<table>
<thead>
<tr>
<th>Title</th>
<th>Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation (Variational Problems and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KADOYA, Atsushi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 951: 111-116</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60357">http://hdl.handle.net/2433/60357</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation

Atsushi KADOYA (Hiroshima Shudo University)

1. Formulation of an optimization problem

This paper is concerned with an optimization problem on the lateral boundary $\partial\Omega$ for a thermodynamical phase separation model formulated in a domain $\Omega$.

$\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N = 2$ or $3$) with smooth boundary $\partial\Omega$ and $T$ is a fixed positive number. Our state problem $SP(\Gamma)$ is of the form

$$
\frac{\partial u}{\partial t} + \lambda(w) - \Delta u = f \quad \text{in } Q := (0, T) \times \Omega,
$$

$$
\xi \in \beta(w) \quad \text{in } Q,
$$

$$
u = h_D \quad \text{on } \Sigma_D := (0, T) \times \Gamma,
$$

$$
\frac{\partial \nu}{\partial n} + n_0 u = h_N \quad \text{on } \Sigma_N := (0, T) \times \Gamma', \quad \Gamma' := \partial\Omega \setminus \Gamma,
$$

$$
\frac{\partial w}{\partial n} + \lambda'(w)u = \lambda'(w)^* \quad \text{in } \Sigma := (0, T) \times \partial\Omega,
$$

$$
u(0, \cdot) = u_0, w(0, \cdot) = w_0 \quad \text{in } \Omega.
$$

Throughout this paper, we use the following notation.

For a general (real) Banach space $Y$, we denote by $|\cdot|_Y$ the norm in $Y$ and by $Y^*$ the dual of $Y$. Also, for a positive finite number $T$, we denote by $C_w([0, T]; Y)$ the space of all weakly continuous functions $u : [0, T] \to Y$, and by definition "$u_n \rightharpoonup u$ in $C_w([0, T]; Y)$ as $n \to +\infty$" means that for each $z^* \in Y^*$, $(z^*, u_n(t))_{Y^*, Y}$ converges to $(z^*, u(t))_{Y^*, Y}$ uniformly in $t \in [0, T]$ as $n \to +\infty$, where $(\cdot, \cdot)_{Y^*, Y}$ is the duality pairing between $Y^*$ and $Y$.

For simplicity we put

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_0 := \{v \in H; \int_{\Omega} vzdx = 0\}, \quad V_0 := V \cap H_0,$

and

$$\Pi := \{\Gamma \subset \partial\Omega; \Gamma \text{ is compact in } \partial\Omega, \sigma(\Gamma) > 0\}.$$

For each $\Gamma \in \Pi$, we put

$$V(\Gamma) := \{z \in V; z = 0 \text{ a.e. on } \Gamma\}.$$
which is a closed subspace of $V$, and

\[ (v, w) := \int_{\Omega} vw \, dx \quad \text{for } v, w \in H, \]

\[ (v, w)_{\partial \Omega} := \int_{\partial \Omega} vw \, d\sigma \quad \text{for } v, w \in L^{2}(\partial \Omega), \]

\[ a(v, w) := \int_{\Omega} \nabla v \cdot \nabla w \, dx \quad \text{for } v, w \in V. \]

In general, given a subset $E$ of $\overline{\Omega}$, $\chi_{E}$ denotes the characteristic function of $E$ defined on $\overline{\Omega}$.

We now introduce a notion of convergence in $H$. By definition, a sequence $\{\Gamma_{n}\} \subset \Pi$ converges to $\Gamma \in \Pi$ as $n \to +\infty$, if the following conditions (C1) - (C3) are satisfied:

(C1) If $\{n_{k}\}$ is a subsequence of $\{n\}$, $z_{k} \in V(\Gamma_{n_{k}})$ and $z_{k} \rightharpoonup z$ weakly in $V$ as $k \to +\infty$, then $z \in V(\Gamma)$.

(C2) For any $z \in V(\Gamma)$, there is a sequence $\{z_{n}\} \subset V$ such that $z_{n} \in V(\Gamma_{n})$, $n = 1, 2, \ldots$, and $z_{n} \rightharpoonup z$ in $V$ as $n \to +\infty$.

(C3) $\chi_{\Gamma_{n}} \rightharpoonup \chi_{\Gamma}$ in $L^{1}(\partial \Omega)$ as $n \to +\infty$.

Also, a subset $\Pi'$ of $\Pi$ is said to have property (C), if $\Pi'$ is compact in the sense of (C1) - (C3), namely, any sequence $\{\Gamma_{n}\}$ of $\Pi'$ contains a subsequence convergent to a certain $\Gamma \in \Pi'$.

We suppose precise assumptions on the data as follows.

(H1) $\rho$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbb{R}$, and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$, and there are constants $A_{0} > 0$ and $\alpha$ with $1 < \alpha < 2$ such that

\[ |\rho(r_{1}) - \rho(r_{2})| \geq A_{0}|r_{1} - r_{2}| \left|\frac{1}{r_{1}r_{2}^\alpha} + 1 \right| \quad \text{for all } r_{1}, r_{2} \in D(\rho). \]

(H2) $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [\sigma_{*}, \sigma^{*}]$ for constants $\sigma_{*}$, $\sigma^{*}$ with $-\infty < \sigma_{*} < \sigma^{*} < +\infty$.

(H3) $\lambda$ is a $C^{2}$-function from $\mathbb{R}$ into itself and $g$ is a $C^{1}$-function from $\mathbb{R}$ into itself; $\lambda'$ is the derivative of $\lambda$.

(H4) (i) $f \in W^{1,2}(0, T; H)$;

(ii) $h_{D} \in W^{1,2}(0, T; H^{1/2}(\partial \Omega))$ such that there is a function $\tilde{h}_{D} \in W^{1,2}(0, T; V)$ with $\rho(\tilde{h}_{D}) \in W^{1,2}(0, T; V);$
Corresponding to functions \( h_D, h_N \) and \( \Gamma \in \Pi \), we consider the function \( h_\Gamma : [0, T] \to V \) given by

\[
\begin{cases}
  h_\Gamma(t) = h_D(t) & \text{a.e. on } \Gamma, \\
  a(h_\Gamma(t), z) + (n_0 h_\Gamma(t) - h_N(t), z)_{\partial\Omega} = 0 & \text{for all } z \in V(\Gamma); \\
\end{cases}
\]

Note under condition (H4) and \( \sigma(\Gamma) \geq \sigma_0 \) for a positive constant \( \sigma_0 \) that such a function \( h_\Gamma \) exists in \( W^{1,2}(0, T; V) \) and \( |h_\Gamma|_{W^{1,2}(0, T; V)} \leq K \) for a certain constant \( K \) depending only on quantities in (H4) and \( \sigma_0 \). Moreover, if \( \Gamma_n \to \Gamma \) in \( \Pi \) as \( n \to +\infty \), then \( h_{\Gamma_n} \to h_\Gamma \) in \( C([0, T]; V) \) as \( n \to +\infty \) (cf. [6]).

We now give the weak formulation for state problem \( SP(\Gamma) \) for each \( \Gamma \in \Pi \).

**Definition 1.1.** A couple \( \{u, w\} \) of functions \( u : [0, T] \to V \) and \( w : [0, T] \to H^2(\Omega) \) is called a (weak) solution of \( SP(\Gamma) \), if the following properties (w1) - (w4) are fulfilled:

(w1) \( u - h_\Gamma \in C_w([0, T]; V(\Gamma)), \rho(u) \in C_w([0, T]; H), \rho(u') \in L^2(0, T; V(\Gamma)'), w \in C_w([0, T]; H^2(\Omega)) \) with \( \frac{\partial w(t)}{\partial n} = 0 \) a.e. on \( \partial\Omega \) for all \( t \in [0, T], \) and \( w' \in L^2(0, T; H). \)

(w2) \( u(0) = u_0 \) and \( w(0) = w_0. \)

(w3) For all \( z \in V(\Gamma) \) and a.e. \( t \in [0, T], \)

\[
\frac{d}{dt}(\rho(u(t) + \lambda(w)(t), z) + a(u(t), z) + n_0 u(t) - h_\Gamma(t), z)_{\partial\Omega} = (f(t), z).
\]

(w4) There exists a function \( \xi \in L^2(0, T; H) \) such that \( \xi \in \beta(w) \) a.e. in \( Q \) and

\[
\frac{d}{dt}(w(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \xi(t) - \lambda'(w(t)) u(t), \Delta \eta) = 0
\]

for all \( \eta \in H^2(\Omega) \) with \( \frac{\partial \eta}{\partial n} = 0 \) a.e. on \( \partial\Omega \) and a.e. \( t \in [0, T]. \)
According to a result [5, Theorem 2.2], problem \(SP(\Gamma)\) has an unique solution \(\{u, w\}\) for each \(\Gamma \in \Pi\). Based on the solvability of \(SP(\Gamma)\), we now propose an optimization problem.

For a given non-empty subset \(\Pi_c\) of \(\Pi\) having property \((C)\), our optimization problem, denoted by \(P(\Pi_c)\), is to find a set \(\Gamma_* \in \Pi_c\) such that

\[
J(\Gamma_*) = \inf_{\Gamma \in \Pi_c} J(\Gamma),
\]

where

\[
J(\Gamma) := A \int_Q |u_{\Gamma} - u_d|^2 dx dt + B |w_{\Gamma} - w_d|_{C(\overline{Q})}^2 + C \int_{\Sigma(\Gamma)} |h_d|^2 d\sigma dt \quad \Gamma \in \Pi_c,
\]

\(A, B, C\) are positive constants, \(u_d, w_d, h_d\) are given in \(L^2(Q), C(\overline{Q}), L^2(\Sigma)\), respectively, and \(\{u_{\Gamma}, w_{\Gamma}\}\) is the solution of state problem \(SP(\Gamma)\); \(d\sigma\) stands for the surface element on \(\partial\Omega\).

Our main results are stated as follows.

**Theorem 1.1.** Let \(\Pi_c\) be a non-empty subset of \(\Pi\) having property \((C)\). Then, optimization problem \(P(\Pi_c)\) has at least one solution \(\Gamma_* \in \Pi_c\).

The above existence result is obtained from the following theorem on the continuous dependence of the solution \(\{u_{\Gamma}, w_{\Gamma}\}\) of \(SP(\Gamma)\) upon \(\Gamma \in \Pi\).

**Theorem 1.2.** Let \(\{\Gamma_n\}\) be a sequence in \(\Pi\) such that \(\Gamma_n \to \Gamma\) in \(\Pi\) as \(n \to +\infty\), and \(\{u_n, w_n\}\) and \(\{u, w\}\) be the solutions of \(SP(\Gamma_n)\) and \(SP(\Gamma)\), respectively. Then

\[u_n \to u \text{ in } C_w([0, T]; V), \quad w_n \to w \text{ in } C_w([0, T]; H^2(\Omega))\]

as \(n \to +\infty\).

For a detailed proofs, see a forthcoming paper [3].

It is easily seen from Theorem 1.2 that any minimizing sequence \(\{\Gamma_n\} \subset \Pi_c\) of the cost functional \(J(\cdot)\) on \(\Pi_c\) contains a subsequence convergent to a solution of \(P(\Pi_c)\).

**2. Regular approximation for \(P(\Pi_c)\)**

In this section, from the numerical point of view we discuss regular approximation of \(SP(\Gamma)\) and \(P(\Pi_c)\).

At first, we introduce the approximation \(\rho^\nu, \beta^\varepsilon\) and \(\chi^F\) for \(\rho, \beta\) and \(\chi_F\), respectively, which are defined below.

(a) Let \(D(\rho) := (r_\ast, r^\ast)\) for \(-\infty \leq r_\ast < r^\ast \leq +\infty\), and choose two families \(\{a_\nu; 0 < \nu \leq 1\}\) and \(\{b_\nu; 0 < \nu \leq 1\}\) in \(D(\rho)\) such that

\[r_\ast < a_\nu < a_\nu' < a_1 < b_1 < b_\nu < b_\nu' < r^\ast\text{ if }0 < \nu < \nu' < 1\]

and

\[a_\nu \downarrow r_\ast, \quad b_\nu \uparrow r^\ast \text{ as } \nu \downarrow 0.\]
Then, \( \rho^\nu : \mathbb{R} \rightarrow \mathbb{R} \) is defined for each \( \nu \in (0, 1] \) by

\[
\rho^\nu(r) := \begin{cases} 
\rho(b_\nu) + r - b_\nu & \text{for } r > b_\nu, \\
\rho(r) & \text{for } a_\nu \leq r \leq b_\nu, \\
\rho(a_\nu) + r - a_\nu & \text{for } r < a_\nu.
\end{cases}
\]

(b) For each \( 0 < \varepsilon \leq 1 \), \( \beta^\varepsilon \) is the Yosida-approximation of \( \beta \), namely,

\[
\beta^\varepsilon(r) := \frac{r - (I + \varepsilon \beta)^{-1}r}{\varepsilon}, \quad r \in \mathbb{R}.
\]

(c) Let \( \{\chi_1^\tau\} := \{\chi_1^\tau; 0 < \tau \leq 1, \Gamma \in \Pi_c\} \) be a family of smooth functions on \( \partial \Omega \) and suppose that it satisfies the following properties (\( \chi_1 \) - (\( \chi_3 \)):

(\( \chi_1 \)) \( 0 \leq \chi_r \leq \chi_1^\tau \leq 1; \mathrm{supp}(\chi_1^\tau) \subset \{x \in \partial \Omega; \mathrm{dist}(x, \Gamma) \leq \tau\} \) for all \( \tau \in (0, 1]\) and \( \Gamma \in \Pi_c \).

(\( \chi_2 \)) For each \( \tau \in (0, 1]\), \( \{\chi_1^\tau; \Gamma \in \Pi_c\} \) is compact in \( L^1(\partial \Omega) \).

(\( \chi_3 \)) Let \( V(\tau, \Gamma) := \{z \in V; \chi_1^\tau z = 0 \mathrm{a.e. on } \Gamma\} \) for each \( \tau \in (0, 1]\) and \( \Gamma \in \Pi_c \). If \( \tau_n \downarrow 0 \) and \( \Gamma_n \in \Pi_c \), then there are a subsequence \( \{n_k\} \) of \( \{n\} \) and \( \Gamma \in \Pi_c \) such that \( \chi_1^\tau_{\Gamma_n} \rightarrow \chi_1^\tau_{\Gamma} \) in \( L^1(\partial \Omega) \) as \( k \rightarrow \infty \), and \( V(\tau_n, \Gamma_n) \rightarrow V(\Gamma) \) in \( V \) as \( k \rightarrow \infty \) in the sense of Mosco [6].

Now we propose a regular approximation for \( SP(\Gamma) \), referred as \( \text{SP}(\Gamma)^{\nu, \varepsilon, \tau, \delta} \), \( \nu, \varepsilon, \tau, \delta \in (0, 1] \), by the penalty method:

\[
\begin{align*}
\rho^\nu(u_t) + \lambda(w)_t - \Delta u &= f \mathrm{ in } Q, \\
w_t - \Delta(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) &= 0 \mathrm{ in } Q, \\
\frac{\partial u}{\partial n} &= -\frac{\chi_1^\tau}{\delta}(u - h_D) + (1 - \chi_1^\tau)(h_N - n_0u) \mathrm{ on } \Sigma, \\
\frac{\partial w}{\partial n} &= 0, \quad \frac{\partial}{\partial n}(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 \mathrm{ on } \Sigma, \\
u(0) &= u_0, := \min\{\max\{u_0, a_\nu\}, b_\nu\}, \quad w(0) = w_0 \mathrm{ in } \Omega.
\end{align*}
\]

The notion of a weak solution of \( \text{SP}(\Gamma)^{\nu, \varepsilon, \tau, \delta} \) is given below.

**Definition 2.1.** A couple \( \{u, w\} \) of functions \( u : [0, T] \rightarrow V \) and \( w : [0, T] \rightarrow H^2(\Omega) \) is called a solution of \( \text{SP}(\Gamma)^{\nu, \varepsilon, \tau, \delta} \) if the following conditions (w1)' - (w4)' are satisfied:

(w1)' \( u \in W^{1,2}(0, T; H) \cap C([0, T]; V) \),

\[
w \in W^{1,2}(0, T; H) \cap C([0, T]; H^2(\Omega)) \mathrm{ with } \frac{\partial w(t)}{\partial n} = 0 \mathrm{ a.e. on } \partial \Omega \mathrm{ for all } t \in [0, T].
\]

(w2)' \( u(0) = u_{0\nu}, u(0) = u_0, w(0) = w_0 \).

(w3)' For all \( z \in V \) and a.e. \( t \in [0, T], \)

\[
(\rho^\nu(u'(t)) + \lambda(w)'(t), z) + a(u(t), z)
\]

\[
+ (\frac{\chi_1^\tau}{\delta}(u(t) - h_D(t)) - (1 - \chi_1^\tau)(h_N(t) - n_0u(t)), z)_{\partial \Omega} = (f(t), z).
\]

(w4)' For all \( \eta \in H^2(\Omega) \) with \( \frac{\partial \eta}{\partial n} = 0 \mathrm{ a.e. on } \partial \Omega \mathrm{ and a.e. } t \in [0, T], \)

\[
(w'(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta \eta) = 0.
\]
According to a result in [4], \( SP(\Gamma)^{\nu\epsilon\tau\delta} \) has a unique solution \( \{u, w\} \). Our regular approximate optimization problem \( P(\Pi_c)^{\nu\epsilon\tau\delta} \) is to find \( \Gamma^{
u\epsilon\tau\delta}_* \in \Pi_c \) such that
\[
J^{\nu\epsilon\tau\delta}(\Gamma^{
u\epsilon\tau\delta}_*) = \inf_{\Gamma \in \Pi_c} J^{\nu\epsilon\tau\delta}(\Gamma),
\]
where
\[
J^{\nu\epsilon\tau\delta}(\Gamma) := A \int_Q |u - u_d|^2 dx dt + B |w - w_d|^2_{C(Q)} + C \int_\Sigma (1 - \chi_\Gamma^\tau)^2 |h_d|^2 d\sigma dt,
\]
\( \{u, w\} \) is the solution of \( SP(\Gamma)^{\nu\epsilon\tau\delta} \).

Finally, we show a convergence result.

**Theorem 2.1.** Let \( \Pi_c, \{\rho^\nu\}, \{\beta^\tau\}, \{\chi_\tau^\nu\} \) be as above. Then:

1. For \( \nu, \varepsilon, \tau, \delta \in (0, 1], P(\Pi_c)^{\nu\epsilon\tau\delta} \) has at least one solution \( \Gamma^{
u\epsilon\tau\delta}_* \in \Pi_c \).
2. Let \( \{\nu_n\}, \{\varepsilon_n\}, \{\tau_n\} \) and \( \{\delta_n\} \) be any null sequences and let \( \{\Gamma_n := \Gamma^{
u\varepsilon\tau\delta}_*\} \) be a sequence of solutions of \( P(\Pi_c)^{\nu\varepsilon\tau\delta\delta} \). Then, \( \{\Gamma_n\} \) contains a subsequence convergent in \( \Pi \) and any limit \( \Gamma_* \) is a solution of \( P(\Pi_c) \).

For a detailed proof, see a forthcoming paper [3].

**References**


