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<th>SEMIALGEBRAIC VERSION OF THOM'S SECOND ISOTOPY LEMMA</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 952: 31-32</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60385">http://hdl.handle.net/2433/60385</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
SEMIALGEBRAIC VERSION OF
THOM'S SECOND ISOTOPY LEMMA

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In real singularities the most important maps are polynomial ones. Moreover, even if a specialist states a theorem by $C^\infty$ maps, he actually consider polynomial maps in mind. So it is natural to restrict our interest to polynomial maps. There are two kinds of equivalence relations on polynomial maps: $C^\infty$ equivalence and $C^0$ equivalence. Let us consider $C^0$ equivalence. It is said that $C^0$ equivalence is visual. But this is not correct, and means only that we consider problems without worrying about differentiability. $C^0$ equivalence is artificial and unnatural. By unnaturalness there are many strange phenomena. For example, recall the King's example of polynomial function germs $f, g: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularities such that $(\mathbb{R}^n, f^{-1}(0))$ and $(\mathbb{R}^n, g^{-1}(0))$ are $C^0$ equivalent but $f$ and $g$ are not $R - L C^0$ equivalent [K]. The homeomorphism germ of $C^0$ equivalence is constructed by infinite process, and since the process cannot be finitely controlled we can not extend the equivalence to $R - L C^0$ equivalence of $f$ and $g$. The example is a counter-example to a Thom's conjecture. We can not expect a beautiful theory on $C^0$ equivalence.

I propose semialgebraic equivalence in place of $C^0$ equivalence, which is defined by a homeomorphism with semialgebraic graph. Semialgebraic equivalence is strictly stronger than $C^0$ equivalence. Namely, 

(1) there exist two polynomial function germs which are $C^0$ equivalent but not semialgebraically equivalent [S].

On the other hand, semialgebraic equivalence is weaker than $C^1$ equivalence. Indeed, 

(2) two polynomial function germs are semialgebraically equivalent if they are $C^1$ equivalent [S].

A good property is the following, which is a positive answer to the above Thom's conjecture. 

(3) For two polynomial function germs $f, g: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, if $(\mathbb{R}^n, f^{-1}(0))$ and $(\mathbb{R}^n, g^{-1}(0))$ are semialgebraically equivalent, $f$ and $g$ are semialgebraically equivalent up to $\pm$, namely, $|f|$ and $|g|$ are semialgebraically equivalent [S].

Behavior of semialgebraic functions at infinity is strongly restricted. This is a reason why I expect a good theory of semialgebraic equivalence. Here we note only that 

(4) there exist two polynomial functions on $\mathbb{R}^8$ which are $C^\infty$ equivalent but not semialgebraically equivalent [S].
Almost all the known positive results on $C^0$ equivalence were proved only by the Thom's second isotopy lemma. Hence the first step to construct a theory of semialgebraic equivalence is to prove its semialgebraic version.

**Theorem** [S]. Let $\{X_i\}$ and $\{Y_j\}$ be semialgebraic $C^1$ Whitney stratifications of closed semialgebraic sets $X$ and $Y$, respectively, in $\mathbb{R}^n$, and let $f: X \to Y$ be a proper semialgebraic $C^1$ map such that for each $i$, $f(X_i)$ equals some $Y_j$ and $f|_{X_i}$ is a $C^1$ submersion onto $Y_j$. Let $p: Y \to \mathbb{R}^m$ be a proper semialgebraic $C^1$ map such that for each $j$, $p|_{Y_j}$ is a $C^1$ submersion onto $\mathbb{R}^m$. Assume $f$ is sans éclatement. Set

$$X(0) = (p \circ f)^{-1}(0), \quad Y(0) = p^{-1}(0).$$

There exist semialgebraic $C^0$ maps $\rho: X \to X(0)$ and $\xi: Y \to Y(0)$ such that $(\rho, p \circ f): X \to X(0) \times \mathbb{R}^m$ and $(\xi, p): Y \to Y(0) \times \mathbb{R}^m$ are homeomorphisms and the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(\rho, p \circ f)} & X(0) \times \mathbb{R}^m \\
| & f \downarrow & | f \times \text{id} \\
Y & \xrightarrow{(\xi, p)} & Y(0) \times \mathbb{R}^m 
\end{array}
$$

is commutative.

One of the corollaries is a version of Mather's $C^0$ Stability Theorem.

**Corollary.** Let $M \subset \mathbb{R}^n$ be a compact nonsingular algebraic variety. The family of semialgebraically stable polynomial maps is dense in the polynomial maps from $M$ to $\mathbb{R}^m$.

Let $r$ be a large integer and let $M_1$ and $M_2$ be semialgebraic $C^r$ manifolds in $\mathbb{R}^n$. The family of semialgebraically stable semialgebraic $C^r$ maps is dense in the semialgebraic $C^r$ maps from $M_1$ to $M_2$. (See [S] for the topology.)

**References**
