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NVNF-sequentiality of Left-linear Term Rewriting Systems

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Abstract

This paper introduces NVNF-sequentiality which is an extension of NV-sequentiality defined by Oyamaguchi. It is shown that the class of NVNF-sequential systems properly includes the class of NV-sequential systems, and the indices with respect to NVNF-sequentiality are computed for a given term when a term rewriting system is NVNF-sequential. NVNF-sequentiality is decidable, and the index reduction is normalizing strategy for NVNF-sequential orthogonal term rewriting systems.

1 Introduction

Term rewriting systems can be regarded as a model for computation in which terms are reduced using a set of directed equations, called rewrite rules. Term rewriting systems play an important role in various fields of computer science such as abstract data type specifications, implementations of functional programming languages and automated deduction.

In a non-terminating term rewriting system, there are possibilities that a term having normal forms has infinite reduction sequences starting with it. We require some strategies telling us which redex should be contracted in order to get the desired result. Therefore, it is important to have a normalizing strategy which guarantees to find the normal form of terms whenever their normal forms exist. Huet and Lévy [3] showed that the needed reduction strategy is normalizing for every orthogonal (i.e., left-linear and non-overlapping) term rewriting system. The needed reduction strategy always rewrites one of needed redexes which have to be rewritten in order to reach a normal form. Unfortunately, it is undecidable in general whether a redex is needed or not. However, they showed that for strong sequential orthogonal term rewriting systems, at least one of the needed redexes in a term not in normal form can be efficiently computed. The work of Huet and Lévy was extended to several kinds of systems. Toyama [9] extended the notion of strong sequentiality to left-linear term rewriting systems. Decidability of strong sequentiality was showed for left-linear systems by Jouannaud and Sadfi [4]. Oyamaguchi [7] introduced NV-sequentiality which is also decidable.

In this paper, we introduce an extension of NV-sequentiality. This sequentiality is called NVNF-sequentiality [6]. Like NV-sequentiality, NVNF-sequentiality is based on the analysis of left-hand sides and the non-variable parts of the right-hand side of rewrite rules. However, the reachability to the normal form is considered in NVNF-sequentiality. We first show that the class of NVNF-sequential systems properly includes the class of NV-sequential systems. Next we prove that, for a given term t, it is decidable whether an occurrence u in t is an index with respect to NVNF-sequentiality. NVNF-sequentiality is decidable, and the index reduction is normalizing strategy for NVNF-sequential orthogonal term rewriting systems.

2 Definition

We mainly follow the notation of [3, 5]. Let $\mathcal{F}$ be a finite set of function symbols and let $\mathcal{V}$ be a countably infinite set of variables where $\mathcal{F} \cap \mathcal{V} = \phi$. The set of all terms built from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is sometimes denoted by $\mathcal{T}$. Terms not containing variables are called ground terms. Identity of terms is denoted by $\equiv$.

The set of occurrences in a term t is denoted by $O(t)$. t/u is the subterm of t at u. $t[u \leftarrow s]$ is the term obtained by replacing t/u with s in t. If s is a subterm of t then we write $s \subseteq t$. Occurrences are partially ordered by the prefix ordering $\leq$, i.e. $u \leq v$ if there exists w such that $u.w = v$. In this case we define v/u as w. If $u \not\leq v$ and $v \not\leq u$ then we say that u and v are disjoint, and write $u \perp v$. If $u_1, \ldots, u_n$ are pairwise disjoint, we use $t\,[\,u_i \leftarrow s_i \mid 1 \leq i \leq n\,]$ to denote $t\,[u_1 \leftarrow s_1] \cdots [u_n \leftarrow s_n]$.

A substitution θ is a mapping from $\mathcal{V}$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Substitutions are extended into homomorphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We write $t\theta$ instead of $\theta(t)$.

A term rewriting system is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a set $\mathcal{F}$ of function symbols and a finite set $\mathcal{R}$ of rewrite rules. A rewrite rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \rightarrow r$ for $\langle l, r \rangle$. An instance of the left-hand side of a rewrite rule is a redex. The rewrite rules of a term rewriting system $(\mathcal{F}, \mathcal{R})$ define a reduction relation $\rightarrow_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $t \rightarrow_{\mathcal{R}} s$ if there exists a rewrite rule $l \rightarrow r \in \mathcal{R}$, a occurrence $u \in O(t)$ and a substitution θ such that $t/u \equiv l\theta$ and $s \equiv t[u \leftarrow r\theta]$. When we want to specify the redex occurrence u of t in this reduction, we write $t \xrightarrow{u} s$. The transitive-reflexive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\rightarrow^*_{\mathcal{R}}$. $\rightarrow^+_{\mathcal{R}}$ is the transitive closure of $\rightarrow_{\mathcal{R}}$ and $\rightarrow^{\equiv}_{\mathcal{R}}$ is the reflexive closure of $\rightarrow_{\mathcal{R}}$. A normal form is a term without redexes. We say t has a normal form if $t \rightarrow^*_{\mathcal{R}} s$ for some normal form s. $\mathrm{NF}_{\mathcal{R}}$ is the set of normal forms of a term rewriting system $\mathcal{R}$. When no confusion can arise, we omit the subscript $\mathcal{R}$. A term rewriting system $\mathcal{R}$ is left-linear if for any $l \rightarrow r \in \mathcal{R}$, every variable in l occurs only once. $\mathcal{R}$ is non-overlapping if for any $l \rightarrow r$, $l' \rightarrow r' \in \mathcal{R}$ and $u \in O(l)$ such that $l/u \notin \mathcal{V}$, there are no substitutions θ, θ' such that $(l/u)\theta \equiv l'\theta'$, except in the case where $l \rightarrow r$, $l' \rightarrow r'$ are the same rewrite rule and $u = \varepsilon$. $\mathcal{R}$ is orthogonal if $\mathcal{R}$ is left-linear and non-overlapping. In this paper we restrict ourselves to the class of left-linear term rewriting systems.

3 NVNF-sequentiality

In this section we will explain NVNF-sequentiality. In order to define this concept, we need some preliminaries.

Let Ω be a new constant symbol representing an unknown part of a term. The set $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ is abbreviated to $\mathcal{T}_\Omega$. Elements of $\mathcal{T}_\Omega$ are called Ω-terms. An Ω-normal form is an Ω-term without redexes, and the set of all Ω-normal forms is denoted by NF_Ω. Only terms containing neither redexs nor Ω's are called normal forms. t_Ω denotes the Ω-term obtained from t by replacing all variables in t by Ω, and t_x denotes the term obtained from t by replacing all Ω by x. $O_\Omega(t)$ denotes the set of Ω-occurrences of t, i.e., $O_\Omega(t) = \{u \in O(t) \mid t/u \equiv \Omega\}$. The prefix ordering $\leq$ on $\mathcal{T}_\Omega$ is defined as follows:

(i) $\Omega \leq t$ for all $t \in \mathcal{T}_\Omega$,

(ii) $f(s_1, \ldots, s_n) \leq f(t_1, \ldots, t_n)$ if $s_i \leq t_i$ $(1 \leq i \leq n)$.

Two Ω-terms t and s are compatible, written $t \uparrow s$, if there exists an Ω-term r such that $t \leq r$ and $s \leq r$. In this case the least upper bound of t and s is denoted by $t \sqcup s$.

Definition 3.1 ([3]) *Let P be a predicate on $\mathcal{T}_\Omega$. An Ω-occurrence u of t is an index with respect to P if for all Ω-term s, $s \geq t$ and $P(s)$ imply $s/u \not\equiv \Omega$.*

The set of indices of t with respect to P is denoted by $I_P(t)$.

Definition 3.2 ([7])

(1) *The reduction relation $\rightarrow_{nv}$ on $\mathcal{T}_\Omega$ is defined as follows: $t \rightarrow_{nv} s$ iff there exists $l \rightarrow r \in \mathcal{R}$, $u \in O(t)$ such that $t/u \geq l_\Omega$ and $s \equiv t[u \leftarrow s']$ for some $s' \geq r_\Omega$.*

(2) *The predicate $nvnf$ on $\mathcal{T}_\Omega$ is defined as follows: $nvnf(t)$ holds iff $t \rightarrow^*_{nv} s$ for some s in normal form.*

Definition 3.3 *A left-linear term rewriting system is NVNF-sequential if every Ω-normal form containing at least one occurrence of Ω has an index with respect to nvnf.*

Oyamaguchi [7] introduced NV-sequentiality, by using the predicate *term*: $term(t)$ holds iff $t \rightarrow^*_{nv} s$ for some $s \in \mathcal{T}$. Note that s may be not in normal form.

Example 3.4 Let

$$\mathcal{R} = \begin{cases} f(a,b,x) \rightarrow a \\ f(b,x,a) \rightarrow b \\ f(x,a,b) \rightarrow c \\ c \rightarrow c. \end{cases}$$

Consider the Ω-term $t \equiv f(\Omega,\Omega,\Omega)$. $I_{nvnf}(t) = \{1\}$ because there does not exist an Ω-term s such that $s \geq t$, $s/1 \equiv \Omega$ and $s \rightarrow^*_{nv} s'$ for some normal form s'. Note that $\mathcal{R}$ is not NV-sequential since $f(\Omega,\Omega,\Omega)$ has no indices with respect to the predicate *term*.

We now show that $\mathcal{R}$ in Example 3.4 is NVNF-sequential system. For this purpose we need the following lemma.

Lemma 3.5 *Let $t \in \mathcal{T}_\Omega$. If $u \in I_{nvnf}(t)$, $t \leq s$ and $s/u \equiv \Omega$ then $u \in I_{nvnf}(s)$.*

Proof. If $u \notin I_{nvnf}(s)$ then there exists $s' \geq s$ such that $s'/u \equiv \Omega$ and $nvnf(s')$ is *true*. Since $s' \geq t$, $u \notin I_{nvnf}(t)$. □

Lemma 3.6 *$\mathcal{R}$ of Example 3.4 is NVNF-sequential system.*

Proof. We first prove the following claim: If $u \in I_{nvnf}(t)$ and $v \in I_{nvnf}(s)$ then $u.v \in I_{nvnf}(t[u \leftarrow s])$.

Proof of the claim. Suppose $u.v \notin I_{nvnf}(t[u \leftarrow s])$. Then there exists $t' \geq t[u \leftarrow s]$ such that $t'/u.v \equiv \Omega$ and $nvnf(t')$ is *true*. Hence there exists a reduction

$$t' \equiv t_0 \xrightarrow{u_0}_{nv} t_1 \xrightarrow{u_1}_{nv} \cdots\cdots \xrightarrow{u_{n-1}}_{nv} t_n \in \mathrm{NF}.$$

We distinguish two case:

(1) $u_i \not< u$ for all i $(0 \leq i \leq n-1)$. Because there exists no u_i such that $u_i < u$, we have $t'/u \rightarrow^*_{nv} t_n/u \in \mathrm{NF}$. Hence $nvnf(t'/u)$ is *true*. Clearly $t'/u \geq s$ and $(t'/u)/v \equiv \Omega$. This contradicts the assumption $v \in I_{nvnf}(s)$.

(2) $u_i < u$ for some i. Let j be the smallest number satisfying $u_j < v$. Note that $t'/u \rightarrow^*_{nv} t_j/u$. t_j/u_j is a redex but $t_j/u_j \not\equiv c$. Moreover $t_j \not\equiv a$ and $t_j \not\equiv b$ because $t'/u \geq s$, $t'/u.v \equiv \Omega$ and $v \in I_{nvnf}(s)$. Thus $t_j[u \leftarrow \Omega] \xrightarrow{u_j}_{nv} t_{j+1}$. We can obtain the following reduction: $t'[u \leftarrow \Omega] \rightarrow^*_{nv} t_j[u \leftarrow \Omega] \rightarrow_{nv} t_{j+1} \rightarrow^*_{nv} t_n$. Hence $t'[u \leftarrow \Omega] \geq t$, $nvnf(t'[u \leftarrow \Omega])$ is *true*. But this is contradictory to $u \in I_{nvnf}(t)$.

Therefore the claim follows.

Let t be an Ω-normal form containing at least one occurrence of Ω. We prove, by induction on the size of t, that t has an index with respect to $nvnf$. When $t \equiv \Omega$, it is clear that t has an index. Induction step:

(1) $t \equiv f(t_1, t_2, t_3)$. If t_1 contains Ω then by induction hypothesis, t_1 has an index. By $1 \in I_{nvnf}(f(\Omega, \Omega, \Omega))$ and Lemma 3.5, $1 \in I_{nvnf}(f(\Omega, t_2, t_3))$. Therefore, from the claim it follows that t has an index. Otherwise we distinguish three cases:

  (1-1) $t_1 \equiv a$. If t_2 contains Ω then by induction hypothesis, t_2 has an index. By $2 \in I_{nvnf}(f(a, \Omega, \Omega))$ and Lemma 3.5, $2 \in I_{nvnf}(f(a, \Omega, t_3))$. From the claim, t has an index. Otherwise t_3 contains Ω. By induction hypothesis, t_3 has an index. We can obtain $3 \in I_{nvnf}(f(a, t_2, \Omega))$. Therefor by the claim, t has an index.

  (1-2) $t_1 \equiv b$. Similar to (1-1).

  (1-3) Otherwise we have $I_{nvnv}(f(t_1, \Omega, \Omega)) = \{2, 3\}$. By induction hypothesis, t_2 or t_3 has an index. By Lemma 3.5 and the claim, t has an index.

(2) $t \equiv g(t_1, \cdots, t_n)$ $(g \neq f)$. Suppose t_i contains Ω. Then by induction hypothesis, t_i has an index. Because $i \in I_{nvnf}(g(t_1, \cdots, t_{i-1}, \Omega, t_{i+1}, \cdots, t_n))$, t has an index from the claim. □

By Example 3.4 and Lemma 3.6 we have the following theorem.

Theorem 3.7 *The class of NVNF-sequential term rewriting systems properly includes the class of NV-sequential systems.*

Proof. NV-sequential system is NVNF-sequential because an index with respect to *term* is also an index with respect to *nvnf*. $\mathcal{R}$ of Example 3.4 is NVNF-sequential but not NV-sequential. Therefore the inclusion is proper. □

$t \xrightarrow{u} s$ is the index reduction if $u \in I_{nvnf}(t[u \leftarrow \Omega])$. Let $\mathcal{R}$ be NVNF-sequential orthogonal term rewriting system. Then we can apply the index reduction to a term which is not a normal form. It is clear that if u is a redex occurrence of t and $u \in I_{nvnf}(t[u \leftarrow \Omega])$ then t/u is a needed redex. Huet and Lévy [3] proved that repeated contraction of needed redexes leads to the normal form, if it exists. Thus we have the following theorem.

Theorem 3.8 *The index reduction is normalizing strategy for NVNF-sequential orthogonal systems.*

The decidability of NVNF-sequentiality was proven by Comon.

Theorem 3.9 ([1]) *NVNF-sequentiality of left-linear term rewriting systems (which may have overlapping rules) is decidable.*

4 Indices with respect to NVNF-sequentiality

In this section we show that for a given $t \in \mathcal{T}_\Omega$, it is decidable whether $u \in O_\Omega(t)$ is an index with respect to $nvnf$ in t. We introduce the reduction $\rightarrow_\omega$ which is given in [7].

Definition 4.1 ([7]) *The reduction relation $\rightarrow_\omega$ is defined as follows: $t \rightarrow_\omega s$ iff there exists $l \rightarrow r \in \mathcal{R}$, $u \in O(t)$ such that $t/u \uparrow l_\Omega$, $t/u \not\equiv \Omega$ and $s \equiv t[u \leftarrow r_\Omega]$.*

We explain a relationship between this reduction $\rightarrow_\omega$ and $\rightarrow_{nv}$, and show the condition for ensuring that Ω-occurrence in a term is an index with respect to $nvnf$.

Lemma 4.2

(1) *If $t \rightarrow^*_{nv} s$ and $t' \leq t$ then $t' \rightarrow^*_\omega s'$ for some $s' \leq s$.*

(2) *If $t \rightarrow^*_\omega s$ then $t' \rightarrow^*_{nv} s$ for some $t' \geq t$.*

Proof.

(1) We prove that if $t \rightarrow_{nv} s$ and $t' \leq t$ then $t' \rightarrow^{\equiv}_\omega s'$ for some $s' \leq s$. If $t \rightarrow_{nv} s$ then there exist $l \rightarrow r \in \mathcal{R}$, $u \in O(t)$ such that $t/u \geq l_\Omega$ and $s \equiv t[u \leftarrow s_1]$ for some $s_1 \geq r_\Omega$. If $u \in O(t')$, $t'/u \not\equiv \Omega$ then $t'/u \uparrow l_\Omega$. Hence $t' \rightarrow_\omega t'[u \leftarrow r_\Omega]$ and $t'[u \leftarrow r_\Omega] \leq s$. Otherwise it is clear that $t' \leq s$. Using this fact, we can prove (1) by induction on the length of $t \rightarrow^*_{nv} s$.

(2) This is proved by induction on the length of $t \rightarrow^*_\omega s$. The case of zero is trivial. Assume that $t \rightarrow_\omega s_1 \rightarrow^*_\omega s$ where $t/u \uparrow l_\Omega$, $t/u \not\equiv \Omega$ and $s_1 \equiv t[u \leftarrow r_\Omega]$ for some $l \rightarrow r \in \mathcal{R}$ and $u \in O(t)$. From induction hypothesis, for some $s_2 \geq s_1$, $s_2 \rightarrow^*_{nv} s$. Let $t_1 \equiv t/u \sqcup l_\Omega$ and $t' \equiv s_2[u \leftarrow t_1]$. Because $s_2 \geq s_1 \equiv t[u \leftarrow r_\Omega]$, $t' \equiv s_2[u \leftarrow t_1] \geq t$. We have $t' \rightarrow_{nv} s_2$ by $s_2/u \geq r_\Omega$. Therefore $t' \rightarrow^*_{nv} s$. □

Lemma 4.3 *Let $t \in \mathcal{T}_\Omega$ and $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. Then $u \notin I_{nvnf}(t)$ iff there exists $s \in \mathrm{NF}_\Omega$ such that $t[u \leftarrow \bullet] \rightarrow^*_\omega s$ and $\bullet \not\subseteq s$.*

Proof. *only-if part.* If $u \notin I_{nvnf}(t)$ then there exists $t' \geq t$ such that $t'/u \equiv \Omega$ and $nvnf(t')$ is *true*. Thus $t' \rightarrow^*_{nv} s'$ for some normal from s'. From $\Omega \not\subseteq s'$ and left-linearity, we can obtain $t'[u \leftarrow \bullet] \rightarrow^*_{nv} s'$ and $\bullet \not\subseteq s'$. Using Lemma 4.2 (1), we obtain $s \leq s'$ such that $t[u \leftarrow \bullet] \rightarrow^*_\omega s$. Because s' is a normal form, s is an Ω-normal form and $\bullet \not\subseteq s$.

if part. If $t[u \leftarrow \bullet] \rightarrow^*_\omega s \in \mathrm{NF}_\Omega$ and $\bullet \not\subseteq s$, then there exists $t' \geq t[u \leftarrow \bullet]$ such that $t' \rightarrow^*_{nv} s$ by Lemma 4.2 (2). Let $t'' \equiv t'_x[u \leftarrow \Omega]$, $s' \equiv s_x$. We can transform the reduction $t' \rightarrow^*_{nv} s$ into $t'' \rightarrow^*_{nv} s'$. Because s is an Ω-normal from, s' is a normal form and hence $nvnf(t'')$ is *true*. Clearly $t'' \geq t$ and $t''/u \equiv \Omega$. Therefore $u \notin I_{nvnf}(t)$. □

We next show that for any $t \in \mathcal{T}_\Omega$, there exists an upper bound of the least hight of Ω-normal form obtained from t by $\rightarrow_\omega$ when it exists.

For given a term rewriting system $\mathcal{R}$, $Rh_\mathcal{R}$ is defined with $Rh_\mathcal{R} = \{r_\Omega \mid l \rightarrow r \in \mathcal{R}\}$, and $Rhn_\mathcal{R}$ is the smallest set such that $Rh_\mathcal{R} \subseteq Rhn_\mathcal{R}$ and if $t \in Rhn_\mathcal{R}$, $u \in O(t)$ and $r \in Rh_\mathcal{R}$ then $t[u \leftarrow r] \in Rhn_\mathcal{R}$. It is clear that if $r \in Rh_\mathcal{R}$ and $r \rightarrow^*_\omega t$ then $t \in Rhn_\mathcal{R}$. In the sequel we often omit the subscript $\mathcal{R}$.

Lemma 4.4 *If $t \rightarrow^+_\omega s$ then there exist $u_1, \ldots, u_n \in O(t)$ which are pairwise disjoint, and the following conditions hold.*

(i) $s \equiv t[\, u_i \leftarrow s/u_i \mid 1 \leq i \leq n\,]$,

(ii) $t/u_i \rightarrow^+_\omega r_i \rightarrow^*_\omega s/u_i$ *for some $r_i \in Rh_\mathcal{R}$, $1 \leq i \leq n$.*

Proof. By $t \to_\omega^+ s$, there exists a reduction

$$t \equiv t_0 \xrightarrow{u_0}_\omega t_1 \xrightarrow{u_1}_\omega \cdots\cdots \xrightarrow{u_{n-1}}_\omega t_n \equiv s \quad (n > 0)\,.$$

Let $\{u_{i_1}, \ldots, u_{i_k}\}$ be the set of minimal redex occurrences of $\{u_0, \cdots, u_{n-1}\}$. Then $u_{i_1}, \ldots, u_{i_k} \in O(t)$ are pairwise disjoint. By minimality of $u_{i_1}, \ldots, u_{i_k}$, (i), (ii) hold. □

We use $|u|$ for the length of a word u. The height $|t|$ of an Ω-term t is defined by $|t| = \max\{\,|u| \mid u \in O(t)\,\}$. The maximum hight of the left-hand sides and right-hand sides of a given $\mathcal{R}$ is denoted by $\rho_{\mathcal{R}}$. We write ρ when confusion does not occur. $(t)_\rho$ is a prefix term of t whose hight is ρ, i.e., $(t)_\rho \equiv t\,[u \leftarrow \Omega \mid u \in O(t) \wedge |u| = \rho]$.

Lemma 4.5 *Let $r \in Rh_{\mathcal{R}}$, $r \to_\omega^* s$ where $|s| > \rho \times n$ $(n > 0)$. Then there exists $\varepsilon < u_0 < \cdots < u_{n-1} \in O(s)$ and for any i $(0 \le i \le n-1)$, the following condition holds: $r \to_\omega^* s[u_i \leftarrow r_i]$ and $r_i \to_\omega^* s/u_i$ for some $r_i \in Rh_{\mathcal{R}}$.*

Proof. The proof is by induction on n. By $r \to_\omega^* s$, there exists a reduction

$$r \equiv t_0 \xrightarrow{v_0}_\omega t_1 \xrightarrow{v_1}_\omega \cdots\cdots \xrightarrow{v_{m-1}}_\omega t_m \equiv s.$$

Because $|s| > \rho \times n$ $(n > 0)$, we can obtain $j < m$ such that $t_j \in Rh_{\mathcal{R}}$ and $v_i \neq \varepsilon$ for all i $(j \le i \le m-1)$. When $n = 1$, using Lemma 4.4, we can easily show that there exists $u \in O(s)$ such that $u \neq \varepsilon$ and $t_j \to_\omega^* s[u \leftarrow r']$, $r' \to_\omega^* s/u$ for some $r' \in Rh_{\mathcal{R}}$. By $r \to^* t_j$, we have $r \to_\omega^* s[u \leftarrow r']$. Suppose $n > 1$. Using Lemma 4.4, we can obtain $u \neq \varepsilon$ such that $t_j \to_\omega^* s[u \leftarrow r']$, $r' \to_\omega^* s/u$ and $|s/u| > \rho \times (n-1)$ for some $r' \in Rh_{\mathcal{R}}$. By induction hypothesis, there exist $\varepsilon < u'_1 < \cdots < u'_{n-1} \in O(s)$, and for any i $(1 \le i \le n-1)$ the following conditions hold: $r' \to_\omega^* (s/u)[u'_i \leftarrow r_i] \equiv s[u.u'_i \leftarrow r_i]/u$ and $r_i \to_\omega^* (s/u)/u'_i \equiv s/u.u'_i$ for some $r_i \in Rh_{\mathcal{R}}$. Let $u_0 = u$, $u_i = u.u'_i$ $(1 \le i \le n-1)$. Clearly $\varepsilon < u_0 < \cdots < u_{n-1} \in O(s)$ and $r \to_\omega^* s[u \leftarrow r']$, $r' \to_\omega^* s/u$. For all i $(1 \le i \le n-1)$, we have $t_j \to_\omega^* s[u \leftarrow r'] \to_\omega^* s[u \leftarrow s[u.u'_i \leftarrow r_i]/u] \equiv s[u_i \leftarrow r_i]$ and $r_i \to_\omega^* s/u_i$. By $r \to_\omega^* t_j$, we have $r \to_\omega^* s[u_i \leftarrow r_i]$. Therefore the lemma holds. □

Lemma 4.6 *Let $t, s \in \mathrm{NF}_\Omega$. If $(t/u)_\rho \equiv (s)_\rho$ then $t\,[u \leftarrow s] \in \mathrm{NF}_\Omega$.*

For given a term rewriting system $\mathcal{R}$, let τ , σ, and $k_{\mathcal{R}}$ be constants defined as follows: $\tau = \|\,\{\,(t)_\rho \mid t \in Rhn_{\mathcal{R}}\,\}\,\|$, $\sigma = \|\,\mathcal{R}\,\|$ and $k_{\mathcal{R}} = \rho_{\mathcal{R}} \times (\tau \times \sigma + 1)$, where $\|\,A\,\|$ is the cardinality of a set A.

Lemma 4.7 *Let $t \in \mathcal{T}_\Omega$ and $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. Then $u \notin I_{nvnf}(t)$ iff there exists $s \in \mathrm{NF}_\Omega$ such that $|s| \le |t| + k_{\mathcal{R}}$, $t[u \leftarrow \bullet] \to_\omega^* s$ and $\bullet \not\subseteq s$.*

Proof. *if part.* By Lemma 4.3.

only-if part. If $u \notin I_{nvnf}(t)$ then by Lemma 4.3 there exists $s \in \mathrm{NF}_\Omega$ such that $t[u \leftarrow \bullet] \to_\omega^* s$ and $\bullet \not\subseteq s$. Let s be an Ω-normal form with the least size satisfying this condition. Suppose $|s| > |t| + k_{\mathcal{R}}$. By Lemma 4.5 and the definition of $k_{\mathcal{R}}$, we can show that there exists $r \in Rh$, $u_1, u_2 \in O(s)$ $(u_1 < u_2)$ such that $(s/u_1)_\rho \equiv (s/u_2)_\rho$, $t\,[u \leftarrow \bullet] \to_\omega^* s[u_i \leftarrow r]$ and $r \to_\omega^* s/u_i$ for $i = 1, 2$. Let $s' \equiv s[u_1 \leftarrow s/u_2]$. By Lemma 4.6, $s' \in \mathrm{NF}_\Omega$ and $t[u \leftarrow \bullet] \to_\omega^* s'$, $\bullet \not\subseteq s'$. Because the size of s' is smaller than the size of s, we obtain a contradiction. □

By Lemma 4.7, in order to determine whether an Ω-occurrence is an index w.r.t. $nvnf$, we need to check the reachability to a finite number of Ω-normal form. For a term rewriting system $\mathcal{R}$, we define $\mathcal{R}^\Omega$ as follows [7]: $\mathcal{R}^\Omega = \{\, l \to r_\Omega \mid l \to r \in \mathcal{R}\,\} \cup \{\,\Omega \to t \mid t \subseteq l_\Omega,\ l \to r \in \mathcal{R}\,\}$. We can prove that in the condition of Lemma 4.7, $\to_\omega^*$ can be replaced with $\to_{\mathcal{R}^\Omega}^*$.

Lemma 4.8 ([7])

(1) *If* $t \to_{\omega}^{*} s$ *then* $t \to_{\mathcal{R}^{\Omega}}^{*} s$.

(2) *If* $t \to_{\mathcal{R}^{\Omega}}^{*} s$ *and* $t' \le t$ *then* $t' \to_{\omega}^{*} s'$ *for some* $s' \le s$.

Lemma 4.9 *Let* $t \in \mathcal{T}_{\Omega}$ *and* $u \in O_{\Omega}(t)$. *Then* $u \notin I_{nvnf}(t)$ *iff there exists* $s \in \mathrm{NF}_{\Omega}$ *such that* $|s| \le |t| + k_{\mathcal{R}}$, $t[u \leftarrow \bullet] \to_{\mathcal{R}^{\Omega}}^{*} s$ *and* $\bullet \not\subseteq s$.

Proof. By Lemma 4.7 and Lemma 4.8. □

We assume that $\mathcal{R}$ is left-linear, so $\mathcal{R}^{\Omega}$ is left-linear and right-ground (i.e. r is ground term for any $l \to r \in \mathcal{R}^{\Omega}$). It is show that the reachability problem is decidable for left-linear and right-ground systems [2, 8]. Thus we obtain the following theorem.

Theorem 4.10 *It is decidable, for* $t \in \mathcal{T}_{\Omega}$, $u \in O_{\Omega}(t)$, *whether* u *is an index with respect to nvnf in* t.

References

[1] H. Comon, Sequentiality, second order monadic logic and tree automata, in *Theory of Rewriting Systems and Its Applications*, RIMS Kokyuroku, Vol. 918, Kyoto University, pp.118-138, 1995.

[2] M. Dauchet, S. Tison, T. Heuillard, and P. Lescanne, Decidability of The Confluence of Finite Ground Term Rewriting Systems and of Other Related Term Rewriting Systems, *Information and Computation* **88**, pp.187-201, 1990.

[3] G. Huet and J.-J. Lévy, Call by Need Computations in Non-Ambiguous Linear Term Rewriting Systems, Report 359, INRIA, 1979; Computations in Orthogonal Rewriting Systems, I and II, Computational Logic, Essays in Honor of Alan Robinson (eds. J.-L.Lassez, G.Plotkin), MIT Press, pp.396-443, 1991.

[4] J.-P. Jouannaud and W. Sadfi, Strong Sequentiality of Left-linear Overlapping Rewrite Systems, in *Proceedings, 4th International Workshop on Conditional Term Rewriting Systems, Jerusalem*, LNCS, vol. 968, pp.235-246, Springer-Verlag, 1995.

[5] J.W. Klop and A. Middeldorp, Sequentiality in Orthogonal Term Rewriting Systems, *Journal of Symbolic Computation* **12**, pp.161-195, 1991.

[6] T. Nagaya, M. Sakai and Y. Toyama, Decidability of Indices on NVNF-sequentiality Term Rewriting Systems, *IEICE Technical Report* **COMP94**-63, Vol.94, No.354, pp.87-94, 1994, *in Japanese.*

[7] M. Oyamaguchi, NV-Sequentiality: A Decidable Condition for Call-by-Need Computations in Term Rewriting Systems, *SIAM Journal on Computation* **22**(1), pp.112-135, 1993.

[8] M. Oyamaguchi, The Reachability Problem for Quasi-Ground Term Rewriting Systems, *Journal of Information Processing* **9**, pp.232-236, 1986.

[9] Y. Toyama, Strong Sequentiality of Left-Linear Overlapping Term Rewriting Systems, *Proceedings of the 7th IEEE Symposium on Logic in Computer Science*, pp.274-284, 1992.