<table>
<thead>
<tr>
<th>Page</th>
<th>サブヘッダー</th>
<th>コンテント</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Title</td>
<td>Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation (Variational Problems and Related Topics)</td>
</tr>
<tr>
<td>2</td>
<td>Author(s)</td>
<td>KADOYA, Atsushi</td>
</tr>
<tr>
<td>3</td>
<td>Citation</td>
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<td>Textversion</td>
<td>publisher</td>
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</tbody>
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Kyoto University
Shape Optimization Problem on the Lateral Boundary
for Thermodynamical Phase Separation

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1. Formulation of an optimization problem

This paper is concerned with an optimization problem on the lateral boundary $\partial \Omega$ for a thermodynamical phase separation model formulated in a domain $\Omega$.

$\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N = 2$ or $3$) with smooth boundary $\partial \Omega$ and $T$ is a fixed positive number. Our state problem $SP(\Gamma)$ is of the form

$$
\begin{aligned}
\rho(u)_t + \lambda(w)_t - \Delta u &= f & \text{in } Q := (0, T) \times \Omega, \\
\xi \in \beta(w) &\quad \text{in } Q, \\
u_0 + n_0 u &= h_N & \text{on } \Sigma_N := (0, T) \times \Gamma', \\
\frac{\partial u}{\partial n} + n_0 u &= h_N & \text{on } \Sigma_N := (0, T) \times \Gamma', \\
\frac{\partial w}{\partial n} &= 0, & \text{on } \Sigma := (0, T) \times \partial \Omega,
\end{aligned}
$$

Throughout this paper, we use the following notation.

For a general (real) Banach space $Y$, we denote by $|\cdot|_Y$ the norm in $Y$ and by $Y^*$ the dual of $Y$. Also, for a positive finite number $T$, we denote by $C_w([0, T]; Y)$ the space of all weakly continuous functions $u : [0, T] \to Y$, and by definition "$u_n \to u$ in $C_w([0, T]; Y)$" as $n \to +\infty"$ means that for each $z^* \in Y^*$, $(z^*, u_n(t))_{Y^*, Y}$ converges to $(z^*, u(t))_{Y^*, Y}$ uniformly in $t \in [0, T]$ as $n \to +\infty$, where $(\cdot, \cdot)_{Y^*, Y}$ is the duality pairing between $Y^*$ and $Y$.

For simplicity we put

$$
H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_0 := \{v \in H; \int_\Omega z dx = 0\}, \quad V_0 := V \cap H_0,
$$

and

$$
\Pi := \{\Gamma \subset \partial \Omega; \ \Gamma \text{ is compact in } \partial \Omega, \ M(\Gamma) > 0\}.
$$

For each $\Gamma \in \Pi$, we put

$$
V(\Gamma) := \{z \in V; \ z = 0 \ \text{a.e. on } \Gamma\}.
$$
which is a closed subspace of $V$, and

$$(v, w) := \int_{\Omega} v w \, dx \quad \text{for } v, w \in H,$$

$$(v, w)_{\partial \Omega} := \int_{\partial \Omega} v w \, d\sigma \quad \text{for } v, w \in L^2(\partial \Omega),$$

$$a(v, w) := \int_{\Omega} \nabla v \cdot \nabla w \, dx \quad \text{for } v, w \in V.$$

In general, given a subset $E$ of $\overline{\Omega}$, $\chi_E$ denotes the characteristic function of $E$ defined on $\overline{\Omega}$.

We now introduce a notion of convergence in $\Pi$. By definition, a sequence $\{\Gamma_n\} \subset \Pi$ converges to $\Gamma \in \Pi$ as $n \to +\infty$, if the following conditions (C1) - (C3) are satisfied:

(C1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in V(\Gamma_{n_k})$ and $z_k \rightharpoonup z$ weakly in $V$ as $k \to +\infty$, then $z \in V(\Gamma)$.

(C2) For any $z \in V(\Gamma)$, there is a sequence $\{z_n\} \subset V$ such that $z_n \in V(\Gamma_n)$, $n = 1, 2, \ldots$, and $z_n \rightharpoonup z$ in $V$ as $n \to +\infty$.

(C3) $\chi_{\Gamma_n} \rightharpoonup \chi_{\Gamma}$ in $L^1(\partial \Omega)$ as $n \to +\infty$.

Also, a subset $\Pi'$ of $\Pi$ is said to have property (C), if $\Pi'$ is compact in the sense of (C1) - (C3), namely, any sequence $\{\Gamma_n\}$ of $\Pi'$ contains a subsequence convergent to a certain $\Gamma \in \Pi'$.

We suppose precise assumptions on the data as follows.

(H1) $\rho$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbb{R}$, and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$, and there are constants $A_0 > 0$ and $\alpha$ with $1 \leq \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \geq \frac{A_0|\gamma_1 - \gamma_2|}{|\gamma_1 \gamma_2|^\alpha + 1} \quad \text{for all } r_1, r_2 \in D(\rho).$$

(H2) $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for constants $\sigma_*$, $\sigma^*$ with $-\infty < \sigma_* < \sigma^* < +\infty$.

(H3) $\lambda$ is a $C^2$-function from $\mathbb{R}$ into itself and $g$ is a $C^1$-function from $\mathbb{R}$ into itself; $\lambda'$ is the derivative of $\lambda$.

(H4) (i) $f \in W^{1,2}(0, T; H)$;

(ii) $h_D \in W^{1,2}(0, T; H^{1/2}(\partial \Omega))$ such that there is a function $\overline{h}_D \in W^{1,2}(0, T; V)$ with $\rho(\overline{h}_D) \in W^{1,2}(0, T; V)$.
(iii) $h_N \in W^{1,2}(0, T; H^2(\Omega)) \cap L^\infty(\Sigma)$ such that

$$n_0 \inf D(\rho) \leq h_N(t, x) \leq n_0 \sup D(\rho) \text{ for a.e. } (t, x) \in \Sigma$$

and there are positive constants $A_1$ and $A_1'$ such that

$$\rho(r)(n_0 r - h_N(t, x)) \geq -A_1|r| - A_1' \text{ for all } r \in D(\rho) \text{ and a.e. } (t, x) \in \Sigma.$$  

(H5) (i) $u_0 \in V$ such that $\rho(u_0) \in H$ and $u_0 = h_D(0, \cdot)$ a.e. on $\partial\Omega$;

(ii) $w_0 \in H^2(\Omega)$ such that

$$\sigma_0 < \frac{1}{|\Omega|} \int_{\Omega} w_0 \, dx =: m < \sigma^*$$

and $\frac{\partial w_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and there is $\xi_0 \in H$ satisfying

$$\xi_0 \in \beta(w_0) \text{ a.e. in } \Omega, \quad -\kappa \Delta w_0 + \xi_0 \in V.$$

Corresponding to functions $h_D$, $h_N$ and $\Gamma \in \Pi$, we consider the function $h_{\Gamma} : [0, T] \rightarrow V$ given by

$$\begin{cases}
   h_{\Gamma}(t) = h_D(t) & \text{a.e. on } \Gamma, \\
   a(h_{\Gamma}(t), z) + (n_0 h_{\Gamma}(t) - h_N(t), z)_{\partial\Omega} = 0 & \text{for all } z \in V(\Gamma);
\end{cases}$$

note under condition (H4) and $\sigma(\Gamma) \geq \sigma_0$ for a positive constant $\sigma_0$ that such a function $h_{\Gamma}$ exists in $W^{1,2}(0, T; V)$ and $|h_{\Gamma}|_{W^{1,2}(0, T; V)} \leq K$ for a certain constant $K$ depending only on quantities in (H4) and $\sigma_0$. Moreover, if $\Gamma_n \rightarrow \Gamma$ in $\Pi$ as $n \rightarrow +\infty$, then $h_{\Gamma_n} \rightarrow h_{\Gamma}$ in $C([0, T]; V)$ as $n \rightarrow +\infty$ (cf. [6]).

We now give the weak formulation for state problem $SP(\Gamma)$ for each $\Gamma \in \Pi$.  

**Definition 1.1.** A couple $\{u, w\}$ of functions $u : [0, T] \rightarrow V$ and $w : [0, T] \rightarrow H^2(\Omega)$ is called a (weak) solution of $SP(\Gamma)$, if the following properties (w1) - (w4) are fulfilled:

(w1) $u - h_{\Gamma} \in C_w([0, T]; V(\Gamma))$, $\rho(u) \in C_w([0, T]; H)$, $\rho(u') \in L^2(0, T; V(\Gamma)^*)$;

(w2) $u(0) = u_0$ and $w(0) = w_0$.

(w3) For all $z \in V(\Gamma)$ and a.e. $t \in [0, T],$

$$\frac{d}{dt}(\rho(u)(t) + \lambda(w)(t), z) + a(u(t), z) + n_0(u(t) - h_{\Gamma}(t), z)_{\partial\Omega} = (f(t), z).$$

(w4) There exists a function $\xi \in L^2(0, T; H)$ such that $\xi \in \beta(w)$ a.e. in $Q$ and

$$\frac{d}{dt}(w(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t), \Delta \eta) = 0$$

for all $\eta \in H^2(\Omega)$ with $\frac{\partial \eta}{\partial n} = 0$ a.e. on $\partial\Omega$ and a.e. $t \in [0, T]$.  


According to a result [5, Theorem 2.2], problem $SP(\Gamma)$ has an unique solution $\{u, w\}$ for each $\Gamma \in \Pi$. Based on the solvability of $SP(\Gamma)$, we now propose an optimization problem.

For a given non-empty subset $\Pi_c$ of $\Pi$ having property (C), our optimization problem, denoted by $P(\Pi_c)$, is to find a set $\Gamma_* \in \Pi_c$ such that

$$J(\Gamma_*) = \inf_{\Gamma \in \Pi_c} J(\Gamma),$$

where

$$J(\Gamma) := A \int_Q |u_{\Gamma} - u_d|^2dxdt + B|w_{\Gamma} - w_d|^2_{C(\overline{Q})} + C \int_{\Sigma(\Gamma)} |h_d|^2 d\sigma dt \quad \Gamma \in \Pi_c,$$

$A, B, C$ are positive constants, $u_d, w_d, h_d$ are given in $L^2(Q), C(\overline{Q}), L^2(\Sigma)$, respectively, and $\{u_{\Gamma}, w_{\Gamma}\}$ is the solution of state problem $SP(\Gamma)$; $d\sigma$ stands for the surface element on $\partial\Omega$.

Our main results are stated as follows.

**Theorem 1.1.** Let $\Pi_c$ be a non-empty subset of $\Pi$ having property (C). Then, optimization problem $P(\Pi_c)$ has at least one solution $\Gamma_* \in \Pi_c$.

The above existence result is obtained from the following theorem on the continuous dependence of the solution $\{u_{\Gamma}, w_{\Gamma}\}$ of $SP(\Gamma)$ upon $\Gamma \in \Pi$.

**Theorem 1.2.** Let $\{\Gamma_n\}$ be a sequence in $\Pi$ such that $\Gamma_n \to \Gamma$ in $\Pi$ as $n \to +\infty$, and $\{u_n, w_n\}$ and $\{u, w\}$ be the solutions of $SP(\Gamma_n)$ and $SP(\Gamma)$, respectively. Then

$$u_n \to u \text{ in } C_w([0, T]; V), \quad w_n \to w \text{ in } C_w([0, T]; H^2(\Omega))$$

as $n \to +\infty$.

For a detailed proofs, see a forthcoming paper [3].

It is easily seen from Theorem 1.2 that any minimizing sequence $\{\Gamma_n\} \subset \Pi_c$ of the cost functional $J(\cdot)$ on $\Pi_c$ contains a subsequence convergent to a solution of $P(\Pi_c)$.

2. Regular approximation for $P(\Pi_c)$

In this section, from the numerical point of view we discuss regular approximation of $SP(\Gamma)$ and $P(\Pi_c)$.

At first, we introduce the approximation $\rho^\nu, \beta^\varsigma$ and $\chi^\tau_\Gamma$ for $\rho, \beta$ and $\chi_\Gamma$, respectively, which are defined below.

(a) Let $D(\rho) := (r_*, r^*)$ for $-\infty \leq r_* < r^* \leq +\infty$, and choose two families $\{a_\nu; 0 < \nu \leq 1\}$ and $\{b_\nu; 0 < \nu \leq 1\}$ in $D(\rho)$ such that

$$r_* < a_\nu < a_\nu' < a_1 < b_1 < b_\nu < b_\nu' < r^* \quad \text{if } 0 < \nu < \nu' < 1$$

and

$$a_\nu \downarrow r_*, \quad b_\nu \uparrow r^* \quad \text{as } \nu \downarrow 0.$$
Then, $\rho^\nu : \mathbb{R} \to \mathbb{R}$ is defined for each $\nu \in (0,1]$ by
\[
\rho^\nu(r) := \begin{cases} 
\rho(b_\nu) + r - b_\nu & \text{for } r > b_\nu, \\
\rho(r) & \text{for } a_\nu \leq r \leq b_\nu, \\
\rho(a_\nu) + r - a_\nu & \text{for } r < a_\nu.
\end{cases}
\]

(b) For each $0 < \varepsilon \leq 1$, $\beta^\varepsilon$ is the Yosida-approximation of $\beta$, namely,
\[
\beta^\varepsilon(r) := \frac{r - (I + \varepsilon\beta)^{-1}r}{\varepsilon}, \quad r \in \mathbb{R}.
\]

(c) Let $\{\chi^1_{T} \} := \{\chi^1_{T}; 0 < \tau \leq 1, \Gamma \in \Pi_c\}$ be a family of smooth functions on $\partial \Omega$ and suppose that it satisfies the following properties (\chi 1) – (\chi 3):

(\chi 1) $0 \leq \chi_T \leq \chi^1_T \leq 1$; $\text{supp}(\chi^1_T) \subset \{x \in \partial \Omega; \text{dist}(x, \Gamma) \leq \tau\}$ for all $\tau \in (0, 1]$ and $\Gamma \in \Pi_c$.

(\chi 2) For each $\tau \in (0,1]$, $\{\chi^1_T; \Gamma \in \Pi_c\}$ is compact in $L^1(\partial \Omega)$.

(\chi 3) Let $V(\tau, \Gamma) := \{z \in V; \chi_T = 0$ a.e. on $\Gamma\}$ for each $\tau \in (0,1]$ and $\Gamma \in \Pi_c$. If $\tau_n \downarrow 0$ and $\Gamma_n \in \Pi_c$, then there are a subsequence $\{n_k\}$ of $\{n\}$ and $\Gamma \in \Pi_c$ such that $\chi_{T_n} \rightarrow \chi_{T'}$ in $L^1(\partial \Omega)$ as $k \to \infty$, and $V(\tau_n, \Gamma_n) \to V(\Gamma)$ in $V$ as $k \to \infty$ in the sense of Mosco [6].

Now we propose a regular approximation for $SP(\Gamma)$, referred as $SP(\Gamma)^{\nu, \varepsilon, \tau, \delta}$, in $(0,1]$, by the penalty method:

\[
\begin{align*}
\rho^\nu(w)_t + \lambda(w)_t - \Delta u &= f \quad \text{in } Q, \\
\rho^\nu(w)_t - \Delta(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) &= 0 \quad \text{in } Q, \\
\frac{\partial u}{\partial \nu} &= \frac{-\chi^1_{T}(u - h_D)}{\delta} + (1 - \chi^1_{T})(h_N - n_0 u) \quad \text{on } \Sigma, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \frac{\partial}{\partial \nu}\left((-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 \quad \text{on } \Sigma, \\
u(0) &= u_0, := \min\{\max\{u_0, a_\nu\}, b_\nu\}, \quad w(0) = w_0 \quad \text{in } \Omega.
\end{align*}
\]

The notion of a weak solution of $SP(\Gamma)^{\nu, \varepsilon, \tau, \delta}$ is given below.

**Definition 2.1.** A couple $\{u, w\}$ of functions $u : [0, T] \to V$ and $w : [0, T] \to H^2(\Omega)$ is called a solution of $SP(\Gamma)^{\nu, \varepsilon, \tau, \delta}$, if the following conditions (w1)' - (w4)' are satisfied:

(w1)' $u \in W^{1,2}(0, T; H) \cap C([0, T]; V)$,

(w2)' $u(0) = u_{0w}, \quad w(0) = w_0$.

(w3)' For all $z \in V$ and a.e. $t \in [0, T]$,

\[
\left(\rho^\nu(u)'(t) + \lambda(w)'(t), \Delta \eta(t) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta \eta) = 0.
\]

(w4)' For all $\eta \in H^2(\Omega)$ with $\frac{\partial \eta}{\partial \nu} = 0$ a.e. on $\partial \Omega$ and a.e. $t \in [0, T]$,

\[
(w', \eta - \mu\Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta \eta) = 0.
\]
According to a result in [4], $SP(\Gamma)^{\nu\tau\delta}$ has a unique solution $\{u, w\}$. Our regular approximate optimization problem $P(\Pi_c)^{\nu\tau\delta}$ is to find $\Gamma^{\nu\tau\delta}_* \in \Pi_c$ such that

$$J^{\nu\tau\delta}(\Gamma^{\nu\tau\delta}_*) = \inf_{\Gamma \in \Pi_c} J^{\nu\tau\delta}(\Gamma),$$

where

$$J^{\nu\tau\delta}(\Gamma) := A \int_Q |u - u_d|^2 dx dt + B |w - w_d|^2_{C(Q)} + C \int_\Sigma (1 - \chi^T_{\Gamma}) |h_d|^2 d\sigma dt,$$

$\{u, w\}$ is the solution of $SP(\Gamma)^{\nu\tau\delta}$.

Finally, we show a convergence result.

**Theorem 2.1.** Let $\Pi_c$, $\{\rho^\nu\}$, $\{\beta^\tau\}$, $\{\chi^T\}$ be as above. Then:

1. For $\nu, \epsilon, \tau, \delta \in (0, 1]$, $P(\Pi_c)^{\nu\tau\delta}$ has at least one solution $\Gamma^{\nu\tau\delta}_* \in \Pi_c$.
2. Let $\{\nu_n\}$, $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\delta_n\}$ be any null sequences and let $\{\Gamma_n := \Gamma^{\nu_n\epsilon_n\tau_n\delta_n}_*\}$ be a sequence of solutions of $P(\Pi_c)^{\nu_n\epsilon_n\tau_n\delta_n}$. Then, $\{\Gamma_n\}$ contains a subsequence convergent in $\Pi$ and any limit $\Gamma_*$ is a solution of $P(\Pi_c)$.

For a detailed proof, see a forthcoming paper [3].

**References**


