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The second pluri-genus of surface singularities

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1 Preliminary

Let \((X, x)\) be a normal \(n\)-dimensional isolated singularity over \(\mathbb{C}\) and \(f : (M, E) \to (X, x)\) a resolution of the singularity \((X, x)\) with exceptional locus \(E = f^{-1}(x)\). We say a resolution \(f\) is good if \(E\) is a divisor of normal crossings. Then the \(m\)-th \(L^2\)-plurigenus of \((X, x)\) is an integer \(\delta_m(X, x)\) which was introduced in [5] and can be computed as

\[
\delta_m(X, x) = \dim_{\mathbb{C}} H^0(M - E, \mathcal{O}_M(mK))/H^0(M, \mathcal{O}_M(mK + (m - 1)E)),
\]

where \(K\) denotes the canonical divisor on \(M\).

The invariant \(p_g(X, x)\) called the geometric genus of a singularity \((X, x)\) is defined by

\[
p_g(X, x) = \dim_{\mathbb{C}}(R^{n-1}f_*\mathcal{O}_M)_x.
\]

We note that \(p_g(X, x) = \delta_1(X, x)\).
Normal surface singularities are classified as the following theorem.

(1.1) **Theorem**[1, Theorem 5.4]. For a normal surface singularity $(X, x)$, we have the following:

1. $\delta_m(X, x) = 0$ for all $m \in \mathbb{N}$, when $(X, x)$ is a quotient singularity;
2. $\delta_m(X, x) \leq 1$ for all $m \in \mathbb{N}$ and $\delta_n(X, x) = 1$ for some $n \in \mathbb{N}$, in one of the following cases;
   a) $(X, x)$ is a simple elliptic singularity;
   b) $(X, x)$ is a cusp singularity;
   c) $(X, x)$ is the quotient by a finite group of a singularity of type (a) or (b);
3. $0 < \limsup_{m \to \infty} \delta_m(X, x)/m^2 < \infty$, in other cases.

(1.2) **Definition.** A normal surface singularity $(X, x)$ is of **general type** if which is in the class (3) above.

(1.3) **Definition.** A normal surface singularity $(X, x)$ is **minimally elliptic** if $p_g(X, x) = 1$ and $(X, x)$ is a Gorenstein singularity.

2 The minimally elliptic singularities

Throughout this section, we assume that $(X, x)$ is a minimally elliptic singularity, $f : (M, E) \to (X, x)$ the minimal good resolution with the irreducible decomposition $E = \bigcup_{i=1}^k E_i$ and $K$ the canonical divisor on $M$
which is supported on E.

Then $-K \geq E$, and $-K = E$ if and only if $(X, x)$ is a simple elliptic or a cusp singularity. If $(X, x)$ is not a simple elliptic singularity, then $E_i \cong \mathbb{P}^1$ for all $i$.

(2.1) DEFINITION. Let $Z$ be the fundamental cycle on $M$. We say the dual graph of $E$ is obtained from another singularity $(X', x')$ if the self-intersection number of the fundamental cycle of $(X', x')$ is $-1$ and the weighted dual graphs of $(X, x)$ and $(X', x')$ are same except for self-intersection numbers of the components of $E$ with multiplicity 1 in $Z$.

In [6], we have followings.

(2.2) THEOREM. For a minimally elliptic singularity $(X, x)$,

$$\delta_2(X, x) = \dim \mathcal{N} \mathcal{H}^1(M, \mathcal{O}_M(2K + E)).$$

If $(X, x)$ is of general type, $\delta_2(X, x) = KD + 2$, where $D = -K - E$.

(2.3) COROLLARY. If $(X, x)$ is a hypersurface (resp. complete intersection), then $\delta_2(X, x) \leq 4$ (resp. 5).

(2.4) PROPOSITION. Let $(X, x)$ be a minimally elliptic singularity of general type. Then $\delta_2(X, x) = 1$ (resp. 2) if and only if $(X, x)$ is obtained from a unimodal (resp. bimodal) singularity, and $f$ is good if and only if
\[ \delta_2(X, x) \geq 2. \]

(2.5) **Question.** Does the inequality \( \delta \geq m(X, x) \) (\( m(X, x) \) denotes the modality of a hypersurface singularity) hold?

That holds for quasi-homogeneous hypersurface singularities (see [7]).

3 The equisingular deformations

We follow the notation and terminology of the second section. We always assume that \((X, x)\) is a minimally elliptic singularity.

Let \( ES \) be the equisingular deformation functor in the sense of [3]. By [2], an equisingular deformation of \( M \) induces a topologically constant deformation of a singularity \((X, x)\).

By [3], \( ES \) is smooth and the tangent space of \( ES \) is \( H^1(M, S) \), where \( S = \mathcal{H}_{\mathcal{O}_M\mathcal{O}_M}(\Omega^1_M{\text{log}}E), \mathcal{O}_M) \) (which is a locally free sheaf of rank 2).

(3.1) **Definition.** We define an invariant \( q(X, x) \) called *irregularity* by

\[
q(X, x) = \dim_\mathbb{C}H^0(M - E, \Omega^1_M{E})/H^0(M, \Omega^1_M).
\]

If \((X, x)\) is a simple elliptic or not a quasi-homogeneous singularity, then \( q(X, x) = 0 \), and \( q(X, x) = 1 \) for every other (cf. [4, Theorem 1.9]).

We denote by \( h^i(\cdot) \) (resp. \( h^1_E(\cdot) \)) the dimension of \( \mathbb{C} \)-vector space \( H^i(\cdot) \) (resp. \( H^1_E(\cdot) \)).
(3.2) Proposition. If \((X, x)\) is of general type, then

\[ h^1(S) = q(X, x) + h^0(D, \Omega^1_M(\log E) \otimes O_D(-E)), \]

where \(D = -K - E\).

Proof. By the duality, \(h^1(S) = h^1_E(\Omega^1_M(\log E) \otimes O(K))\). By Wahl's vanishing theorem, \(h^1(\Omega^1_M(\log E) \otimes O(K)) = 0\). Hence we have

\[ h^1_E(\Omega^1_M(\log E) \otimes O(K)) = \dim_C H^0(M - E, \Omega^1_M(\log E) \otimes O(K))/H^0(\Omega^1_M(\log E) \otimes O(K)) \]

and

\[ h^0(D, \Omega^1_M(\log E) \otimes O_D(-E)) = \dim_C H^0(\Omega^1_M(\log E) \otimes O(-E))/H^0(\Omega^1_M(\log E) \otimes O(K)). \]

From the exact sequence

\[ 0 \to H^0(\Omega^1_M(\log E) \otimes O(-E)) \to H^0(\Omega^1_M) \to H^0(\Omega^1_E) \to 0, \]

and inclusions

\[ H^0(\Omega^1_M(\log E) \otimes O(K)) \subseteq H^0(\Omega^1_M(\log E) \otimes O(-E)) \subseteq H^0(\Omega^1_M) \subseteq H^0(M - E, \Omega^1_M), \]

we have the assertion of the proposition.

In a similar way as above, we have \(h^1(S) = 1\) for simple elliptic singularities and \(h^1(S) = 0\) for cusp singularities.
(3.3) Example. Let \((X, x)\) be of general type and \(E = \bigcup_{i=0}^{r} E_i\) the irreducible decomposition. Assume that dual graph of \(E\) is star-shaped such that \(E_0 E_i = 1\) for \(i = 1, \ldots, r\), \(E_i E_j = 0\) if \(1 \leq i < j \leq r\) and \(E_0^2 = -r + 2\).

Then \(K = -2E_0 - \sum_{i=1}^{r} E_i\), and \(D = -K - E = E_0\). Hence \(\delta_2(X, x) = KD + 2 = r - 2\).

There is an isomorphism

\[ \Omega^1_M(\log E) \otimes \mathcal{O}_{E_0}(-E) \cong \mathcal{O}_{E_0}(-4 + r) \oplus \mathcal{O}_{E_0}(-2). \]

By (3.2), \(h^1(S) = q(X, x) + r - 3\). Hence \(\delta_2(X, x) = h^1(S) - q(X, x) + 1\).

If \((X, x)\) is a quasi-homogeneous hypersurface singularity, and moreover if the invariance of Milnor's number implies the invariance of the topological type, then \(\delta_2(X, x) = m(X, x)\).

By [4, (1.5),(1.6)], the exterior differentiation gives an exact sequence

\[ 0 \rightarrow d\mathcal{O}(K) \rightarrow \Omega^1_M(\log E) \otimes \mathcal{O}(K) \rightarrow \Omega^2_M \otimes \mathcal{O}(K + E) \rightarrow 0 \]

and isomorphisms \(H^i(\mathcal{O}(K)) \cong H^i(d\mathcal{O}(K))\) for all \(i\), where we consider \(\mathcal{O}(K)\) an ideal sheaf of \(\mathcal{O}_M\).

There is an exact sequence

\[ 0 \rightarrow H^0(d\mathcal{O}(K)) \rightarrow H^0(M - E, d\mathcal{O}) \rightarrow H^1_E(d\mathcal{O}(K)) \rightarrow H^1(d\mathcal{O}(K)) \cong H^1(\mathcal{O}(K)) = 0. \]

If \((X, x)\) is of general type, then \(H^1_{[x]}(d\mathcal{O}_X) = 0\) by [4,(1.13.4)]. Hence
$H^0(X,d\mathcal{O}_X) \cong H^0(M - E, d\mathcal{O}),$ and the map $H^0(d\mathcal{O}(K)) \to H^0(M - E, d\mathcal{O})$ is surjective. Hence $H^1_E(d\mathcal{O}(K)) = 0.$

By the exact sequence

$$
H^1_E(d\mathcal{O}(K)) \to H^1_E(\Omega^1_M(\log E) \otimes \mathcal{O}(K)) \to H^1_E(\mathcal{O}(2K + E))
$$

$$
\to H^2_E(d\mathcal{O}(K)) \to H^2_E(\Omega^1_M(\log E) \otimes \mathcal{O}(K)),
$$

we have $\delta_2(X, x) = h^1(S) + \alpha$ for the singularity $(X, x)$ of general type, where

$$\alpha = \dim \text{Ker}(H^2_E(d\mathcal{O}(K)) \to H^2_E(\Omega^1_M(\log E) \otimes \mathcal{O}(K))).$$

**References**


