Ergodic control in a single product manufacturing system

(1) Introduction

This paper deals with the following 1st order differential equation:

$$\lambda = F\left(\frac{\partial v}{\partial x}(x,i)\right) - i\frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \quad x \in R^1, \quad i = 1, 2, \ldots, d. \quad (1)$$

Here $\lambda$ is a constant, $F(x) = kx$ if $x < 0$, $= 0$ if $x \geq 0$ for some positive constant $k > 0$, $h$ is convex function, and $A$ denotes the infinitesimal generator of an irreducible Markov chain $(z(t), P)$ with state space $Z = \{1, 2, \ldots, d\}$:

$$Av(x,i) = \sum_{j \neq i} q_{ij}[v(x,j) - v(x,i)], \quad (2)$$

where $q_{ij}$ is the jump rate of $z(t)$ from $i$ to $j$. The unknown are the pair $(v, \lambda)$, where $v(.,i) \in C^1(R^1)$ for every $i \in Z$.

Equation (1) arises in the ergodic control problem of stochastic production planning in a single product manufacturing system and is called the Bellman equation. The inventory level $x(t)$ of stochastic production planning modeled by Sethi and Zhang [11] is governed by the differential equation

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \quad z(0) = i, \quad P\text{-a.s.}, \quad (3)$$
for production rate $0 \leq p(t) \leq k$, in which $z(t)$ is interpreted as the demand rate. For ergodic control, the cost $J(p(\cdot) : x, i)$ associated with $p(\cdot)$ is given by

$$J(p(\cdot) : x, i) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[\int_{0}^{T} h(x(t)) dt \mid x(0) = x, z(0) = i],$$

(4)

where $h(x)$ represents the convex inventory cost.

The purpose of this paper is to show the existence of a solution of Bellman equation (1) and to present an optimal control minimizing the cost $J(p(\cdot) : x, i)$ subject to (3). In the control problem of manufacturing systems [5], [12] with discounted rate $\alpha > 0$, many authors have investigated the Bellman equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i \frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x).$$

(5)

Our method consists in studying the limit of (5) as $\alpha$ tends to 0. This approach develops the technique of Bensoussan-Frehse [2] concerning non-degenerate 2nd order partial differential equations to our degenerate case. We also refer to Ghosh et al. [7], [8] in the case that the Brownian motion is added to (3) as sales returns and a bounded restriction on production rate $p$ is made.

Section 2 is devoted to the existence problem of (1) under the convexity assumption and others on $h$, and properties of the solution are shown in § 3. In § 4 an optimal control for the ergodic control problem and the value are given. In § 5 we present an example of the solution to (1).

## 2 Existence

We are concerned with the equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i \frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x) \quad x \in \mathbb{R}^1, \; i \in \mathbb{Z},$$

(6)

and make the following assumptions:

- $h(x)$ is nonnegative and convex on $\mathbb{R}^1$,
- $\exists C > 0; \; 0 \leq h(x) \leq C(1 + |x|^{\kappa})$ for some positive integer $\kappa$,
- $k - d > 0$.

### Theorem 2.1

We assume (7), (8) and (9). Then there exists a unique convex solution $u_\alpha(\cdot, i) \in C^1(\mathbb{R}^1), i \in \mathbb{Z}$ of equation (6) such that

$$\alpha \|u_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r,$$

(10)

$$\|\frac{\partial u_\alpha}{\partial x}(\cdot, i)\|_{L^\infty(I_r)} \leq K_r,$$

(11)

$$\|Au_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \; \; i \in \mathbb{Z},$$

(12)
where $K_r$ is a positive constant depending only on $r$ of $I_r = (-r, r)$.

Proof. According to [11, Theorem 3.1], equation (6) has a viscosity solution [6] given by
\[
u(x, i) = \inf_{p(t) \in P(x, i)} \{ E \left[ \int_0^\infty e^{-\alpha t} h(x(t)) \, dt \mid x(0) = x, z(0) = i \right] \},
\]
where $x(t)$ is as in (3), and the infimum is taken over the class $P(x, i)$ of control processes $p(\cdot)$ such that $0 \leq p(t) \leq k$ and $p(t)$ is adapted to $\mathcal{F}_t = \sigma(z(s), s \leq t)$. Moreover, $u_\alpha(x, i)$ is convex and hence a classical solution of (6) in $C^1(R^1)$. As is well-known [9], for the irreducible Markov chain $(z(t), P)$ there exists a unique equilibrium distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_d) > 0$ such that
\[
\pi A = 0 \quad \text{and} \quad \sum_{i \in Z} \pi_i = 1.
\]
(13)

Now, multiplying (6) by $\pi_i$ and summing up, we have
\[
\alpha \sum_i \pi_i \nu(x, i) = \sum_i \pi_i \left\{ F\left( \frac{\partial \nu}{\partial x}(x, i) \right) - i \frac{\partial \nu}{\partial x}(x, i) \right\} + h(x).
\]
(14)

Since $F(x) - ix \leq 0$ under (9), we have
\[
\alpha \sum_i \pi_i \nu(x, i) \leq h(x) \leq K_r \quad \text{on } I_r.
\]
Thus we can obtain (10) by $u_\alpha(x, i) \geq 0$.

Next, note that
\[
F(x) - ix \leq -a|x|,
\]
(15)
where $a = \min\{k - d, 1\} > 0$. Hence, we have by (14)
\[
a \sum_i \pi_i \left| \frac{\partial \nu}{\partial x}(x, i) \right| \leq h(x) - \alpha \sum_i \pi_i \nu(x, i).
\]
Thus we deduce $|\frac{\partial \nu}{\partial x}(x, i)| \leq K_r$ on $I_r$ by (10) and then (11). Finally, (12) follows from (6), (10) and (11) immediately.

Next we show the behavior of a solution to equation (6) as $\alpha \to 0$.

**Theorem 2.2** Under the assumptions of Theorem 2.1, there exists a subsequence $\alpha \to 0$ such that
\[
\nu_\alpha(x, i) := u_\alpha(x, i) - u_\alpha(0, i) \to v_0(x, i) \in C^1(R^1),
\]
\[
\mu_\alpha := \alpha u_\alpha(0, i) - Au_\alpha(0, i) \to \mu \in R^1,
\]
uniformly on each $\overline{I}_r$. The limit $(v_0(\cdot, i), \mu_i)$, $i \in Z$, satisfies

$$\mu_i = F\left(\frac{\partial v_0}{\partial x}(x, i)\right) - i \cdot \frac{\partial v_0}{\partial x}(x, i) + Av_0(x, i) + h(x), \quad x \in \mathbb{R}^1. \tag{16}$$

Proof. Let us note that $(v_\alpha(\cdot, i), \mu(\alpha))$ satisfies

$$\alpha v_\alpha(x, i) + \mu(\alpha) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \cdot \frac{\partial v_\alpha}{\partial x}(x, i) + Av_\alpha(x, i) + h(x). \tag{17}$$

By (11) it is obvious that

$$\|v_\alpha(\cdot, i)\|_{L^\infty(I_r)} + \|\frac{\partial v_\alpha}{\partial x}(\cdot, i)\|_{L^\infty(\overline{I}_r)} \leq K_r, \quad i \in Z. \tag{18}$$

Hence $\{v_\alpha(\cdot, i)\}$ is equicontinuous on $\overline{I}_r$.

Let us define

$$B_\alpha(x) = \alpha v_\alpha(x, i) + \mu(\alpha) - Av_\alpha(x, i) - h(x).$$

We recall that, by assumption, $h(x)$ is Lipschitz continuous on $\overline{I}_r$. Then, by (18)

$$|B_\alpha(x) - B_\alpha(y)| \leq C|x - y|, \quad (C > 0 : \text{indep. of } \alpha),$$

From (17) it follows that

$$B_\alpha(x) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \cdot \frac{\partial v_\alpha}{\partial x}(x, i),$$

Then we have

$$\frac{\partial v_\alpha}{\partial x} = \begin{cases} \frac{B_\alpha(x)}{k-i} & \text{if } \frac{\partial v_\alpha}{\partial x} < 0 \\ -\frac{B_\alpha(x)}{i} & \text{if } \frac{\partial v_\alpha}{\partial x} \geq 0, \end{cases}$$

Since $\frac{\partial v_\alpha}{\partial x}$ is nondecreasing, we can see

$$|\frac{\partial v_\alpha}{\partial x}(x, i) - \frac{\partial v_\alpha}{\partial x}(y, i)| \leq C|x - y|.$$

Thus $\{\frac{\partial v_\alpha}{\partial x}(\cdot, i)\}$ is also equicontinuous on $\overline{I}_r$. By the Ascoli-Arzelà theorem, there exists a subsequence $\alpha \to 0$ such that

$$v_\alpha(x, i) \to v_0(x, i), \quad (19)$$

$$\frac{\partial v_\alpha}{\partial x}(x, i) \to \frac{\partial v_0}{\partial x}(x, i), \quad \text{uniformly on } \overline{I}_r. \tag{20}$$

By a standard argument, we can choose a subsequence $\alpha \to 0$, independent of $r$, such that (19) and (20) are fulfilled on every $\overline{I}_r$. Further, by (10) and (12)

$$\mu(\alpha) \to \mu_i.$$ 

Letting $\alpha \to 0$ in (17), we deduce (16). The proof is complete.

Now let us show the existence of a solution to equation (1).
Theorem 2.3 We assume (7), (8) and (9). Then there exists a solution \((v, \lambda)\) of equation (1) such that \(v(x, i)\) is convex on \(R^1\) and \(v(\cdot, i) \in C^1(R^1)\).

Proof. Let us define

\[
v(x, i) = v_0(x, i) + f(i),
\]
\[
\lambda = \sum_i \pi_i \mu_i,
\]

where \((v_0(\cdot, i), \mu_i)\) is as in (16) and \(f(i)\) is a solution of

\[Af(i) = -\mu_i + \lambda, \quad i \in Z.\]  

Then it is easily seen that \((v, \lambda)\) satisfies (1). The convexity of \(v(x, i)\) and \(v(\cdot, i) \in C^1(R^1)\) are immediate from Theorems 2.1 and 2.2.

To complete the proof, it is sufficient to check the existence of \(f(i)\). By the irreducible Markov chain \((z(t), P)\) it follows that for any \(g \in R^d\)

\[
E[g(z(s/\alpha))] \rightarrow \sum_i \pi_i g(i) \quad \text{as} \quad \alpha \rightarrow 0.
\]

Hence

\[
\alpha G_\alpha g(i) = \alpha E[\int_0^\infty e^{-\alpha t} g(z(t))dt] \\
= \int_0^\infty e^{-s} E[g(z(s/\alpha))]ds \\
\rightarrow \sum_i \pi_i g(i) = \pi g,
\]

where \(G_\alpha\) denotes the resolvent operator of the Markov chain \((z(t), P)\). According to [4, Lemma 7.3(c,d), p.39], we can obtain the relation:

\[
\{g \in R^d \mid \pi g = 0\} = \{Ag \in R^d \mid g \in R^d\}.
\]

We notice by (13) that

\[
\pi (-\mu + \lambda) = 0.
\]

Therefore we conclude that equation (21) admits a solution \(f(i)\).

3 Properties

We investigate properties of a solution to the Bellman equation (1). Now we make the assumption:

\[
h(x)/|x| \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty.
\]  

(22)
Lemma 3.1 Under (22), the convex solution \( v(\cdot, i) \in C^1(\mathbb{R}^1) \) of equation (1) satisfies
\[
\frac{\partial v}{\partial x}(x, i) \rightarrow \infty \text{ as } |x| \rightarrow \infty. \tag{23}
\]

Proof. It is sufficient to show (23) in the case \( x \rightarrow -\infty \). By the convexity of \( v(x, i) \), we can define \( M_i \) by
\[
M_i = \lim_{x \rightarrow -\infty} \frac{\partial v}{\partial x}(x, i).
\]
For any sequence \( x_n \rightarrow -\infty \), we can easily see
\[
\frac{v(x_n, i)}{|x_n|} \rightarrow M_i.
\]
Suppose that \( M_i < \infty \) for some \( i \in Z \). Then, dividing (1) by \( |x_n| \) and passing to the limit, we have by (22)
\[
\lambda/|x_n| = \left[ F\left( \frac{\partial v}{\partial x}(x_n, i) \right) - i \frac{\partial v}{\partial x}(x_n, i) + \sum_{j \neq i} q_{ij}v(x_n, j) \right] \\
- \sum_{j \neq i} q_{ij}v(x_n, i) + h(x_n)/|x_n| \rightarrow \infty,
\]

since \( v(x, j) \geq ax + b \) for some constants \( a \) and \( b \). This is a contradiction. Hence \( M_i = \infty \) for all \( i \in Z \), and thus the assertion follows.

Lemma 3.2 For the convex solution \( v(\cdot, i) \in C^1(\mathbb{R}^1) \) of equation (1), there is a constant \( C > 0 \) such that
\[
|v(x, i)| \leq C(1 + |x|^\kappa+1). \tag{24}
\]

Proof. From (1) and (15) it follows that
\[
\lambda \leq -a \left| \frac{\partial v}{\partial x}(x, i) \right| + Av(x, i) + h(x).
\]
If \( \frac{\partial v}{\partial x}(x, i) < 0 \) on some interval \((-\infty, x_1)\) with \( x_1 < 0 \), then by (8)
\[
- \frac{\partial v}{\partial x}(x, i) \leq \frac{1}{a} Av(x, i) + C(1 + |x|^\kappa) \tag{25}
\]
Multiplying (25) by \( \pi_i \) and summing up, we get by (13)
\[
- \sum_i \pi_i \frac{\partial v}{\partial x}(x, i) \leq C(1 + |x|^\kappa).
Integrating over \((x, x_1)\), we have
\[
\sum_i \pi_i (v(x, i) - v(x_1, i)) \leq C(1 + |x|^{k+1}).
\]

This relation can be obtained in the case that \(\frac{\partial v}{\partial x} \geq 0\) on some interval \((x_2, \infty)\) with \(x_2 > 0\). Therefore we can obtain the desired result by \(\pi > 0\).

Next, we consider the equation
\[
\frac{dx^*(t)}{dt} = p^*(x^*(t), z(t)) - z(t), \quad x^*(0) = x, \quad z(0) = i, \quad P - a.s.,
\]
where
\[
p^*(x, i) = \begin{cases} 
  k & \text{if } \frac{\partial v}{\partial x}(x, i) < 0 \\
  i & \text{if } \frac{\partial v}{\partial x}(x, i) = 0 \\
  0 & \text{if } \frac{\partial v}{\partial x}(x, i) > 0.
\end{cases}
\]

**Lemma 3.3** Equation (26) admits a unique solution \(x^*(t)\), which satisfies
\[
\sup_t \|x^*(t)\|_\infty < \infty.
\]

**Proof.** Since \(p^*(x, i)\) is nonincreasing in \(x\), the differential equation (26) has a unique solution by [6, Theorem 6.2].

To complete the proof, let \(\bar{x} = \sup\{x \in R^1 : p^*(x, i) \geq i \text{ for some } i \in Z\}\). Obviously, \(\bar{x}\) is finite, because \(p^*(x, i)\) is nonnegative. Similarly, let \(\bar{x} = \inf\{x \in R^1 : p^*(x, i) \leq i \text{ for some } i \in Z\}\). Suppose that \(\bar{x}\) is not finite. Then there exists \(i \in Z\) such that \(\frac{\partial v}{\partial x}(x, i) \geq -2i\) for all \(x \in R^1\). On the other hand, by Lemma 3.1, \(\frac{\partial v}{\partial x}(x, i) \to -\infty\) as \(x \to -\infty\). This is a contradiction.

Now, if \(x^*(t) > \bar{x}\) (resp. \(x^*(t) < \bar{x}\)), then \(\frac{dx^*}{dt}(t) < 0\) (resp. \(\frac{dx^*}{dt}(t) > 0\)). Hence the interval \([\bar{x}, \bar{x}]\) is an attracting set for (26). Thus the boundedness of \(x^*(t)\) is immediate.

**Lemma 3.4** The constant solution \(\lambda\) of equation (1) satisfies
\[
\lambda = \inf_{p(i) \in P(x,i)} \limsup_{\alpha \to 0} \alpha E\left[\int_0^\infty e^{-\alpha t} h(x(t)) \, dt \mid x(0) = x, z(0) = i\right].
\]

**Proof.** For the convex solution \(v(\cdot, i) \in C^1(R^1)\), let us apply an elementary rule and Dynkin’s formula to the first and the second variables of \(v(x(t), z(t))\) respectively. Then we have the relation:
\[
E[e^{-\alpha t}v(x(t), z(t)) \mid x(0) = x, z(0) = i]
= v(x, i) - \alpha E\left[\int_0^t e^{-\alpha s} v(x(s), z(s)) \, ds \mid x(0) = x, z(0) = i\right]
+ E\left[\int_0^t e^{-\alpha s} \frac{\partial v}{\partial x}(x(s), z(s)) \, dz(s) \mid x(0) = x, z(0) = i\right]
+ E\left[\int_0^t e^{-\alpha s} Av(x(s), z(s)) \, ds \mid x(0) = x, z(0) = i\right]
\]
We notice that the minimum of
\[
\min_{0 \leq P \leq k} p \frac{\partial v}{\partial x} = F\left( \frac{\partial v}{\partial x} \right)
\]
is attained by \( p^*(x, i) \). By (1), we have
\[
\lambda \leq \frac{\partial v}{\partial x}(x, i)(p - i) + Av(x, i) + h(x),
\]
and the equality holds for \( p = p^*(x, i) \). Clearly, by (3)
\[
|x(t)| \leq C(t + 1) \quad \text{for all } \ p(\cdot) \in \mathcal{P}(x, i).
\]

By Lemma 3.2
\[
E[e^{-\alpha t}|v(x(t), z(t))| \mid x(0) = x, z(0) = i] \leq CE[e^{-\alpha t}(1 + |x(t)|^{\kappa+1}) | x(0) = x, z(0) = i]
\]
\[
\leq Ce^{-\alpha t}(1 + (t + 1)^{\kappa+1}) \rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]
Hence, substituting (30) into (29), we get
\[
\frac{\lambda}{\alpha} \leq -v(x, i) + \alpha E[\int_0^\infty e^{-\alpha s}v(x(s), z(s))ds \mid x(0) = x, z(0) = i]
\]
\[
+ E[\int_0^\infty e^{-\alpha s}h(x(s))ds \mid x(0) = x, z(0) = i].
\]
We note that by Lemma 3.2 and 3.3
\[
\alpha^2 E[\int_0^\infty e^{-\alpha s}|v(x(s), z(s))|ds \mid x(0) = x, z(0) = i] 
\]
\[
\leq \alpha^2 C \int_0^\infty e^{-\alpha s}(1 + |x^*(s)|^{\kappa+1})ds \rightarrow 0 \quad \text{as } \alpha \rightarrow 0.
\]
Thus we deduce
\[
\lambda \leq \inf_{p(\cdot) \in \mathcal{P}(x, i)} \limsup_{\alpha \rightarrow 0} \alpha E[\int_0^\infty e^{-\alpha s}h(x(s))ds \mid x(0) = x, z(0) = i],
\]
and the equality holds for \( p(t) = p^*(x^*(t), z(t)) \) of (27).

4 An application to ergodic control

We shall study the ergodic control problem to minimize the cost:
\[
J(p(\cdot) : x, i) = \limsup_{T \rightarrow \infty} \frac{1}{T} E[\int_0^T h(x(t))dt \mid x(0) = x, z(0) = i]
\]
over all $p(\cdot) \in U$ subject to

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \; z(0) = i, \quad \text{P-a.s.,}$$

where $U$ is the set of all nonnegative progressively measurable processes $p(t)$ such that

- $p(t)$ is adapted to $\mathcal{F}_t$,
- $0 \leq p(t) \leq k$,
- $\sup_t E[|x(t)|^{\kappa+1} | x(0) = x, z(0) = i] < \infty$ for $\kappa$ in (8).

**Theorem 4.1** We assume (7), (8), (9) and (22). Then the optimal control $p^*(t)$ is given by

$$p^*(t) = p^*(x^*(t), z(t)),$$

and the value by

$$J(p^*(\cdot) : x, i) = \lambda,$$

where $p^*(x^*(t), z(t))$ is as in (27).

**Proof.** From the same formula as (29) it follows that

$$E[v(x(T), z(T)) | x(0) = x, z(0) = i]$$

$$= v(x, i) + E[\int_0^T \frac{\partial v}{\partial x}(x(s), z(s))dx(s) | x(0) = x, z(0) = i]$$

$$+ E[\int_0^T Av(x(s), z(s))ds | x(0) = x, z(0) = i].$$

We recall (30) to obtain

$$E[v(x(T), z(T)) | x(0) = x, z(0) = i]$$

$$\geq v(x, i) + E[\int_0^T (\lambda - h(x(s)))ds | x(0) = x, z(0) = i],$$

where the equality holds for $x = x^*$ and $p = p^*$ of (27). By Lemma 3.2 and the definition of $U$

$$\frac{1}{T} E[|v(x(T), z(T))| | x(0) = x, z(0) = i]$$

$$\leq \frac{C}{T} E[1 + |x(T)|^{\kappa+1} | x(0) = x, z(0) = i] \to 0 \quad \text{as} \quad T \to \infty.$$

Also, by Lemma 3.3, $p^*(t)$ belongs to $U$. Thus we deduce

$$J(p(\cdot) : x, i) = \lim_{T \to \infty} \sup_{T} E[\int_0^T h(x(s))ds | x(0) = x, z(0) = i]$$

$$\geq \lambda = J(p^*(\cdot) : x, i).$$

The proof is complete.
5 An Example

In this section we present the example of a solution to the Bellman equation:

$$
\lambda = F(\frac{\partial v}{\partial x}(x,i)) - i \frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \quad x \in \mathbb{R}^1, \ i \in \mathbb{Z},
$$

(31)

in the case that

$$
\begin{align*}
    h(x) &= x^2, k = 3, \\
    Z &= \{1, 2\}, q_{12} = q_{21} = 1.
\end{align*}
$$

(32, 33)

Figure Solution $v(x,i), i=1,2$, to the Bellman Equation (31)
We remark that the matrix induced by $A$ is given by

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

and the equilibrium distribution $\pi$ is

$$\pi = \left( \frac{1}{2}, \frac{1}{2} \right).$$

Therefore the assumptions of Theorem 4.1 are fulfilled.

Now, recalling the form of optimal control $p^*$ and solving the Bellman equation (31) with (32) and (33), we have

$$\lambda = 0,$$

$$v(x, 1) = \begin{cases} \frac{1}{81} (18x^3 + 18x^2 - 24x + 16 - 16e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81} (18x^3 + 9x^2 + 12x + 8 - 8e^{\frac{3}{2}x}) & \text{if } x < 0, \end{cases}$$

$$v(x, 2) = \begin{cases} \frac{1}{81} (18x^3 - 9x^2 + 12x - 8 + 8e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81} (18x^3 - 18x^2 - 24x - 16 + 16e^{\frac{3}{2}x}) & \text{if } x < 0. \end{cases}$$

Then the optimal control $p^*$ is given by

$$p^*(x, i) = \begin{cases} 0 & \text{if } x > 0 \\ i & \text{if } x = 0 \\ 3 & \text{if } x < 0. \end{cases}$$

The solution $v(x, i)$ with (23) - (24), $i = 1, 2$ can be shown in Figure.

References


