The second pluri-genus of surface singularities

筑波大学数学系 渡辺公夫 (Kimio Watanabe)

筑波大学数学研究科 奥間智弘 (Tomohiro Okuma)

1 Preliminary

Let \((X, x)\) be a normal \(n\)-dimensional isolated singularity over \(\mathbb{C}\) and \(f : (M, E) \to (X, x)\) a resolution of the singularity \((X, x)\) with exceptional locus \(E = f^{-1}(x)\). We say a resolution \(f\) is good if \(E\) is a divisor of normal crossings. Then the \(m\)-th \(L^2\)-plurigenus of \((X, x)\) is an integer \(\delta_m(X, x)\) which was introduced in [5] and can be computed as

\[
\delta_m(X, x) = \dim_{\mathbb{C}} H^0(M - E, \mathcal{O}_M(mK))/H^0(M, \mathcal{O}_M(mK + (m-1)E)),
\]

where \(K\) denotes the canonical divisor on \(M\).

The invariant \(p_\delta(X, x)\) called the geometric genus of a singularity \((X, x)\) is defined by

\[
p_\delta(X, x) = \dim_{\mathbb{C}}(R^{n-1}f_*\mathcal{O}_M)_x.
\]

We note that \(p_\delta(X, x) = \delta_1(X, x)\).
Normal surface singularities are classified as the following theorem.

(1.1) **Theorem**[1, Theorem 5.4], For a normal surface singularity $(X, x)$, we have the following:

1. $\delta_m(X, x) = 0$ for all $m \in \mathbb{N}$, when $(X, x)$ is a quotient singularity;
2. $\delta_m(X, x) \leq 1$ for all $m \in \mathbb{N}$ and $\delta_n(X, x) = 1$ for some $n \in \mathbb{N}$, in one of the following cases;
   a. $(X, x)$ is a simple elliptic singularity;
   b. $(X, x)$ is a cusp singularity;
   c. $(X, x)$ is the quotient by a finite group of a singularity of type (a) or (b);
3. $0 < \limsup_{m \to \infty} \delta_m(X, x)/m^2 < \infty$, in other cases.

(1.2) **Definition.** A normal surface singularity $(X, x)$ is of general type if which is in the class (3) above.

(1.3) **Definition.** A normal surface singularity $(X, x)$ is minimally elliptic if $p_g(X, x) = 1$ and $(X, x)$ is a Gorenstein singularity.

2 The minimally elliptic singularities

Throughout this section, we assume that $(X, x)$ is a minimally elliptic singularity, $f : (M, E) \to (X, x)$ the minimal good resolution with the irreducible decomposition $E = \bigcup_{i=1}^k E_i$ and $K$ the canonical divisor on $M$. 
which is supported on $E$.

Then $-K \geq E$, and $-K = E$ if and only if $(X, x)$ is a simple elliptic or a cusp singularity. If $(X, x)$ is not a simple elliptic singularity, then $E_i \cong \mathbb{P}^1$ for all $i$.

(2.1) Definition. Let $Z$ be the fundamental cycle on $M$. We say the dual graph of $E$ is obtained from another singularity $(X', x')$ if the self-intersection number of the fundamental cycle of $(X', x')$ is $-1$ and the weighted dual graphs of $(X, x)$ and $(X', x')$ are same except for self-intersection numbers of the components of $E$ with multiplicity 1 in $Z$.

In [6], we have followings.

(2.2) Theorem. For a minimally elliptic singularity $(X, x)$,

$$\delta_2(X, x) = \dim \mathcal{C}H^1(M, \mathcal{O}_M(2K + E))$$

If $(X, x)$ is of general type, $\delta_2(X, x) = KD + 2$, where $D = -K - E$.

(2.3) Corollary. If $(X, x)$ is a hypersurface (resp. complete intersection), then $\delta_2(X, x) \leq 4$ (resp. 5).

(2.4) Proposition. Let $(X, x)$ be a minimally elliptic singularity of general type. Then $\delta_2(X, x) = 1$ (resp. 2) if and only if $(X, x)$ is obtained from a unimodal (resp. bimodal) singularity, and $f$ is good if and only if
\[ \delta_2(X, x) \geq 2. \]

\textbf{(2.5) QUESTION.} Does the inequality \[ \delta_2 \geq m(X, x) \] (\(m(X, x)\) denotes the modality of a hypersurface singularity) hold?

That holds for quasi-homogeneous hypersurface singularities (see [7]).

3 The equisingular deformations

We follow the notation and terminology of the second section. We always assume that \((X, x)\) is a minimally elliptic singularity.

Let \(ES\) be the equisingular deformation functor in the sense of [3]. By [2], an equisingular deformation of \(M\) induces a topologically constant deformation of a singularity \((X, x)\).

By [3], \(ES\) is smooth and the tangent space of \(ES\) is \(H^1(M, S)\), where \(S = \mathcal{H}_{\mathcal{O}_M}(\Omega^1_{\mathcal{H}}(\log E), \mathcal{O}_M)\) (which is a locally free sheaf of rank 2).

\textbf{(3.1) DEFINITION.} We define an invariant \(q(X, x)\) called irregularity by

\[ q(X, x) = \dim_{\mathbb{C}}H^0(M - E, \Omega^1_{\mathcal{M} - E})/H^0(M, \Omega^1_{\mathcal{M}}). \]

If \((X, x)\) is a simple elliptic or not a quasi-homogeneous singularity, then \(q(X, x) = 0\), and \(q(X, x) = 1\) for every other (cf. [4, THEOREM 1.9]).

We denote by \(h^i(\cdot)(\text{resp. } h^1_{E}(\cdot))\) the dimension of \(\mathbb{C}\)-vector space \(H^i(\cdot)(\text{resp. } H^1_{E}(\cdot))\).
(3.2) PROPOSITION. If \((X, x)\) is of general type, then

\[ h^1(S) = q(X, x) + h^0(D, \Omega^1_M(\log E) \otimes \mathcal{O}_D(-E)), \]

where \(D = -K - E\).

Proof. By the duality, \(h^1(S) = h^1_E(\Omega^1_M(\log E) \otimes \mathcal{O}(K)).\) By Wahl's vanishing theorem, \(h^1(\Omega^1_M(\log E) \otimes \mathcal{O}(K)) = 0.\) Hence we have

\[ h^1_E(\Omega^1_M(\log E) \otimes \mathcal{O}(K)) = \dim_C H^0(M - E, \Omega^1_{M - E})/H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(K)) \]

and

\[ h^0(D, \Omega^1_M(\log E) \otimes \mathcal{O}_D(-E)) = \dim_C H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(E))/H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(K)). \]

From the exact sequence

\[ 0 \rightarrow H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(-E)) \rightarrow H^0(\Omega^1_M) \rightarrow H^0(\oplus \Omega^1_{E_i}) = 0, \]

and inclusions

\[ H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(K)) \subseteq H^0(\Omega^1_M(\log E) \otimes \mathcal{O}(-E)) \subseteq H^0(\Omega^1_M) \subseteq H^0(M - E, \Omega^1_M), \]

we have the assertion of the proposition.

In a similar way as above, we have \(h^1(S) = 1\) for simple elliptic singularities and \(h^1(S) = 0\) for cusp singularities.
(3.3) Example. Let \((X, x)\) be of general type and \(E = \bigcup_{i=0}^{r} E_i\) the irreducible decomposition. Assume that dual graph of \(E\) is star-shaped such that \(E_0 E_i = 1\) for \(i = 1, \ldots, r\), \(E_i E_j = 0\) if \(1 \leq i < j \leq r\) and \(E_0^2 = -r + 2\).

Then \(K = -2E_0 - \sum_{i=1}^{r} E_i\), and \(D = -K - E = E_0\). Hence \(\delta_2(X, x) = KD + 2 = r - 2\).

There is an isomorphism

\[ \Omega^1_M(\log E) \otimes \mathcal{O}_{E_0}(-E) \cong \mathcal{O}_{E_0}(-4 + r) \oplus \mathcal{O}_{E_0}(-2). \]

By (3.2), \(h^1(S) = q(X, x) + r - 3\). Hence \(\delta_2(X, x) = h^1(S) - q(X, x) + 1\).

If \((X, x)\) is a quasi-homogeneous hypersurface singularity, and moreover if the invariance of Milnor's number implies the invariance of the topological type, then \(\delta_2(X, x) = m(X, x)\).

By [4, (1.5),(1.6)], the exterior differentiation gives an exact sequence

\[ 0 \to d\mathcal{O}(K) \to \Omega^1_M(\log E) \otimes \mathcal{O}(K) \to \Omega^2_M \otimes \mathcal{O}(K + E) \to 0 \]

and isomorphisms \(H^i(\mathcal{O}(K)) \cong H^i(d\mathcal{O}(K))\) for all \(i\), where we consider \(\mathcal{O}(K)\) an ideal sheaf of \(\mathcal{O}_M\).

There is an exact sequence

\[ 0 \to H^0(d\mathcal{O}(K)) \to H^0(M - E, d\mathcal{O}) \to H^1_E(d\mathcal{O}(K)) \]

\[ \to H^1(d\mathcal{O}(K)) \cong H^1(\mathcal{O}(K)) = 0. \]

If \((X, x)\) is of general type, then \(H^1_{[x]}(d\mathcal{O}_X) = 0\) by [4,(1.13.4)]. Hence
$H^0(X, dO_X) \cong H^0(M - E, dO)$, and the map $H^0(dO(K)) \to H^0(M - E, dO)$ is surjective. Hence $H^1_E(dO(K)) = 0$.

By the exact sequence

$$H^1_E(dO(K)) \to H^1_E(\Omega^1_M(log E) \otimes O(K)) \to H^1_E(O(2K + E))$$

$$\to H^2_E(dO(K)) \to H^2_E(\Omega^1_M(log E) \otimes O(K)),$$

we have $\delta_2(X, x) = h^1(S) + \alpha$ for the singularity $(X, x)$ of general type, where

$$\alpha = \dim_C \text{Ker}(H^2_E(dO(K)) \to H^2_E(\Omega^1_M(log E) \otimes O(K))).$$

REFERENCES


