Global structure of Brezis-Nirenberg type equations on the unit ball

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1 Introduction

We consider the existence and uniqueness of radial solutions of

\[
\begin{cases}
\Delta u + \lambda u + u^{(n+2)/(n-2)} = 0, & \text{in } B = \{x \in \mathbb{R}^n : |x| < 1\}, \\
u > 0, & \text{in } B, \\
\kappa \frac{\partial u}{\partial \nu} + u = 0, & \text{on } \partial B,
\end{cases}
\]

where \( \nu \) is the outward unit normal vector on \( \partial B \), \( \lambda < \lambda_*^2 \) (\( \lambda_*^2 \) is the first eigenvalue of \( -\Delta \) with 0-Dirichlet condition on \( B \)) and \( \kappa \geq 0 \).

In the case \( \kappa = 0 \), it is well-known that any solution of (1) is radially symmetric by Gidas-Ni-Nirenberg [5]. Moreover, Brezis-Nirenberg [2] proved that (1) with \( n = 3 \) has a solution if and only if \( \lambda \in (\pi^2/4, \pi^2) \) and that (1)
with $n \geq 4$ does if and only if $\lambda \in (0, \lambda_*)$. Later, Kwong-Li [4] and Zhang [15] proved that the solution obtained by [2] is unique.

Even though the 0-Dirichlet problem has no positive solution for the case $\lambda < 0$, the homogeneous Neumann problem has a positive one. There is also a nonradial solution which has a peak on the boundary at least if $\lambda$ is near $-\infty$ by Ni-Takagi [7] [9].

As for the third boundary conditions, X.-J. Wang [11] treated more general problems than (1) under the “least energy” condition. Recently, X.-B. Pan [10] treated the asymptotic behavior of solutions to (1) as $\lambda \to -\infty$ in analogous to [7].

Though there seems to be no results similar to Gidas-Ni-Nirenberg [5] for the third boundary conditions for small $|\lambda|$, we restrict our attention only to radial solutions.

We consider the initial value problem

$$
\begin{align*}
&u_{rr} + \frac{n-1}{r}u_r + \lambda u + u_+^{(n+2)/(n-2)} = 0, \quad 0 < r < 1, \\
&u(0) = \alpha, \quad u_r(0) = 0
\end{align*}
$$

and seek a suitable number $\alpha > 0$ satisfying $u(r) > 0$ on $(0, 1)$ and

$$
\kappa u_r(1) + u(1) = 0,
$$

where $u_+ = \max\{u, 0\}$. Note that (2) has a solution (denoted by $u(r; \lambda, \alpha)$) for any $\alpha > 0$ and $\lambda$.

The main purpose of this article is to make clear the range of $\lambda$ in which (1) has a unique solution and find out the relation between $\lambda$ and $\alpha$.

Hereafter we restrict ourselves to the case $n = 3$.

## 2 Results

To state our theorems, we introduce four numbers. Let $\lambda_\kappa \in (0, \pi]$ satisfy $\tan \lambda_\kappa = \kappa \lambda_\kappa/(\kappa - 1)$ if $\kappa \neq 1$ and $\lambda_\kappa = \pi/2$ if $\kappa = 1$. Define $\lambda_2$ by $\tan(\lambda_2 - \pi/2) = \kappa \lambda_2/(\kappa - 1)$ if $0 < \kappa \leq 1$. As we see in Theorem 2, $\lambda_2$ is a
blow-up point. Set \( \lambda_3 \tanh \lambda_3 = (\kappa - 1)/\kappa \) if \( \kappa \geq 1 \). For \( \kappa \in [0, 1) \), we define \( \lambda_4 > 0 \) by \( \tanh \lambda_4 = \kappa \lambda_4 \) if \( 0 < \kappa < 1 \) and \( \lambda_4 = \infty \) if \( \kappa = 0 \).

Note that \( \lambda_\kappa^2 \) is the first eigenvalue of \(-\Delta\) with the boundary condition \( \kappa \partial u/\partial \nu + u = 0 \) on \( \partial B \). Our methods are based on Yotsutani [12] and Yanagida-Yotsutani [13], [14]

**Theorem 1** Let \( n = 3 \).

Case (I): \( 0 \leq \kappa \leq 1 \). If \( \lambda_2^2 < \lambda < \lambda_\kappa^2 \), then (1) has a unique radial solution.

Case (II): \( 1 < \kappa \). If \( -\lambda_3^2 < \lambda < \lambda_\kappa^2 \), then (1) has a unique radial solution.

**Remark.** If \( -\lambda_4^2 \leq \lambda \leq \lambda_2^2 \), then (1) has no radial solution. Moreover, for such \( \lambda \) the inequality \( \kappa u_r(1; \lambda, \alpha) + u(1; \lambda, \alpha) > 0 \) holds for any \( \alpha > 0 \). For \( \lambda < -\lambda_4 \), there may be at least two solutions, while \( \lambda_4 \) may not be sharp.

By this theorem, there is a one-to-one mapping from \( \lambda \) to \( \alpha \), that is, \( \alpha \) is a function of \( \lambda \). So we can draw the graph of \( \alpha = \alpha(\lambda; \kappa) \).

Let

\[
C = \{ (\lambda, \alpha(\lambda)) | \alpha \text{ satisfies (2) - (3)} \},
\]

\[
D_C = \{ (\lambda, y) | y > \alpha(\lambda) \},
\]

\[
D_S = \{ (\lambda, y) | y < \alpha(\lambda) \}.
\]

Note that for \( (\lambda, \alpha) \in D_C, \kappa u_r(1; \lambda, \alpha) + u(1; \lambda, \alpha) < 0 \) or \( u(r; \lambda, \alpha) \) has a zero in \( (0, 1) \). Similarly, for \( (\lambda, \alpha) \in D_S, \kappa u_r(1; \lambda, \alpha) + u(1; \lambda, \alpha) > 0 \).

**Theorem 2** \( \alpha(\lambda) \) is a continuous function of \( \lambda \) satisfying \( \alpha(\lambda) \to 0 \) as \( \lambda \to \lambda_\kappa^2 - 0 \) and \( \alpha(\lambda) \to \infty \) as \( \lambda \to \lambda_2^2 + 0 \). More precisely, \( \alpha(\lambda) \) satisfies

\[
\lim_{\lambda \to \lambda_2^2 + 0} (\lambda - \lambda_2^2) \alpha(\lambda)^2 = \frac{2\sqrt{3}\pi \lambda_2^2 \{(1 - \kappa) \sin \lambda_2 + \kappa \lambda_2\}}{\sin \lambda_2}.
\]
Remark. As a matter of fact, the curve $C$ must be a $C^1$ curve. As we see from the standard bifurcation theory, $(\lambda_\kappa^2, 0)$ is a bifurcation point.

The blow-up rate of $\alpha(\lambda)$ as $\lambda \to \lambda_\kappa^2 + 0$ is known by Brezis-Peletier [3] for $\kappa = 0$. We show the graph of $\alpha(\lambda)$ in Section 5.

3 Reduction to a Matukuma-type equation

Our idea for the proof of Theorem 1 is to reduce (2)-(3) to an exterior Neumann problem of a Matukuma-type equation. For a solution $\varphi$ to

\[
\begin{aligned}
\varphi_{rr} + \frac{2}{r} \varphi_r + \lambda \varphi &= 0, \quad 0 < r < 1, \\
\varphi(0) &= 1, \quad \varphi_r(0) = 0,
\end{aligned}
\tag{4}
\]

let $u = v \varphi$. Note that

\[
\varphi = \begin{cases} 
\sin(\sqrt{\lambda}r) / \sqrt{\lambda r} & \text{if } \lambda > 0, \\
\sinh(\sqrt{-\lambda}r) / \sqrt{-\lambda r} & \text{if } \lambda < 0.
\end{cases}
\]

If $u$ is a solution to (2)-(3), then $v$ satisfies

\[
\begin{aligned}
v_{rr} + \left(\frac{2}{r} + 2 \frac{\varphi_r}{\varphi}\right) \varphi_r + \varphi^4 v^5 &= 0, \quad 0 < r < 1, \\
v(0) &= 1, \\
\kappa \varphi(1) / (\varphi(1) + \kappa \varphi_r(1)) v_r(1) + v(1) &= 0.
\end{aligned}
\tag{5}
\]
Next, let \( g(r) = r^2 \varphi^2 \). Then \( v \) satisfies

\[
\begin{cases}
\frac{1}{g^2} (g v_r)_r + \varphi^4 v^5 = 0, & 0 < r < 1, \\
v(0) = 1, \\
\frac{\kappa \varphi(1)}{\varphi(1) + \kappa \varphi_r(1)} v_r(1) + v(1) = 0.
\end{cases}
\] (6)

Finally, let

\[
h(r) = g(r) \left( \int_r^1 \frac{ds}{g(s)} + \frac{\kappa \varphi(1)}{g(1)(\varphi(1) + \kappa \varphi_r(1))} \right),
\]

\[
w(\tau) := \frac{g(r)}{h(r)} v(r),
\]

and

\[
\tau := \exp \left( \int_r^1 \frac{ds}{h(s)} \right).
\]

Then \( w(\tau) \) satisfies the exterior Neumann problem

\[
\begin{cases}
\frac{1}{\tau^2} (\tau^2 w_\tau)_\tau + K(\tau) v^5 = 0, & \tau > 1, \\
w_\tau(1) = 0, \\
\lim_{\tau \to \infty} \tau w(\tau) > 0,
\end{cases}
\] (7)

where

\[
K(\tau) := \frac{1}{\tau^2} \frac{h(r)^6}{g(r)^4} \varphi(r)^4.
\]

We can apply the modified version of [13] to obtain Theorem 1.

As for Theorem 2, we may follow the argument in [14]. To prove the blow-up rate, we use the argument as in [3].
4 Concluding remarks

So far, we do not pay much attention to the Neumann problem \((\kappa = \infty)\). However, there are many results on the radial solutions as well as non-radial ones (least-energy solutions). See for instance, Adimurthi-Yadava [1], Ni-Takagi [9] [10] or Ni-Pan-Takagi [7]. According to their results, nonconstant radial solutions bifurcate from the constant solution at \((-\mu_j/4, (\mu_j/4)^{1/4})\) where \(\mu_j\) the eigenvalues of \(-\Delta\) subject to the homogeneous Neumann problem \((0 = \mu_0 < \mu_1 < \mu_2 < \ldots)\). Moreover, the properties of the bifurcation branch are known by [1], [9] etc. In view of the graphs in the case where \(\lambda < 0\) and \(\kappa > 0\) is sufficiently large, our results seems to be a "homotopy bridge" connecting the Dirichlet problem and the Neumann one. We may regard the graph \((\kappa = 1000)\) as an imperfect bifurcation, though we do not have any rigorous proofs. See for instance, Chapter 3 of Golubitsky and Schaeffer [6].
5 Graphs

These computations are due to Mr. H. Morishita of Hyogo University.
References


