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Kyoto University
Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation

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1. Formulation of an optimization problem

This paper is concerned with an optimization problem on the lateral boundary $\partial \Omega$ for a thermodynamical phase separation model formulated in a domain $\Omega$.

$\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N = 2$ or $3$) with smooth boundary $\partial \Omega$ and $T$ is a fixed positive number. Our state problem $SP(\Gamma)$ is of the form

\[
\begin{aligned}
\rho(u)_t + \lambda(w)_t - \Delta u &= f \quad \text{in } Q := (0, T) \times \Omega, \\
w_t - \Delta \{ -\mu \Delta w_t - \kappa \Delta w + \xi + g(w) - \lambda'(w)u \} &= 0 \quad \text{in } Q, \\
\xi &\in \beta(w) \quad \text{in } Q, \\
u &= h_D \quad \text{on } \Sigma_D := (0, T) \times \Gamma, \\
\frac{\partial \nu}{\partial n} + n_0 u &= h_N \quad \text{on } \Sigma_N := (0, T) \times \Gamma', \quad \Gamma' := \partial \Omega \setminus \Gamma, \\
\frac{\partial w}{\partial n} &= 0, \quad \frac{\partial}{\partial n} \{ -\mu \Delta w_t - \kappa \Delta w + \xi + g(w) - \lambda'(w)u \} = 0 \quad \text{on } \Sigma := (0, T) \times \partial \Omega, \\
u(0, \cdot) &= u_0, w(0, \cdot) = w_0 \quad \text{in } \Omega.
\end{aligned}
\]

Throughout this paper, we use the following notation.

For a general (real) Banach space $Y$, we denote by $| \cdot |_Y$ the norm in $Y$ and by $Y^*$ the dual of $Y$. Also, for a positive finite number $T$, we denote by $C_w([0, T]; Y)$ the space of all weakly continuous functions $u : [0, T] \to Y$, and by definition "$u_n \to u$ in $C_w([0, T]; Y)$ as $n \to +\infty$" means that for each $z^* \in Y^*$, $(z^*, u_n(t))_{Y^*, Y}$ converges to $(z^*, u(t))_{Y^*, Y}$ uniformly in $t \in [0, T]$ as $n \to +\infty$, where $(\cdot, \cdot)_{Y^*, Y}$ is the duality pairing between $Y^*$ and $Y$.

For simplicity we put

\[ H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_0 := \{ v \in H; \int_{\Omega} z dx = 0 \}, \quad V_0 := V \cap H_0, \]

and

\[ \Pi := \{ \Gamma \subset \partial \Omega; \Gamma \text{ is compact in } \partial \Omega, \sigma(\Gamma) > 0 \}. \]

For each $\Gamma \in \Pi$, we put

\[ V(\Gamma) := \{ z \in V; z = 0 \text{ a.e. on } \Gamma \}. \]
which is a closed subspace of $V$, and

$$(v, w) := \int_{\Omega} v w d\sigma$$

for $v, w \in H$,

$$(v, w)_{\partial\Omega} := \int_{\partial\Omega} v w d\sigma$$

for $v, w \in L^2(\partial\Omega)$,

$$a(v, w) := \int_{\Omega} \nabla v \cdot \nabla w d\sigma$$

for $v, w \in V$.

In general, given a subset $E$ of $\overline{\Omega}$, $\chi_E$ denotes the characteristic function of $E$ defined on $\overline{\Omega}$.

We now introduce a notion of convergence in $H$. By definition, a sequence $\{\Gamma_n\} \subset \Pi$ converges to $\Gamma \in \Pi$ as $n \to +\infty$, if the following conditions (C1) - (C3) are satisfied:

(C1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_{k} \in V(\Gamma_{n_k})$ and $z_k \rightharpoonup z$ weakly in $V$ as $k \to +\infty$, then $z \in V(\Gamma)$.

(C2) For any $z \in V(\Gamma)$, there is a sequence $\{z_n\} \subset V$ such that $z_n \in V(\Gamma_n)$, $n=1,2,\ldots$, and $z_n \to z$ in $V$ as $n \to +\infty$.

(C3) $\chi_{\Gamma_n} \to \chi_{\Gamma}$ in $L^1(\partial\Omega)$ as $n \to +\infty$.

Also, a subset $\Pi'$ of $\Pi$ is said to have property (C), if $\Pi'$ is compact in the sense of (C1) - (C3), namely, any sequence $\{\Gamma_n\}$ of $\Pi'$ contains a subsequence convergent to a certain $\Gamma \in \Pi'$.

We suppose precise assumptions on the data as follows.

(H1) $\rho$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbb{R}$, and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$, and there are constants $A_0 > 0$ and $\alpha$ with $1 \leq \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \geq \frac{A_0|r_1 - r_2|}{|r_1 r_2|^{\alpha} + 1} \text{ for all } r_1, r_2 \in D(\rho).$$

(H2) $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for constants $\sigma_*$, $\sigma^*$ with $-\infty < \sigma_* < \sigma^* < +\infty$.

(H3) $\lambda$ is a $C^2$-function from $\mathbb{R}$ into itself and $g$ is a $C^1$-function from $\mathbb{R}$ into itself; $\lambda'$ is the derivative of $\lambda$.

(H4) (i) $f \in W^{1,2}(0, T; H)$;

(ii) $h_D \in W^{1,2}(0, T; H^{1/2}(\partial\Omega))$ such that there is a function $\tilde{h}_D \in W^{1,2}(0, T; V)$ with $\rho(\tilde{h}_D) \in W^{1,2}(0, T; V)$. 

(iii) $h_N \in W^{1,2}(0, T; L^2(\partial\Omega)) \cap L^\infty(\Sigma)$ such that

$$n_0 \inf D(\rho) \leq h_N(t, x) \leq n_0 \sup D(\rho) \quad \text{for a.e. } (t, x) \in \Sigma$$

and there are positive constants $A_1$ and $A'_1$ such that

$$\rho(r)(n_0 r - h_N(t, x)) \geq -A_1|r| - A'_1 \quad \text{for all } r \in D(\rho) \text{ and a.e. } (t, x) \in \Sigma.$$

(H5) (i) $u_0 \in V$ such that $\rho(u_0) \in H$ and $u_0 = h_D(0, \cdot)$ a.e. on $\partial\Omega$;
(ii) $w_0 \in H^2(\Omega)$ such that

$$\sigma_* < \frac{1}{|\Omega|} \int_{\Omega} w_0\,dx =: m < \sigma^*$$

and $\frac{\partial w_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and there is $\xi_0 \in H$ satisfying

$$\xi_0 \in \beta(w_0) \quad \text{a.e. in } \Omega, \quad -\kappa \Delta w_0 + \xi_0 \in V.$$

Corresponding to functions $h_D, h_N$ and $\Gamma \in \Pi$, we consider the function $h_\Gamma : [0, T] \rightarrow V$ given by

$$\begin{cases}
h_\Gamma(t) = h_D(t) & \text{a.e. on } \Gamma, \\
a(h_\Gamma(t), z) + (n_0 h_\Gamma(t) - h_N(t), z)_{\partial\Omega} &= 0 \quad \text{for all } z \in V(\Gamma);
\end{cases}$$

note under condition (H4) and $\sigma(\Gamma) \geq \sigma_0$ for a positive constant $\sigma_0$ that such a function $h_\Gamma$ exists in $W^{1,2}(0, T; V)$ and $|h_\Gamma|_{W^{1,2}(0, T; V)} \leq K$ for a certain constant $K$ depending only on quantities in (H4) and $\sigma_0$. Moreover, if $\Gamma_n \rightarrow \Gamma$ in $\Pi$ as $n \rightarrow +\infty$, then $h_{\Gamma_n} \rightarrow h_\Gamma$ in $C([0, T]; V)$ as $n \rightarrow +\infty$ (cf. [6]).

We now give the weak formulation for state problem $SP(\Gamma)$ for each $\Gamma \in \Pi$.

**Definition 1.1.** A couple $\{u, w\}$ of functions $u : [0, T] \rightarrow V$ and $w : [0, T] \rightarrow H^2(\Omega)$ is called a (weak) solution of $SP(\Gamma)$, if the following properties (w1) – (w4) are fulfilled:

(w1) $u - h_\Gamma \in C_w([0, T]; V(\Gamma)), \rho(u) \in C_w([0, T]; H), \rho(u)' \in L^2(0, T; V(\Gamma))$, $w \in C_w([0, T]; H^2(\Omega))$ with $\frac{\partial w(t)}{\partial n} = 0$ a.e. on $\partial\Omega$ for all $t \in [0, T]$, and $w' \in L^2(0, T; H)$.
(w2) $u(0) = u_0$ and $w(0) = w_0$.
(w3) For all $z \in V(\Gamma)$ and a.e. $t \in [0, T]$,

$$\frac{d}{dt}(\rho(u)(t) + \lambda(w)(t), z) + a(u(t), z) + n_0(u(t) - h_\Gamma(t), z)_{\partial\Omega} = (f(t), z).$$

(w4) There exists a function $\xi \in L^2(0, T; H)$ such that $\xi \in \beta(w)$ a.e. in $Q$ and

$$\frac{d}{dt}(w(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t), \Delta \eta) = 0$$

for all $\eta \in H^2(\Omega)$ with $\frac{\partial \eta}{\partial n} = 0$ a.e. on $\partial\Omega$ and a.e. $t \in [0, T]$. 
According to a result [5, Theorem 2.2], problem $SP(\Gamma)$ has an unique solution $\{u, w\}$ for each $\Gamma \in \Pi$. Based on the solvability of $SP(\Gamma)$, we now propose an optimization problem.

For a given non-empty subset $\Pi_c$ of $\Pi$ having property $(C)$, our optimization problem, denoted by $P(\Pi_c)$, is to find a set $\Gamma_* \in \Pi_c$ such that

$$J(\Gamma_*) = \inf_{\Gamma \in \Pi_c} J(\Gamma),$$

where

$$J(\Gamma) := A \int_Q |u_{\Gamma} - u_d|^2 dx dt + B |w_{\Gamma} - w_d|_{C(\overline{Q})}^2 + C \int_{\Sigma} |h_d|^2 d\sigma dt \quad \Gamma \in \Pi_c,$$

$A, B, C$ are positive constants, $u_d, w_d, h_d$ are given in $L^2(Q), C(\overline{Q}), L^2(\Sigma)$, respectively, and $\{u_{\Gamma}, w_{\Gamma}\}$ is the solution of state problem $SP(\Gamma)$; $d\sigma$ stands for the surface element on $\partial \Omega$.

Our main results are stated as follows.

**Theorem 1.1.** Let $\Pi_c$ be a non-empty subset of $\Pi$ having property $(C)$. Then, optimization problem $P(\Pi_c)$ has at least one solution $\Gamma_* \in \Pi_c$.

The above existence result is obtained from the following theorem on the continuous dependence of the solution $\{u_{\Gamma}, w_{\Gamma}\}$ of $SP(\Gamma)$ upon $\Gamma \in \Pi$.

**Theorem 1.2.** Let $\{\Gamma_n\}$ be a sequence in $\Pi$ such that $\Gamma_n \rightarrow \Gamma$ in $\Pi$ as $n \rightarrow +\infty$, and $\{u_n, w_n\}$ and $\{u, w\}$ be the solutions of $SP(\Gamma_n)$ and $SP(\Gamma)$, respectively. Then

$$u_n \rightarrow u \text{ in } C_w([0, T]; V), \quad w_n \rightarrow w \text{ in } C_w([0, T]; H^2(\Omega))$$

as $n \rightarrow +\infty$.

For a detailed proofs, see a forthcoming paper [3].

It is easily seen from Theorem 1.2 that any minimizing sequence $\{\Gamma_n\} \subset \Pi_c$ of the cost functional $J(\cdot)$ on $\Pi_c$ contains a subsequence convergent to a solution of $P(\Pi_c)$.

2. Regular approximation for $P(\Pi_c)$

In this section, from the numerical point of view we discuss regular approximation of $SP(\Gamma)$ and $P(\Pi_c)$.

At first, we introduce the approximation $\rho^\nu$, $\beta^\nu$ and $\chi^\nu_{\Gamma}$ for $\rho$, $\beta$ and $\chi_{\Gamma}$, respectively, which are defined below.

(a) Let $D(\rho) := (r_*, r^*)$ for $-\infty \leq r_* < r^* \leq +\infty$, and choose two families $\{a_\nu; 0 < \nu \leq 1\}$ and $\{b_\nu; 0 < \nu \leq 1\}$ in $D(\rho)$ such that

$$r_* < a_\nu < a_\nu < a_1 < b_1 < b_\nu < b_\nu < r^* \text{ if } 0 < \nu < \nu' < 1$$

and

$$a_\nu \downarrow r_*, \quad b_\nu \uparrow r^* \text{ as } \nu \downarrow 0.$$
Then, $\rho^\nu : \mathbb{R} \rightarrow \mathbb{R}$ is defined for each $\nu \in (0, 1]$ by

$$\rho^\nu(r) := \begin{cases} 
\rho(b_{\nu}) + r - b_{\nu} & \text{for } r > b_{\nu}, \\
\rho(r) & \text{for } a_{\nu} \leq r \leq b_{\nu}, \\
\rho(a_{\nu}) + r - a_{\nu} & \text{for } r < a_{\nu}.
\end{cases}$$

(b) For each $0 < \varepsilon \leq 1$, $\beta^\varepsilon$ is the Yosida-approximation of $\beta$, namely,

$$\beta^\varepsilon(r) := \frac{r - (I + \varepsilon\beta)^{-1}r}{\varepsilon}, \quad r \in \mathbb{R}.$$

(c) Let \( \{\chi^\tau_{\Gamma}\} := \{\chi^\tau_{\Gamma}; 0 < \tau \leq 1, \Gamma \in \Pi_c\} \) be a family of smooth functions on \( \partial \Omega \) and suppose that it satisfies the following properties (\( \chi_1 \)) – (\( \chi_3 \)):

(\( \chi_1 \)) \( 0 \leq \chi_{\tau} \leq \chi^0_{\tau} \leq 1; \supp(\chi^\tau_{\Gamma}) \subset \{x \in \partial \Omega; \text{dist}(x, \Gamma) \leq \tau\} \) for all \( \tau \in (0, 1] \) and \( \Gamma \in \Pi_c \).

(\( \chi_2 \)) For each \( \tau \in (0, 1] \), \( \{\chi^\tau_{\Gamma}; \Gamma \in \Pi_c\} \) is compact in \( L^1(\partial \Omega) \).

(\( \chi_3 \)) Let \( V(\tau, \Gamma) := \{z \in V; \chi_{\tau}^\Gamma z = 0 \ \text{a.e. on } \Gamma\} \) for each \( \tau \in (0, 1] \) and \( \Gamma \in \Pi_c \). If \( \tau_n \downarrow 0 \) and \( \Gamma_n \in \Pi_c \), then there are a subsequence \( \{n_k\} \) of \( \{n\} \) and \( \Gamma \in \Pi_c \) such that \( \chi_{\Gamma_{n_k}}^\tau \rightarrow \chi_{\Gamma}^\tau \) in \( L^1(\partial \Omega) \) as \( k \rightarrow \infty \), and \( V(\tau_{n_k}, \Gamma_{n_k}) \rightarrow V(\Gamma) \) in \( V \) as \( k \rightarrow \infty \) in the sense of Mosco [6].

Now we propose a regular approximation for \( SP(\Gamma) \), referred as \( SP(\Gamma)^{\nu, \varepsilon, \tau, \delta} \), \( \nu, \varepsilon, \tau, \delta \in (0, 1] \), by the penalty method:

\[
\begin{aligned}
\rho^\nu(u), \lambda(w) &- \Delta u = f & &\text{in } Q, \\
u_t - \Delta(-\mu \Delta w_k - \kappa \Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) & = 0 & &\text{in } Q, \\
\frac{\partial u}{\partial n} & = -\frac{\chi^\tau_{\Gamma}}{\delta} (u - h_D) + (1 - \chi^\tau_{\Gamma})(h_N - n_0 u) & &\text{on } \Sigma, \\
\frac{\partial w}{\partial n} & = 0, & &\text{on } \Sigma, \\
u(0) & = u_{0\nu}, & &w(0) = w_0 \text{ in } \Omega.
\end{aligned}
\]

The notion of a weak solution of \( SP(\Gamma)^{\nu, \varepsilon, \tau, \delta} \) is given below.

**Definition 2.1.** A couple \( \{u, w\} \) of functions \( u : [0, T] \rightarrow V \) and \( w : [0, T] \rightarrow H^2(\Omega) \) is called a solution of \( SP(\Gamma)^{\nu, \varepsilon, \tau, \delta} \), if the following conditions (w1)' – (w4)' are satisfied:

(w1)' \( u \in W^{1,2}(0, T; H) \cap C([0, T]; V) \),

(w2)' \( u(0) = u_{0\nu}, w(0) = w_0 \).

(w3)' For all \( \varepsilon \in V \) and a.e. \( t \in [0, T] \),

\[
(\rho^\nu(u)'(t) + \lambda(w)'(t), z) + a(u(t), z) + (\frac{\chi^\tau_{\Gamma}}{\delta} (u(t) - h_D(t))) + (1 - \chi^\tau_{\Gamma})(h_N(t) - n_0 u(t)), z)_{\partial \Omega} = (f(t), z).
\]

(w4)' For all \( \eta \in H^2(\Omega) \) with \( \frac{\partial \eta}{\partial n} = 0 \) a.e. on \( \partial \Omega \) and a.e. \( t \in [0, T] \),

\[
(w'(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta \eta) = 0.
\]
According to a result in [4], \( SP(\Gamma)^{\nu\tau\delta} \) has a unique solution \( \{u, w\} \). Our regular approximate optimization problem \( P(\Pi_c)^{\nu\tau\delta} \) is to find \( \Gamma^*_{\nu\tau\delta} \in \Pi_c \) such that

\[
J^{\nu\tau\delta}(\Gamma^*_{\nu\tau\delta}) = \inf_{\Gamma \in \Pi_c} J^{\nu\tau\delta}(\Gamma),
\]

where

\[
J^{\nu\tau\delta}(\Gamma) := A \int_Q |u - u_d|^2 dx dt + B \int_C |w - w_d|^2 dt + C \int_{\Sigma} (1 - \chi^\tau_{\Gamma})|h_d|^2 d\sigma dt,
\]

\( \{u, w\} \) is the solution of \( SP(\Gamma)^{\nu\tau\delta} \).

Finally, we show a convergence result.

**Theorem 2.1.** Let \( \Pi_c \), \( \{\rho^\nu\} \), \( \{\beta^\tau\} \), \( \{\chi^\tau\} \) be as above. Then:

1. For \( \nu, \varepsilon, \tau, \delta \in (0, 1] \), \( P(\Pi_c)^{\nu\tau\delta} \) has at least one solution \( \Gamma^*_{\nu\tau\delta} \in \Pi_c \).
2. Let \( \{\nu_n\} \), \( \{\varepsilon_n\} \), \( \{\tau_n\} \) and \( \{\delta_n\} \) be any null sequences and let \( \{\Gamma_n := \Gamma^*_{\nu_n\varepsilon_n\tau_n\delta_n}\} \) be a sequence of solutions of \( P(\Pi_c)^{\nu_n\varepsilon_n\tau_n\delta_n} \). Then, \( \{\Gamma_n\} \) contains a subsequence convergent in \( \Pi \) and any limit \( \Gamma \) is a solution of \( P(\Pi_c) \).

For a detailed proof, see a forthcoming paper [3].

**References**


