_\alpha}{\partial x} < 0 \\ -\frac{B_\alpha(x)}{i} & \text{if } \frac{\partial v_\alpha}{\partial x} \geq 0, \end{cases}$$

Since $\frac{\partial v_\alpha}{\partial x}$ is nondecreasing, we can see

$$|\frac{\partial v_\alpha}{\partial x}(x, i) - \frac{\partial v_\alpha}{\partial x}(y, i)| \leq C|x - y|.$$

Thus $\{\frac{\partial v_\alpha}{\partial x}(\cdot, i)\}$ is also equicontinuous on $I_r$. By the Ascoli-Arzelà theorem, there exists a subsequence $\alpha \to 0$ such that

$$v_\alpha(x, i) \to v_0(x, i), \quad (19)$$

$$\frac{\partial v_\alpha}{\partial x}(x, i) \to \frac{\partial v_0}{\partial x}(x, i), \quad \text{uniformly on } I_r. \quad (20)$$

By a standard argument, we can choose a subsequence $\alpha \to 0$, independent of $r$, such that (19) and (20) are fulfilled on every $I_r$. Further, by (10) and (12)

$$\mu(\alpha) \to \mu_i.$$

Letting $\alpha \to 0$ in (17), we deduce (16). The proof is complete.

Now let us show the existence of a solution to equation (1).
Theorem 2.3 We assume (7), (8) and (9). Then there exists a solution $(v, \lambda)$ of equation (1) such that $v(x, i)$ is convex on $R^1$ and $v(\cdot, i) \in C^1(R^1)$.

Proof. Let us define

\[ v(x, i) = v_0(x, i) + f(i), \]
\[ \lambda = \sum_i \pi_i \mu_i, \]

where $(v_0(\cdot, i), \mu_i)$ is as in (16) and $f(i)$ is a solution of

\[ Af(i) = -\mu + \lambda, \quad i \in Z. \tag{21} \]

Then it is easily seen that $(v, \lambda)$ satisfies (1). The convexity of $v(x, i)$ and $v(\cdot, i) \in C^1(R^1)$ are immediate from Theorems 2.1 and 2.2.

To complete the proof, it is sufficient to check the existence of $f(i)$. By the irreducible Markov chain $(z(t), P)$ it follows that for any $g \in R^d$

\[ E[g(z(s/\alpha))] \to \sum_i \pi_i g(i) \quad \text{as} \quad \alpha \to 0. \]

Hence

\[ \alpha G_\alpha g(i) = \alpha E[\int_0^\infty e^{-\alpha t} g(z(t)) dt] = \int_0^\infty e^{-s} E[g(z(s/\alpha))] ds \to \sum_i \pi_i g(i) = \pi g, \]

where $G_\alpha$ denotes the resolvent operator of the Markov chain $(z(t), P)$. According to [4, Lemma 7.3(c,d), p.39], we can obtain the relation:

\[ \{g \in R^d | \pi g = 0\} = \{Ag \in R^d | g \in R^d\}. \]

We notice by (13) that

\[ \pi (-\mu + \lambda) = 0. \]

Therefore we conclude that equation (21) admits a solution $f(i)$.

3 Properties

We investigate properties of a solution to the Bellman equation (1). Now we make the assumption:

\[ h(x)/|x| \to \infty \quad \text{as} \quad |x| \to \infty. \tag{22} \]
Lemma 3.1 Under (22), the convex solution \( v(\cdot, i) \in C^1(R^1) \) of equation (1) satisfies

\[
|\frac{\partial v}{\partial x}(x, i)| \rightarrow \infty \text{ as } |x| \rightarrow \infty. \tag{23}
\]

Proof. It is sufficient to show (23) in the case \( x \rightarrow -\infty \). By the convexity of \( v(x, i) \), we can define \( M_i \) by

\[
M_i = -\lim_{x \rightarrow -\infty} \frac{\partial v}{\partial x}(x, i).
\]

For any sequence \( x_n \rightarrow -\infty \), we can easily see

\[
\frac{v(x_n, i)}{|x_n|} \rightarrow M_i.
\]

Suppose that \( M_i < \infty \) for some \( i \in Z \). Then, dividing (1) by \( |x_n| \) and passing to the limit, we have by (22)

\[
\lambda/|x_n| = \left[F(\frac{\partial v}{\partial x}(x_n, i)) - i\frac{\partial v}{\partial x}(x_n, i) + \sum_{j \neq i} q_{ij}v(x_n, j) - \sum_{j \neq i} q_{ij}v(x_n, i) + h(x_n)/|x_n|\right] \rightarrow \infty,
\]

since \( v(x, j) \geq ax + b \) for some constants \( a \) and \( b \). This is a contradiction. Hence \( M_i = \infty \) for all \( i \in Z \), and thus the assertion follows.

Lemma 3.2 For the convex solution \( v(\cdot, i) \in C^1(R^1) \) of equation (1), there is a constant \( C > 0 \) such that

\[
|v(x, i)| \leq C(1 + |x|^\kappa+1). \tag{24}
\]

Proof. From (1) and (15) it follows that

\[
\lambda \leq -a|\frac{\partial v}{\partial x}(x, i)| + Av(x, i) + h(x).
\]

If \( \frac{\partial v}{\partial x}(x, i) < 0 \) on some interval \((-\infty, x_1)\) with \( x_1 < 0 \), then by (8)

\[
-\frac{\partial v}{\partial x}(x, i) \leq \frac{1}{a}Av(x, i) + C(1 + |x|^\kappa) \tag{25}
\]

Multiplying (25) by \( \pi_i \) and summing up, we get by (13)

\[
-\sum_i \pi_i \frac{\partial v}{\partial x}(x, i) \leq C(1 + |x|^\kappa).
\]
Integrating over \((x, x_1)\), we have
\[
\sum_i \pi_i (v(x, i) - v(x_1, i)) \leq C(1 + |x|^{k+1}).
\]

This relation can be obtained in the case that \(\frac{\partial v}{\partial x} \geq 0\) on some interval \((x_2, \infty)\) with \(x_2 > 0\). Therefore we can obtain the desired result by \(\pi > 0\).

Next, we consider the equation
\[
\frac{dx^*(t)}{dt} = p^*(x^*(t), z(t)) - z(t), \quad x^*(0) = x, \quad z(0) = i, \quad P - a.s.,
\]
where
\[
p^*(x, i) = \begin{cases} 
  k & \text{if } \frac{\partial v}{\partial x}(x, i) < 0 \\
  i & \text{if } \frac{\partial v}{\partial x}(x, i) = 0 \\
  0 & \text{if } \frac{\partial v}{\partial x}(x, i) > 0.
\end{cases}
\]

**Lemma 3.3** Equation (26) admits a unique solution \(x^*(t)\), which satisfies
\[
\sup_t ||x^*(t)||_{L^\infty} < \infty.
\]

**Proof.** Since \(p^*(x, i)\) is nonincreasing in \(x\), the differential equation (26) has a unique solution by [6, Theorem 6.2].

To complete the proof, let \(\bar{x} = \sup \{x \in R^1 : p^*(x, i) \geq i\ \text{for some } i \in Z\}\). Obviously, \(\bar{x}\) is finite, because \(p^*(x, i)\) is nonnegative. Similarly, let \(\tilde{x} = \inf \{x \in R^1 : p^*(x, i) \leq i\ \text{for some } i \in Z\}\). Suppose that \(\tilde{x}\) is not finite. Then there exists \(i \in Z\) such that \(\frac{\partial v}{\partial x}(x, i) \geq -2i\) for all \(x \in R^1\). On the other hand, by Lemma 3.1, \(\frac{\partial v}{\partial x}(x, i) \to -\infty\) as \(x \to -\infty\). This is a contradiction.

Now, if \(x^*(t) > \bar{x}\) (resp. \(x^*(t) < \tilde{x}\)), then \(\frac{dx^*}{dt}(t) < 0\) (resp. \(\frac{dx^*}{dt}(t) > 0\)). Hence the interval \([\tilde{x}, \bar{x}]\) is an attracting set for (26). Thus the boundedness of \(x^*(t)\) is immediate.

**Lemma 3.4** The constant solution \(\lambda\) of equation (1) satisfies
\[
\lambda = \inf_{p(\cdot) \in P(x, i)} \limsup_{\alpha \to 0} \alpha E[\int_0^\infty e^{-\alpha t} h(x(t)) \, dt \mid x(0) = x, z(0) = i].
\]

**Proof.** For the convex solution \(v(\cdot, i) \in C^1(R^1)\), let us apply an elementary rule and Dynkin’s formula to the first and the second variables of \(v(x(t), z(t))\) respectively. Then we have the relation:
\[
E[e^{-\alpha t} v(x(t), z(t)) \mid x(0) = x, z(0) = i]
\]
\[
= v(x, i) - \alpha E[\int_0^t e^{-\alpha s} v(x(s), z(s)) \, ds \mid x(0) = x, z(0) = i]
\]
\[
+ E[\int_0^t e^{-\alpha s} \frac{\partial v}{\partial x}(x(s), z(s)) \, dx(s) \mid x(0) = x, z(0) = i]
\]
\[
+ E[\int_0^t e^{-\alpha s} Av(x(s), z(s)) \, ds \mid x(0) = x, z(0) = i]
\]
We notice that the minimum of

$$\min_{0 \leq p \leq k} p \frac{\partial v}{\partial x} = F(\frac{\partial v}{\partial x})$$

is attained by $p^*(x, i)$. By (1), we have

$$\lambda \leq \frac{\partial v}{\partial x}(x, i)(p - i) + Av(x, i) + h(x),$$

and the equality holds for $p = p^*(x, i)$. Clearly, by (3)

$$|x(t)| \leq C(t + 1) \text{ for all } p(\cdot) \in P(x, i).$$

By Lemma 3.2

$$E[e^{-\alpha t}|v(x(t), z(t))| \mid x(0) = x, z(0) = i]$$

$$\leq CE[e^{-\alpha t}(1 + |x(t)|^{\kappa + 1}) \mid x(0) = x, z(0) = i]$$

$$\leq Ce^{-\alpha t}(1 + (t + 1)^{\kappa + 1}) \rightarrow 0 \text{ as } t \rightarrow \infty.$$ 

Hence, substituting (30) into (29), we get

$$\frac{\lambda}{\alpha} \leq -v(x, i) + \alpha E[\int_0^\infty e^{-\alpha s}v(x(s), z(s))ds \mid x(0) = x, z(0) = i]$$

$$+ E[\int_0^\infty e^{-\alpha s}h(x(s))ds \mid x(0) = x, z(0) = i].$$

We note that by Lemma 3.2 and 3.3

$$\alpha^2 E[\int_0^\infty e^{-\alpha s}|v(x(s), z(s))|ds \mid x(0) = x, z(0) = i]$$

$$\leq \alpha^2 C \int_0^\infty e^{-\alpha s}(1 + |x^*(s)|^{\kappa + 1})ds \rightarrow 0 \text{ as } \alpha \rightarrow 0.$$ 

Thus we deduce

$$\lambda \leq \inf_{p(\cdot) \in P(x, i)} \limsup_{\alpha \rightarrow 0} \alpha E[\int_0^\infty e^{-\alpha s}h(x(s))ds \mid x(0) = x, z(0) = i],$$

and the equality holds for $p(t) = p^*(x^*(t), z(t))$ of (27).

4 An application to ergodic control

We shall study the ergodic control problem to minimize the cost:

$$J(p(\cdot) : x, i) = \limsup_{T \rightarrow \infty} \frac{1}{T} E[\int_0^T h(x(t))dt \mid x(0) = x, z(0) = i]$$
over all \( p(\cdot) \in U \) subject to
\[
\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \ z(0) = i, \text{ P-a.s.,}
\]
where \( U \) is the set of all nonnegative progressively measurable processes \( p(t) \) such that
\[
p(t) \text{ is adapted to } \mathcal{F}_t,
\]
\[
0 \leq p(t) \leq k,
\]
\[
\sup_t E[|x(t)|^{k+1} \mid x(0) = x, z(0) = i] < \infty \text{ for } \kappa \text{ in (8)}.
\]

**Theorem 4.1** We assume (7), (8), (9) and (22). Then the optimal control \( p^*(t) \) is given by
\[
p^*(t) = p^*(x^*(t), z(t)),
\]
and the value by
\[
J(p^*(\cdot) : x, i) = \lambda,
\]
where \( p^*(x^*(t), z(t)) \) is as in (27).

**Proof.** From the same formula as (29) it follows that
\[
E[v(x(T), z(T)) \mid x(0) = x, z(0) = i]
\]
\[
= v(x, i) + E[\int_0^T \frac{\partial v}{\partial x}(x(s), z(s)) dx(s) \mid x(0) = x, z(0) = i]
\]
\[
+ E[\int_0^T A v(x(s), z(s)) ds \mid x(0) = x, z(0) = i].
\]

We recall (30) to obtain
\[
E[v(x(T), z(T)) \mid x(0) = x, z(0) = i]
\]
\[
\geq v(x, i) + E[\int_0^T (\lambda - h(x(s))) ds \mid x(0) = x, z(0) = i],
\]
where the equality holds for \( x = x^* \) and \( p = p^* \) of (27). By Lemma 3.2 and the definition of \( U \)
\[
\frac{1}{T} E[|v(x(T), z(T))| \mid x(0) = x, z(0) = i]
\]
\[
\leq \frac{C}{T} E[1 + |x(T)|^{k+1} \mid x(0) = x, z(0) = i] \to 0 \text{ as } T \to \infty.
\]

Also, by Lemma 3.3, \( p^*(t) \) belongs to \( U \). Thus we deduce
\[
J(p(\cdot) : x, i) = \lim_{T \to \infty} \sup_T E[\int_0^T h(x(s)) ds \mid x(0) = x, z(0) = i]
\]
\[
\geq \lambda = J(p^*(\cdot) : x, i).
\]

The proof is complete.
5 An Example

In this section we present the example of an solution to the Bellman equation:

$$\lambda = F(\frac{\partial v}{\partial x}(x,i)) - i \frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \quad x \in \mathbb{R}^1, \ i \in Z, \quad (31)$$

in the case that

$$h(x) = x^2, \ k = 3, \quad (32)$$
$$Z = \{1, 2\}, \ q_{12} = q_{21} = 1. \quad (33)$$

Figure Solution $v(x,i)$, $i=1,2$, to the Bellman Equation (31)
We remark that the matrix induced by $A$ is given by
\[
A = \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix},
\]
and the equilibrium distribution $\pi$ is
\[
\pi = \left(\frac{1}{2}, \frac{1}{2}\right).
\]
Therefore the assumptions of Theorem 4.1 are fulfilled.

Now, recalling the form of optimal control $p^*$ and solving the Bellman equation (31) with (32) and (33), we have
\[
\begin{align*}
\lambda &= 0, \\
v(x, 1) &= \begin{cases} \\
\frac{1}{81}(18x^3 + 18x^2 - 24x + 16 - 16e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\
-\frac{1}{81}(18x^3 + 9x^2 + 12x + 8 - 8e^\frac{3}{2}x) & \text{if } x < 0,
\end{cases} \\
v(x, 2) &= \begin{cases} \\
\frac{1}{81}(18x^3 - 9x^2 + 12x - 8 + 8e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\
-\frac{1}{81}(18x^3 - 18x^2 - 24x - 16 + 16e^\frac{3}{2}x) & \text{if } x < 0.
\end{cases}
\end{align*}
\]
Then the optimal control $p^*$ is given by
\[
p^*(x, i) = \begin{cases} \\
0 & \text{if } x > 0 \\
3 & \text{if } x < 0.
\end{cases}
\]
The solution $v(x, i)$ with (23) - (24), $i = 1, 2$ can be shown in Figure.

References


